Advice Automatic Structures and Uniformly Automatic Classes

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Abstract

We study structures that are automatic with advice. These are structures that admit a presentation by finite automata (over finite or infinite words or trees) with access to an additional input, called an advice. Over finite words, a standard example of a structure that is automatic with advice, but not automatic in the classical sense, is the additive group of rational numbers $(\mathbb{Q}, +)$.

By using a set of advices rather than a single advice, this leads to the new concept of a parameterised automatic presentation as a means to uniformly represent a whole class of structures. The decidability of the first-order theory of such a uniformly automatic class reduces to the decidability of the monadic second-order theory of the set of advices that are used in the presentation. Such decidability results also hold for extensions of first-order logic by regularity preserving quantifiers, such as cardinality quantifiers and Ramsey quantifiers.

To investigate the power of this concept, we present examples of structures and classes of structures that are automatic with advice but not without advice, and we prove classification theorems for the structures with an advice automatic presentation for several algebraic domains. In particular, we prove that the class of all torsion-free Abelian groups of rank one is uniformly ω-automatic and that there is a uniform ω-tree-automatic presentation of the class of all Abelian groups up to elementary equivalence and of the class of all countable divisible Abelian groups. On the other hand we show that every uniformly ω-automatic class of Abelian groups must have bounded rank.

While for certain domains, such as trees and Abelian groups, it turns out that automatic presentations with advice are capable of presenting significantly more complex structures than ordinary automatic presentations, there are other domains, such as Boolean algebras, where this is provably not the case. Further, advice seems to not be of much help for representing some particularly relevant examples of structures with decidable theories, most notably the field of reals.

Finally we study closure properties for several kinds of uniformly automatic classes, and decision problems concerning the number of non-isomorphic models in uniformly automatic classes with the unique representation property.

1 Introduction

Automatic structures are structures that allow finite presentations by automata. Roughly speaking, a structure is called automatic if its domain can be represented as a regular set in such a way that its relations become recognisable by synchronous multi-tape automata.

The history of automatic structures can be traced back to the early days of automata theory, for instance to the automata theoretic decision procedures by Büchi and Rabin for Presburger arithmetic and other theories. A more systematic investigation has been started by Khoussainov and Nerode [14], who also coined the term automatic structures. In [5] the concept was lifted from finite words to automata that read trees as well as their infinite counterparts. For a more elaborate introduction to the topic we refer the reader to [3, 20].

An important research objective in the field of automatic structures is to determine which structures admit automatic presentations and to characterise all automatic models inside certain classes of structures. For instance, a long standing open problem had been whether the additive group of the rational numbers is automatic, until Tsankov [23] gave a negative answer to the question. It has been noted, however, that $(\mathbb{Q}, +)$ is "almost" automatic in the sense that there is a presentation in which addition is automatic but the domain is not a regular set [19]. Kruckman et al. remarked in [16] that the domain is also recognisable by an automaton, provided that it has access to a specific infinite advice string. Moreover this advice string itself has a decidable monadic second-order theory, which is sufficient to give an automata-based decision procedure for the first-order theory of $(\mathbb{Q}, +)$.

This motivates our study of advice automatic structures. A structure is advice automatic if it has an automatic presentation in the same way as $(\mathbb{Q}, +)$ does: it can be presented by automata that have access to some fixed advice. This setting has appeared occasionally in the literature [7, 13] but to the authors knowledge no systematic investigation has been done so far. Advice automatic structures are interesting, because they generalise the domain of infinite structures that admit automata-based finite presentations while, as we shall prove, preserving the good algorithmic and model-theoretic properties of automatic structures, in particular the decidability of their first-order theories. But there is a further very interesting twist: Automata with advice permit us to lift the notion of an automatic presentation from single structures to classes of structures that can be represented by a single presentation, but with a set of different advices. This will lead us to the concept of uniformly automatic classes of structures.

We shall in fact introduce several variants of this concept. Of course, not all advice sets give us classes of structures with a decidable theory since one can easily encode undecidable problems inside the set of advices, or even in a single advice. But any class of structures that admits an automatic presentation with an advice set that has a decidable monadic second-order theory does indeed have an automata-based decision procedure for its first-order theory, and even for the extension of first-order logic by different variants of cardinality quantifiers and by Ramsey quantifiers. These results show that automatic presentations with advice provide relevant generalisations of the concept of automata-based representations of infinite structures, and that the algorithmic properties, which make automatic structures suitable for applications, survive under these generalisations.

We then investigate the power of this concept. We identify classes of structures, such as trees and Abelian groups, where automatic presentations with advice are capable of presenting significantly more complex structures than ordinary automatic presentations. Among other results we provide a uniformly ω-automatic presentation of the torsion-free Abelian groups of rank one and a uniformly ω-tree automatic presentation for the class of

all countable divisible Abelian groups and the class of all Abelian groups up to elementary equivalence. We also prove limitations of this concept by identifying classes, such as the class of all Boolean algebras, where we do not gain anything essential from the access to an advice. We further show that every uniformly ω-automatic class of countable Abelian groups must have bounded rank and extend known non-automaticity results to the case of advice automatic structures. In particular, it turns out that an advice does not help for representing some particularly relevant examples of structures with decidable theories, most notably the field of reals.

Further we investigate closure properties for uniformly automatic classes. We show that whenever a class of structures is uniformly tree- or ω-tree-automatic then this is also true for the closure under direct products and the closure under disjoint unions, a property that is not shared by the uniformly ω-automatic classes.

Finally, we study decidability issues for counting the number of non-isomorphic models inside uniformly automatic classes that have the unique representation property, i.e. where distinct advices always give non-isomorphic structures. While such counting problems often are decidable, the unique representation property itself turns out to be undecidable even in the simplest conceivable cases of regularly automatic classes.

2   Automatic Presentations with Advice

For two words $v, w \in \Sigma^*$ the **convolution** $v \otimes w$ is a word over the alphabet $(\Sigma \uplus \{\Box\})^2$ of length $\max(|v|, |w|)$ with

$$(v \otimes w)(i) = \begin{cases} (v(i), w(i)) & \text{if } i < \min(|v|, |w|) \\ (\Box, w(i)) & \text{if } |v| \leq i < |w| \\ (v(i), \Box) & \text{if } |w| \leq i < |v|. \end{cases}$$

The convolution of two ω-words is defined analogously with the difference that a padding symbol is not needed. The convolution of trees follows the same idea. For two Σ-labelled trees s, t the convolution $s \otimes t$ is the $(\Sigma \uplus \{\Box\})^2$-labelled tree with $\text{dom}_{s \otimes t} = \text{dom}_s \cup \text{dom}_t$ and the labelling

$$(s \otimes t)(w) = \begin{cases} (s(w), t(w)) & \text{if } w \in \text{dom}_s \cap \text{dom}_t \\ (\Box, t(w)) & \text{if } w \in \text{dom}_t \setminus \text{dom}_s \\ (s(w), \Box) & \text{if } w \in \text{dom}_s \setminus \text{dom}_t. \end{cases}$$

Instead of $w_1 \otimes w_2 \otimes \cdots \otimes w_n$ we will often write $\langle w_1, w_2, \ldots, w_n \rangle$, and for a language L we let $L^{\otimes n} := \{\langle w_1, \ldots, w_n \rangle \mid w_1, \ldots, w_n \in L\}$. For $I \subseteq \mathbb{N}$ let wI denote the subword of w that consists of the letters at positions in I. We denote the **prefix-relation** on words by $<_p$, the **length-lexicographical relation** by $<_{llex}$, and for $m \in \mathbb{N}$ the **m-equal-ends** relation $\sim_e^m$ on infinite words is given by $v \sim_e^m w$ if, and only if, $v[m, \infty) = w[m, \infty)$. The **equal-ends** relation $\sim_e$ is the union of the $\sim_e^m$, i.e. $v \sim_e w$ if and only if $v \sim_e^m w$ for some $m \in \mathbb{N}$.

We assume familiarity with the classical models of finite automata and extend their semantics to define languages that are *regular with advice*.

▶ **Definition 1.** A **parameterised Muller automaton** is a Muller automaton $\mathcal{A}$ over the alphabet $\Sigma \times \Gamma$. For $\alpha \in \Gamma^\omega$ **the language that $\mathcal{A}$ recognises with advice** α is $L(\mathcal{A}[\alpha]) := \{\beta \in \Sigma^\omega \mid \beta \otimes \alpha \in L(\mathcal{A})\}$. In this case we also say L is recognised by $\mathcal{A}[\alpha]$. A language L is called ω-**regular with advice** α if there is a parameterised Muller automaton $\mathcal{A}$ with $L = L(\mathcal{A}[\alpha])$.

Parameterised automata on finite words and finite or infinite trees are defined analogously.

An automaton $\mathcal{A}$ **recognises a relation** (possibly with advice) R if it recognises the language $\{a_1 \otimes \ldots \otimes a_k \mid (a_1, \ldots, a_k) \in R\}$ of all convolutions of tuples in R. Given a relation $R \subseteq A^s \times A^t$ and tuples $\bar{a} \in A^s$, $\bar{b} \in A^t$, we will frequently also use the notation $\bar{a}R$ and $R\bar{b}$ to denote the projections $\{\bar{b} \in A^t \mid (\bar{a}, \bar{b}) \in R\}$ and $\{\bar{a} \in A^s \mid (\bar{a}, \bar{b}) \in R\}$, respectively. We are now ready to introduce the notion of an automatic presentation with advice. It is a general concept but we state it for automata on infinite words.

▶ **Definition 2.** Let τ be a finite relational signature. An ω-**automatic presentation with advice** α is a tuple $\mathfrak{d} = (\mathcal{A}, \mathcal{A}_{\approx}, (\mathcal{A}_R)_{R \in \tau})$ of parameterised Muller automata such that:

- $A_\alpha := L(\mathcal{A}[\alpha])$ presents the universe of a structure.
- For $R \in \tau$ of arity r, $\mathcal{A}_R[\alpha]$ recognises an r-ary relation R_α on A_α.
- $\mathcal{A}_{\approx}[\alpha]$ recognises a binary congruence relation $\approx_\alpha$ on the structure $(A_\alpha, (R_\alpha)_{R \in \tau})$.

The **induced structure of $\mathfrak{d}$ and α** is $\mathcal{S}_{\approx}(\mathfrak{d}[\alpha]) := (A_\alpha, \approx_\alpha, (R_\alpha)_{R \in \tau})$ and we say that $\mathfrak{d}[\alpha]$ **presents the structure** $\mathcal{S}(\mathfrak{d}[\alpha]) := \mathcal{S}_{\approx}(\mathfrak{d}[\alpha]) / \approx_\alpha$. In the case that $\approx_\alpha$ is just the identity we say that **the presentation is injective** and omit $\mathcal{A}_{\approx}$ in our notation.

We say that a τ-structure $\mathfrak{A}$ is ω-**automatic with advice** if there is a parameterised ω-automatic presentation $\mathfrak{d}$ with $\mathfrak{A} \cong \mathcal{S}(\mathfrak{d}[\alpha])$ for some parameter α. In our applications we will often assume that we have also fixed a witnessing isomorphism $\pi : \mathcal{S}(\mathfrak{d}[\alpha]) \to \mathfrak{A}$. Note that such an isomorphism π extends in a natural way to a strong homomorphism $\pi_{\approx} : \mathcal{S}_{\approx}(\mathfrak{d}[\alpha]) \to \mathfrak{A}$. Further we extend $\pi_{\approx}$ to convolutions of words in $L(\mathcal{A}[\alpha])$ in the obvious way.

By changing the automata model we obtain analogous notions of structures that are, for instance, word-automatic with advice or tree-automatic with advice.

3 Uniformly Automatic Classes

Although to our knowledge the concept of a uniformly automatic class has not been explicitly studied in literature there are of course several examples where the underlying idea has been very successfully applied in various areas of computer science. This includes the following insights:

- The class of all *countable linear orders* is regularly ω-tree-automatic.
- For any fixed $d \in \mathbb{N}$, the class of finite graphs of *tree width* at most d and the class of finite graphs of *clique width* at most d are regularly tree-automatic.
- For any fixed $d \in \mathbb{N}$, the class of finite graphs of *path width* at most d and the class of finite graphs of *linear clique width* at most d are regularly automatic.

To make this precise and more general, we introduce the following definitions. As above, we state them in terms of automata on infinite words. The corresponding variants based on automata on finite words, or on finite or infinite trees are completely analogous.

▶ **Definition 3.** A class of τ-structures $\mathcal{C}$ is **uniformly ω-automatic** if there is a parameterised ω-automatic presentation $\mathfrak{c}$ and a set of parameters P, so that $\mathcal{S}(\mathfrak{c}[P]) := \{\mathcal{S}(\mathfrak{c}[\alpha]) \mid \alpha \in P\}$ is equal to $\mathcal{C}$ up to isomorphism. If P has a decidable MSO-theory we say that $\mathcal{C}$ is **strongly ω-automatic**. If P is even regular then we say that $\mathcal{C}$ is **regularly ω-automatic**. In this case we call a tuple $(\mathcal{A}_p, \mathfrak{c})$ with $L(\mathcal{A}_p) = P$ a **regularly ω-automatic presentation** of $\mathcal{C}$.

What makes automatic structures so interesting for applications in computer science is that there is an effective decision procedure for the FO-theory of every automatic structure. We outline how to get an effective decision procedure for the FO-theory of a strongly automatic class. The decision procedure works by recursively building the union, complement or projection automaton from automata that recognise the relations defined by subformulae. Since advice automata are basically ordinary synchronous multi-tape automata with a designated advice tape, advice regular relations are also effectively closed under union, complement and projection and thus the following theorem holds.

▶ **Theorem 4.** *There is an algorithm which, given a parameterised ω-automatic presentation $\mathfrak{d}$ and a FO-formula $\varphi(\overline{x})$ over the signature of $\mathfrak{d}$, constructs an automaton $\mathcal{A}_\varphi$ with $L(\mathcal{A}_\varphi) = \{\overline{a} \otimes \alpha \,|\, \mathcal{S}_\approx(\mathfrak{d}[\alpha]) \models \varphi(\overline{a})\}$.*

Given a sentence φ the algorithm constructs an automaton with $L(\mathcal{A}_\varphi) = \{\alpha \,|\, \mathcal{S}_\approx(\mathfrak{d}[\alpha]) \models \varphi\}$. Deciding whether φ is in the FO-theory of a class $\mathcal{S}(\mathfrak{d}[P])$ thus reduces to deciding the inclusion problem $P \subseteq L(\mathcal{A}_\varphi)$. The well-known correspondence theorems between MSO-definable languages and regular languages imply that there is a MSO-sentence ψ with $\alpha \models \psi$ if, and only if, $\alpha \in L(\mathcal{A}_\varphi)$ and thus the inclusion problem reduces to checking whether ψ holds in every $\alpha \in P$, which proves claims 2 and 3 of the following corollary. Claims 1 and 4 follow from Theorem 4 analogously to the case of automatic structures. We refer to [4] for an introduction to FO-interpretations in the context of automatic structures.

▶ **Corollary 5.** **1.** *The class of ω-automatic structures with advice α is effectively closed under FO-interpretations.*
2. *The FO-theory of a structure that is ω-automatic with advice α is decidable if the MSO-theory of α is decidable.*
3. *The FO-theory of a strongly ω-automatic class is decidable.*
4. *If $\mathcal{C}$ is FO-interpretable in a uniformly ω-automatic class $\mathcal{D}$ then $\mathcal{C}$ is also uniformly ω-automatic.*
The analogous automatic, tree- and ω-tree-automatic versions of these statements hold true as well.

Analogous versions of Theorem 4 further hold for extensions of FO by regularity preserving quantifiers, i.e. evaluation of a formula with regularity preserving quantifiers in an automatic structure yield effectively a regular relation again. We will later make use of the cardinality quantifiers $\exists^\infty/\exists^{>\aleph_0}/\exists^{(k,m)}$, meaning "there exists infinitely/uncountably/k mod m/ many", which are regularity preserving for automatic and ω-automatic structures [12]. FOC denotes the extension of FO by $\exists^\infty, \exists^{>\aleph_0}, \exists^{(k,m)}$.

It is known that every countable ω-automatic structure is automatic [12]. We can generalise this result to uniformly ω-automatic classes of countable structures in the following sense: We say that an ω-automatic presentation is a **presentation over finite words** if the elements of the domain(s) of the structure(s) are encoded in a subset of $\Sigma^*\{\square\}^\omega$. When a finite words presentation is given we will for brevity often write w for $w\square^\omega$.

▶ **Theorem 6.** *A class $\mathcal{C}$ of countable structures has a parameterised ω-automatic presentation $\mathfrak{c}$ with parameter set P, if, and only if, it has an injective parameterised ω-automatic presentation over finite words $\mathfrak{c}'$ with the same parameter set P. Moreover $\mathfrak{c}'$ can be effectively constructed from $\mathfrak{c}$.*

Another example for a quantifier that is regularity preserving for automatic structures is the Ramsey quantifier. For any $k \geq 1$, the *k-Ramsey quantifier* $\exists^{\text{k-ram}}$ is defined by $\mathfrak{A} \models \exists^{\text{k-ram}}\overline{x}\varphi(\overline{x},\overline{c})$ if, and only if, there is an infinite $X \subseteq A$ so that $\mathfrak{A} \models \varphi(a_1,\ldots,a_k,\overline{c})$ for all pairwise different $a_1,\ldots,a_k \in X$.

The Ramsey quantifier is not regularity preserving anymore for ω-automatic structures [17]. As the following lemma implies it is however regularity preserving for countable ω-automatic structures with advice. We will also make use of the following lemma in the proof of Theorem 17.

▶ **Lemma 7.** *Let $\mathfrak{d}$ be an ω-automatic presentation over finite words with advice α, and $P = \{\overline{p} \in A_\alpha^l \mid \mathcal{S}_\approx(\mathfrak{d}[\alpha]) \models \exists^{\text{k-ram}}\overline{x}R_\alpha\overline{x}\overline{p}\}$, where R_α is a $(k+l)$-ary relation of $\mathcal{S}_\approx(\mathfrak{d}[\alpha])$. Then there is a subset $A' \subseteq A_\alpha \otimes P$ that is ω-regular with advice α so that for all $\overline{p} \in P$ the sets $A'\overline{p} := \{a \in A_\alpha \mid a \otimes \overline{p} \in A'\}$ are infinite with $\mathcal{S}_\approx(\mathfrak{d}[\alpha]) \models R_\alpha(a_1, \dots, a_k, \overline{p})$ for all pairwise different $a_1, \dots, a_k \in A'\overline{p}$.*

A parameterised Muller automaton $\mathcal{A}'$ with $L(\mathcal{A}'[\alpha]) = A'$ can be effectively constructed.

Proof. Let Σ be the alphabet of the presentation $\mathfrak{d}$. Consider ω-words of the form $s \otimes t$ with $s = s_0 s_1 \dots$ and $t = t_0 t_1 \dots$ such that $|s_i| = |t_i|$ and $s_i \in \Sigma^* \{\dot{a} \mid a \in \Sigma\}$, $t_i \in \Sigma^*$. Say that a word $x \in \Sigma^*$ is *on* $s \otimes t$, if there is an i so that $x = s_0 \dots s_i t_{i+1}$ (ignoring dots on letters). Let $On(s \otimes t)$ be the set of words that are on $s \otimes t$. It is not hard to construct a parameterised Muller automaton $\mathcal{A}$, so that $\mathcal{A}[\alpha]$ recognises exactly those ω-words of the form $s \otimes t \otimes \overline{p}$ with $\mathcal{S}_\approx(\mathfrak{d}[\alpha]) \models R_\alpha(x_1, \dots, x_k, \overline{p})$ for all pairwise different $x_1, \dots, x_k \in On(s \otimes t)$. Applying the uniformization theorem for ω-regular relations [6] to the $(2 + l + 1)$-ary relation recognised by $\mathcal{A}$, we get a Muller automaton $\mathcal{U}$ so that for every $\overline{p} \in P$ there is *at most* one $s \otimes t$ with $s \otimes t \otimes \overline{p} \otimes \alpha \in L(\mathcal{U}) \Leftrightarrow s \otimes t \otimes \overline{p} \in L(\mathcal{U}[\alpha])$. From $\mathcal{U}$ we can easily construct another parameterised Muller automaton $\mathcal{A}'$ which on input $\overline{p}$ guesses a $s \otimes t$ with $s \otimes t \otimes \overline{p} \in L(\mathcal{U}[\alpha])$ and uses it to recognise $On(s \otimes t) \otimes \overline{p}$, i.e. $L(\mathcal{A}'[\alpha]) = \{w \otimes \overline{p} \mid w \in On(s \otimes t)\}$. It remains to show that for each $\overline{p} \in P$ there is at least one $s \otimes t$ with $s \otimes t \otimes \overline{p} \in L(\mathcal{A})$. Let $\overline{p} \in P$ and $X \subseteq \Sigma^*$ be an infinite set with $\mathcal{S}_\approx(\mathfrak{d}[\alpha]) \models R_\alpha(x_1, \dots, x_k, \overline{p})$ for all pairwise distinct $x_1, \dots, x_k \in X$. Consider the subtree of $(\Sigma^*, \leq_p)$ that is generated by the prefix-closure of X. According to König's Lemma there is an infinite path $\gamma \in \Sigma^\omega$ in this tree so that from every node on the path a node in X is reachable. We define inductively words $s_i, t_i \in \Sigma^*$, so that the following invariants hold: 1. $|s_i| = |t_i|$ for all $i \in \mathbb{N}$, 2. $s_0 \dots s_i$ is a prefix of γ for all $i \in \mathbb{N}$ and 3. $s_0 \dots s_i t_{i+1} \in X$ for all $i \in \mathbb{N}$. Define $s_0 := \varepsilon, t_0 := \varepsilon$, t_{i+1} as a shortest path from $s_0 \dots s_i$ to a node in X and s_{i+1} as the path of length $|t_{i+1}|$ so that $s_0 \dots s_{i+1}$ remains a prefix of γ for all $i \in \mathbb{N}$. ◀

4   Examples and Classifications for Uniformly Automatic Classes

We shall now provide examples of structures that admit automatic presentations with advice, and of uniformly automatic classes. For every concept of structures, and classes of structures, that admit a certain type of finite presentation, it is of course relevant to understand which structures and classes actually fall under this concept. We study this question here for infinite trees, for Abelian groups, for Boolean algebras and some further algebraic domains.

4.1   Infinite Trees

Consider the infinite $|\Sigma|$-ary tree $\mathfrak{S}_\Sigma := (\Sigma^*, \leq_p, (S_a)_{a \in \Sigma}, \text{el})$ with successor relations $S_a := \{(w, wa) : w \in \Sigma^*\}$ and equal-level relation $\text{el} := \{(w, v) \in \Sigma^* \times \Sigma^* : |w| = |v|\}$. It is well known that $\mathfrak{S}_\Sigma$ has an automatic presentation [5]. In the following we want to consider two ways to generalise the tree $\mathfrak{S}_\Sigma$ to obtain two different types of uniformly automatic classes of infinite trees with equal-level relation el and ancestor relation $\leq_p$. In the first generalisation we only consider trees with bounded node degree, but relax the condition that the subtree at each node has the same isomorphism type, which is the property that characterises the

trees $\mathfrak{S}_\Sigma$. Instead we allow that the trees have on each level i at most $|Q|$ many different isomorphism types of subtrees for a constant $|Q|$. Trees of this kind can also be characterised as the unwindings of advice automata, where each path follows at each step i a transition from a new transition relation $\Delta(i) \subseteq Q \times \Sigma \times Q$.

▶ **Example 8.** Let Q, Σ be finite sets, $q_0 \in Q$ and $\Delta \subseteq Q \times \Sigma \times Q \times \mathbb{N}$.

The tree $\mathfrak{S}_{\Sigma,\Delta} := (\tilde{\Delta}, \leq_p, (\tilde{\Delta}_a)_{a \in \Sigma}, \text{el})$ that consists of all Σ-labelled finite paths $\tilde{\Delta} :=$ $\{(q_0, a_0, q_1)(q_1, a_1, q_2)(q_2, a_2, q_3)...(q_n, a_n, q_{n+1}) \in (Q \times \Sigma \times Q)^* : n \in \mathbb{N}, (q_i, a_i, q_{i+1}, i) \in \Delta\}$ with successor relations $\tilde{\Delta}_a := \{(w, v) \in \tilde{\Delta} \times \tilde{\Delta} : v = w(q, a, p) \wedge (q, a, p, |w|) \in \Delta\}$ has an ω-automatic presentation with advice.

Furthermore for any fixed Q, Σ the class $\mathcal{T}_{Q,\Sigma} := \{\mathfrak{S}_{\Sigma,\Delta} : \Delta \subseteq Q \times \Sigma \times Q \times \mathbb{N}\}$ is regularly ω-automatic. To see this, let $\Gamma := \mathcal{P}(Q \times \Sigma \times Q)$ be the advice alphabet. It is easy to construct a uniform ω-automatic presentation that represents $\mathfrak{S}_{\Sigma,\Delta}$ with the advice $\Delta(0)\Delta(1) \ldots \in \Gamma^\omega$, where $\Delta(i) := \{(q, a, p) : (q, a, p, i) \in \Delta\}$ for all $i \in \mathbb{N}$. Moreover the set of parameters Γ^ω is an ω-regular set.

Next we consider the class of trees with the property that each node has finite degree and all nodes of the same depth have the same number of a-successors for each $a \in \Sigma$.

▶ **Example 9.** $\mathcal{C}_\Sigma := \{(T, \leq_p, (S_a)_{a \in \Sigma}, \text{el}) : \forall a \in \Sigma \forall (t, t') \in \text{el} : |tS_a| < \infty \wedge |tS_a| = |t'S_a|\}$, where tS_a denotes the set of a-successors of t, is uniformly ω-automatic.

Choose $\Gamma := \Sigma \cup \{\#\}$ as the advice alphabet. Then the tree $(T, \leq_p, (S_a)_{a \in \Sigma}, \text{el}) \in \mathcal{C}_\Sigma$ can be represented with an advice of the form $\alpha := \alpha_0 \# \alpha_1 \# \ldots$ where $\alpha_i \in \Sigma^*$ such that for all $a \in \Sigma$ $|\alpha_i|_a = |tS_a|$ for any $t \in T$ of depth i. Code the domain of the tree by $D_T := \{w_0 \# w_1 \# \ldots \# w_n : n \in \mathbb{N}, \text{ for all } i \leq n : w_i \in 0^*10^* \wedge |w_i| = |\alpha_i|\}$ which is obviously regular with advice α. $\leq_p$ and el are just the regular prefix-relation and equal-length relation on words and $S_a := \{(x, x\#0^i10^k) \in D_T \times D_T : x = w_0 \# w_1 \# ... \# w_{n-1} \wedge \alpha_n(i) = a\}$ is also regular with advice α.

4.2 Abelian Groups

We recall some standard notions and facts of Abelian group theory. The *order* of an element a in an Abelian group is the smallest positive integer n with $n \cdot a = 0$, or ∞ if no such n exists. A group is *torsion-free*, if the neutral element is the only element of finite order in the group. A group is *periodic*, if all of its elements have finite order. The *rank* of an Abelian group is the cardinality of a maximal subset S of the group that is linearly independent over $\mathbb{Z}$, i.e. such that for any nonempty finite subset $F \subseteq S$ the equation $\sum_{a \in F} z_a \cdot a = 0$ in the variables $(z_a)_{a \in F}$ has over $\mathbb{Z}$ only the trivial solution $z_a = 0$ for all $a \in F$.

The torsion-free Abelian groups of rank n coincide up to isomorphism with the subgroups of $(\mathbb{Q}^n, +)$ [8], so that it is sufficient to consider a classification of those. A complete classification of the subgroups of $(\mathbb{Q}, +)$ has long been known [2]. The classification problem of the torsion-free Abelian groups of rank n for $n \geq 2$ on the other side seems to be much more intricate and is an active research area of infinite Abelian group theory [22]. We recall here Baer's classification of the subgroups of $(\mathbb{Q}, +)$. Let $\mathbb{P}$ be the set of prime numbers. Every sequence $c := (c_p)_{p \in \mathbb{P}}$ with $c_p \in \mathbb{N} \cup \{\infty\}$ for all $p \in \mathbb{P}$ corresponds to the subgroup $(\mathbb{Q}_c, +) := \left(\left\{ \frac{z}{p_1^{d_1} \ldots p_k^{d_k}} \mid z \in \mathbb{Z}, p_i \in \mathbb{P}, d_i \in \mathbb{N}, d_i \leq c_{p_i} \right\}, + \right)$ of $(\mathbb{Q}, +)$ and every subgroup of $(\mathbb{Q}, +)$ is isomorphic to a group of the form $(\mathbb{Q}_c, +)$ for some c.

It has already been noted in [16, 19] that addition of rational numbers is advice automaton recognizable, if rationals are encoded as digit sequences $(d_i)_i$ of their *factorial base*

representation. Every rational number $r \in [0,1)$ can be written as $r = \sum_{i=2}^{n} \frac{d_i}{i!}$ with $d_i < i$. The following lemma generalises the factorial base representation, so that all and only the elements of a subgroup $(\mathbb{Q}_c, +)$ have a representation in a generalised factorial base which depends on c. For this sake we substitute for a given sequence of natural numbers $n = (n_i)_{i \in \mathbb{N}}$ the factorial $i!$ by a generalised factorial $n_{i!} := n_{i-1} n_{i-2} \ldots n_0$. We define the index notation $\underline{z!}$ for any $z \in \mathbb{Z}$ via $n_{\underline{z!}} := (\prod_{i < |z|} n_i)^{\mathrm{sgn}(z)}$, where $\mathrm{sgn}(z)$ is the sign of z. Further let $h_p(n)$ be the exponent of p in the prime factorisation of n where $n \in \mathbb{N}, p \in \mathbb{P}$. The precise conditions under which all and only the elements of $\mathbb{Q}_c$ have a unique presentation as a digit sequence in a suitable generalised factorial base are given in the next lemma.

▶ **Lemma 10.** *Let $(n_i)_{i \in \mathbb{N}}$ be a sequence of natural numbers with $n_i \geq 2$ for all $i \in \mathbb{N}$ and let $c := (c_p)_{p \in \mathbb{P}}$ with $c_p := \sum_{i=0}^{\infty} h_p(n_i)$ for all $p \in \mathbb{P}$ (set $\sum_{i=0}^{\infty} h_p(n_i) = \infty$ if the sum does not converge). Then for every $r \in \mathbb{Q}_c$ with $r \geq 0$ there is a unique sequence $(d_z)_{z=-l}^{k}$ with*

1. $0 \leq d_i < n_i$ for $i = 0, \ldots, k$ and $0 \leq d_{-i} < n_{i-1}$ for $i = 1, \ldots, l$
2. $d_k \neq 0$ or $k = 0, d_k = 0$
   $d_{-l} \neq 0$ or $l = 1, d_{-l} = 0$
3. $r = \sum_{z=-l}^{k} d_z n_{\underline{z!}}$

Proof. Let $r \in \mathbb{Q}_c$ with $r \geq 0$. Decompose r into its fractional part and its integer part $r = \frac{a}{b} + m$ with $m, a, b \in \mathbb{N}, 0 \leq \frac{a}{b} < 1$, and a, b coprime.

First we show how to get $d_{-l}, \ldots, d_{-1}$ with $\frac{a}{b} = \sum_{z=-l}^{-1} d_z n_{\underline{z!}}$. Since $c_p \geq h_p(b)$ for any prime factor p of b there is an $l(p)$ with $h_p(n_0 \ldots n_{l(p)}) = h_p(n_0) + \ldots + h_p(n_{l(p)}) \geq h_p(b)$. Thus for $l - 1 = \max\{l(p) \mid p \in \mathbb{P}, p | b\}$ it holds that $b | n_0 \ldots n_{l-1}$. Consequently there is a q with $\frac{a}{b} = \frac{qa}{n_0 \ldots n_{l-1}}$. If $qa < n_{l-1}$ let $d_{-i} := 0$ for $i < l$ and $d_{-l} := qa$ and be done. Otherwise $qa \geq n_{l-1}$ and $l - 1 > 0$, then $qa = n_{l-1}q' + d_{-l}$ for some $q', d_{-l} \in \mathbb{N}$ with $d_{-l} < n_{l-1}$. Then $\frac{a}{b} = \frac{d_{-l}}{n_0 \ldots n_{l-1}} + \frac{q}{n_0 \ldots n_{l-2}}$ and we can continue the decomposition recursively to obtain $d_{-1}, \ldots, d_{-(l-1)}$ with $\frac{q}{n_0 \ldots n_{l-2}} = \sum_{z=-(l-1)}^{-1} d_z n_{\underline{z!}}$.

Similarly we show how to get $d_0, \ldots, d_k$ with $m = \sum_{z=0}^{k} d_z n_{\underline{z!}}$. Choose the smallest k so that $m < n_0 \ldots n_k$. If $k = 0$ let $d_0 := m$ and be done. Otherwise $n_0 \ldots n_{k-1} \leq m < n_0 \ldots n_k$ and thus $m = d_k n_0 \ldots n_{k-1} + q$ for some $d_k < n_k$ and $q \in \mathbb{N}$ and we can continue the decomposition recursively with $q < n_0 \ldots n_{k-1}$ to obtain $d_0, \ldots, d_{k-1}$ with $q = \sum_{z=0}^{k-1} d_z n_{\underline{z!}}$.

It remains to prove that the representations are unique. First note that $0 \leq \sum_{z=0}^{k} d_z n_{\underline{z!}} \leq \sum_{z=0}^{k} (n_z - 1) n_{\underline{z!}} = \sum_{z=0}^{k} n_{\underline{z+1!}} - n_{\underline{z!}} = n_{\underline{k+1!}} - 1$. The mapping that maps each digit sequence $(d_z)_{z=0}^{k}$ to $\sum_{z=0}^{k} d_z n_{\underline{z!}}$ is, as was shown above, surjective and since there are only $n_{\underline{k+1!}}$ many such digit sequences it must also be injective.

Now suppose $(d_z)_{z=-l}^{-1}$ and $(d'_z)_{z=-l'}^{-1}$ would be two different representations of $0 \leq \frac{a}{b} < 1$. Let $l \geq r \geq 1$ be the smallest number with $d_{-r} \neq d'_{-r}$. Then

$d_{-r} + \frac{d_{-r-1}}{n_r} + \ldots + \frac{d_{-l}}{n_r \ldots n_{l-1}} = d'_{-r} + \frac{d'_{-r-1}}{n_r} + \ldots + \frac{d'_{-l'}}{n_r \ldots n_{l'-1}}$. Since $\frac{d_{-r-1}}{n_r} + \ldots + \frac{d_{-s}}{n_r \ldots n_{s-1}} \leq \frac{n_r - 1}{n_r} + \ldots + \frac{n_{s-1} - 1}{n_r \ldots n_{s-1}} = 1 - \frac{1}{n_r \ldots n_{s-1}} < 1$ for any $s \geq r$ we have $d_{-r} + x = d'_{-r} + y$ for some $0 \leq x, y < 1$ and thus it must hold that $d_{-r} = d'_{-r}$. Contradiction! ◀

▶ **Theorem 11.** *The class of torsion-free Abelian groups of rank 1 is regularly ω-automatic.*

More specifically, there is a parameterised ω-automatic presentation $\mathfrak{c}$, so that for all $(n_i)_{i \in \mathbb{N}}$ with $n_i \geq 2$ we have $\mathcal{S}(\mathfrak{c}[\mathrm{bin}(n_0) \# \mathrm{bin}(n_1) \# \ldots]) \cong (\mathbb{Q}_c, +, <, \mathbb{Z})$ where $c = (c_p)_{p \in \mathbb{P}}$ with $\sum_{i=0}^{\infty} h_p(n_i) = c_p$ for all $p \in \mathbb{P}$.

Proof. It will be sufficient to construct a presentation $\mathfrak{c} = (\mathcal{A}, \mathcal{A}_+, \mathcal{A}_<, \mathcal{A}_{\mathbb{N}})$ for the sub-semigroups $(\mathbb{Q}_c^+, +, <, \mathbb{N})$ of the semigroup $(\mathbb{Q}^+, +, <, \mathbb{N})$ of non-negative rational numbers,

because there is a first-order interpretation that interprets $(\mathbb{Q}_c, +, <, \mathbb{Z})$ in $(\mathbb{Q}_c^+, +, <, \mathbb{N})$ for all c.

For any $(n_i)_{i \in \mathbb{N}}$ that satisfies the conditions of the theorem every $r \in \mathbb{Q}_c^+$ has according to Lemma 10 a presentation of the form $\sum_{z=-l}^{k} d_z n_{\underline{z!}}$ for a unique coefficient sequence $(d_i)_{i=-l}^{k}$. Encode $(d_i)_{i=-l}^{k}$ as a word in the following way:

$$
\begin{array}{llllllll}
\mathrm{bin}(d_0) & \# \, \mathrm{bin}(d_1) & \# \ldots \# \, \mathrm{bin}(d_l) & \# \, \mathrm{bin}(d_{l+1}) & \ldots \# \, \mathrm{bin}(d_k) \# \square \ldots \\
\mathrm{bin}(d_{-1}) & \# \, \mathrm{bin}(d_{-2}) & \# \ldots \# \, \mathrm{bin}(d_{-l-1}) & \# \square \ldots & \ldots \square \ldots \\
\hline
\mathrm{bin}(n_0) & \# \, \mathrm{bin}(n_1) & \# \ldots \# \, \mathrm{bin}(n_l) & \# \, \mathrm{bin}(n_{l+1}) & \ldots \# \, \mathrm{bin}(n_k) \# \ldots
\end{array}
$$

where the binary encoding of the numbers is padded by leading zeros so that the #'s of each row align with the #'s of the advice. With the advice string, the automaton $\mathcal{A}$ can verify that condition 1 of Lemma 10 holds for each triple $(\mathrm{bin}(d_i), \mathrm{bin}(d_{-i-1}), \mathrm{bin}(n_i))$ in a #-seperated segment and that $d_{-l} \neq 0 \lor (l = 1 \land d_{-l} = 0)$ and $d_k \neq 0 \lor k = 0 = d_k$ hold which ensures condition 2. The automaton $\mathcal{A}_{\mathbb{N}}$ merely has to check that the fractional part of r is zero, i.e. that the second row has the form $\mathrm{bin}(0) \# \square^{\omega}$. $\mathcal{A}_{<}$ recognises the length-lexicographical ordering on the digit sequences. To verify that addition is performed correctly the automaton has to check $\sum_{z=-l}^{k} d_z n_{\underline{z!}} + \sum_{z=-l'}^{k'} e_z n_{\underline{z!}} = \sum_{z=-l''}^{k''} s_z n_{\underline{z!}}$ given the encodings of the sequences $(d_z)_{-l \leq z \leq k}$, $(e_z)_{-l' \leq z \leq k'}$, and $(s_z)_{-l'' \leq z \leq k''}$. This can be verified by computing the sum of every #-separated segment i modulo n_i while passing a carry bit from the lower significant segments to the higher significant segments. Note that $\frac{n_i + s_{-i-1}}{n_0 n_1 \cdots n_i} = \frac{1}{n_0 n_1 \cdot n_{i-1}} + \frac{s_{-i-1}}{n_0 n_1 \cdots n_i}$ and $(n_i + s_i) n_0 n_1 \ldots n_{i-1} = s_i n_0 n_1 \ldots n_{i-1} + n_0 \cdots n_i$. It is routine to construct an automaton that implements this idea on the described encoding. Note finally that the set of parameters $\{\mathrm{bin}(n_0) \# \mathrm{bin}(n_1) \# \ldots \mid n_i \geq 2\} = \{1\{0,1\}^+ \#\}^{\omega}$ is clearly ω-regular. ◀

Next we show that every uniformly ω-automatic class $\mathcal{C}$ of Abelian groups has bounded rank. This also implies that every Abelian group that is ω-automatic with advice has finite rank. We need the following combinatorial fact about parameterised ω-automatic presentations.

▶ **Lemma 12.** *Let $\mathfrak{c}$ be a parameterised ω-automatic presentation. There is a constant $c \in \mathbb{N}$ such that whenever $\mathfrak{c}[\alpha]$ presents some countable structure $\mathfrak{A}$ for some advice α and f is a binary function of $\mathfrak{A}$ then for every substructure $\mathfrak{B} \subseteq \mathfrak{A}$ and for every finite subset C of $\mathfrak{B}$ there is a finite subset $D \supseteq C$ of $\mathfrak{B}$ with $|f(D, D)| \leq c \cdot |D|$.*

Proof. Due to Theorem 6 we can assume without loss of generality that $\mathfrak{c}$ is an injective presentation over finite words, so that C can be identified with a finite set over finite words, i.e. $C \subseteq \Sigma^* \{\square\}^{\omega}$ for a finite alphabet Σ. Let m be the maximal length of words in C. Then all words in C are $\sim_e^m$-equivalent. Let $C' \subseteq B$ be a $\sim_e^m$-class over B of maximal cardinality. Then there are ω-suffixes γ_0, γ_1 so that $C = C_0 \gamma_0$ and $C' = C_1 \gamma_1$ for some sets $C_0, C_1 \subseteq \Sigma_{\square}^m$. Let $D := C \cup C'$. Then $f(D, D) = \bigcup_{i,j \in \{0,1\}} f(C_i \gamma_i, C_j \gamma_j)$. It suffices to show that $f(C_i \gamma_i, C_j \gamma_j)$ is contained in the union of no more than $q \sim_e^m$-equivalence classes over B. Due to the maximality of C' it then follows that $|f(C_i \gamma_i, C_j \gamma_j)| \leq q|C'| \leq q|D|$ for all $i, j \in \{0, 1\}$ and thus $|f(D, D)| \leq 4q|D|$, so that $c := 4q$ is the constant we are looking for. For this matter let $\mathcal{A}_f$ be the parameterised ω-automaton in the presentation that recognises the graph of f and let q be the number of states of $\mathcal{A}_f$. Towards a contradiction suppose there were $q + 1$ words $\beta_1, \ldots, \beta_{q+1} \in f(C_i \gamma_i, C_j \gamma_j)$ that are pairwise not $\sim_e^m$-equivalent. By the pigeonhole principle there must then also be $i_0 \neq i_1$ with $\beta_{i_0} = f(c_{i_0} \gamma_i, c_{j_0} \gamma_j)$, $\beta_{i_1} = f(c_{i_1} \gamma_i, c_{j_1} \gamma_j)$ for some $c_{i_0}, c_{i_1} \in C_i$ and $c_{j_0}, c_{j_1} \in C_j$, so that $\mathcal{A}_f[\alpha]$ reaches the same state p after reading the m-prefix $\langle c_{i_0}, c_{j_0}, \beta_{i_0} \rangle [0, m)$ as after reading the m-prefix

$\langle c_{i_1}, c_{j_1}, \beta_{i_1} \rangle [0, m)$. Since $\mathcal{A}_f[\alpha]$ accepts $\langle c_{i_k} \gamma_i, c_{j_k} \gamma_j, \beta_k \rangle$ for $k \in \{0, 1\}$ it accepts beginning in state p and from position m on the input tape both suffixes $\langle \gamma_i, \gamma_j, \beta_k \rangle [m, \infty)$ and thus both words $\langle c_{i_0} \gamma_i, c_{j_0} \gamma_j, \beta_0 [0, m) \beta_k [m, \infty) \rangle$ for $k \in \{0, 1\}$. Since f can map the pair $(c_i \gamma_i, c_j \gamma_j)$ only to one word, this implies $\beta_0 [m, \infty) = \beta_1 [m, \infty)$ in contradiction to $\beta_{i_0} \not\sim_e^m \beta_{i_1}$.   ◀

Similar restrictions for classical automatic structures are well known and follow more or less directly from the pumping lemma for regular languages. However, in the presence of an advice string a pumping argument is not possible, because a manipulation of the elements of the structure via pumping would inevitably alter the advice. Therefore we need to employ a different combinatorial analysis, similar to techniques from [1].

We shall make use of Freiman's theorem, which has also been applied in the non-automaticity proof for $(\mathbb{Q}, +)$ [23]. A *generalised arithmetic progression* P of rank $d \geq 1$ in a torsion-free Abelian group $(G, +)$ is a set of the form $P := \{a_0 + \sum_{i=1}^d z_i \cdot a_i : z_i \in \mathbb{Z}\}$ for $a_0, a_1, \ldots, a_d \in G$. A simplified version of Freiman's theorem reads as follows.

▶ **Theorem 13** (Freiman). *Let $(G, +)$ be a torsion-free Abelian group. There exists a function $r : \mathbb{Q}_c^+ \to \mathbb{N}$ such that for all finite subsets $A \subseteq G$ there is a generalised arithmetic progression P of rank $r\left(\frac{|A+A|}{|A|}\right)$ that contains A.*

▶ **Lemma 14.** *There exists a function $f : \mathbb{Q}^+ \to \mathbb{N}$ such that for every torsion-free Abelian group $(G, +)$ the following is true: For every finite set $X \subseteq G$ the rank of the subgroup generated by X is bounded by $\mathrm{rank}(\langle X \rangle) \leq f\left(\frac{|X+X|}{|X|}\right)$.*

Proof. Consider the torsion-free Abelian group $(\mathbb{Z}^\omega, +)$. By Freiman's Theorem there is a function $g : \mathbb{Q}^+ \to \mathbb{N}$ such that for every finite subset $X \subseteq \mathbb{Z}^\omega$ the following holds: Let $n := r\left(\frac{|X+X|}{|X|}\right)$, then $\langle X \rangle$ is contained in an n-dimensional generalised arithmetic progression $P = \{a_0 + k_1 a_1 + \cdots + k_n a_n \mid k_1, \ldots, k_n \in \mathbb{Z}\}$ for some $a_0, a_1, \ldots, a_n \in \mathbb{Z}^\omega$. Hence, $\langle X \rangle \subseteq \langle \{a_0, a_1, \ldots, a_n\} \rangle$ and therefore $\mathrm{rank}(\langle X \rangle) \leq \mathrm{rank}(\langle \{a_0, a_1, \ldots, a_n\} \rangle) \leq n + 1$.

Let G be a torsion-free Abelian group and let X be finite subset of G. Then $\langle X \rangle_G$ is a finitely generated torsion-free Abelian group and therefore, by the classification of finitely generated Abelian groups, isomorphic to $(\mathbb{Z}^n, +)$ for some $n \in \mathbb{N}$. Consequently $\langle X \rangle$ is also isomorphic to a subgroup of $(\mathbb{Z}^\omega, +)$. Fix some embedding $\iota : \langle X \rangle \to (\mathbb{Z}^\omega, +)$. We can bound the rank of $\langle X \rangle_G$ by $\mathrm{rank}(\langle X \rangle) = \mathrm{rank}(\langle \iota(X) \rangle) \leq r\left(\frac{|X+X|}{|X|}\right) + 1 =: f\left(\frac{|X+X|}{|X|}\right)$.   ◀

We are prepared to prove our claim. In fact, we show a slightly stronger result. A semigroup $\mathfrak{S}$ is cancellative if for all $x, y, z \in S$ it holds that $xy = xz$ implies $y = z$ and $yx = zx$ implies $y = z$. Every commutative cancellative semigroup $\mathfrak{S}$ can be embedded into an Abelian group in very much the same way as $(\mathbb{N}, +)$ can be embedded into $(\mathbb{Z}, +)$. More precisely there exists a unique Abelian group $G(\mathfrak{S})$ such that $\mathfrak{S}$ embeds into $G(\mathfrak{S})$ in the sense that whenever $\iota : \mathfrak{S} \to \mathfrak{G}$ is an embedding into some Abelian group $\mathfrak{G}$ then $\langle \iota(S) \rangle$ is isomorphic to $G(\mathfrak{S})$. For instance $G((\mathbb{N}^k, +)) \cong (\mathbb{Z}^k, +)$ for every $k \geq 1$ and $G(\mathfrak{G}) \cong \mathfrak{G}$ for all Abelian groups $\mathfrak{G}$. For more information we refer to [10].

▶ **Theorem 15.** *Let $\mathcal{C}$ be a uniformly ω-automatic class of countable commutative cancellative semigroups. Then the class $\mathcal{D} = \{G(\mathfrak{S}) \mid \mathfrak{S} \in \mathcal{C}\}$ has bounded rank.*

Proof. Let $\mathcal{C}$ be presented by a uniformly ω-automatic presentation $\mathfrak{c}$ over some parameter set P. Then let c be the constant from Lemma 12 with respect to $\mathfrak{c}$ and $+$, and let f be the function from Lemma 14. We claim that the rank of $\mathcal{D}$ is bounded by $f(c)$. Consider $\mathfrak{S} \in \mathcal{C}$ with $\mathfrak{G} := G(\mathfrak{S})$ and fix an embedding $\iota : \mathfrak{S} \to \mathfrak{G}$. Let $\mathfrak{H} \subseteq \mathfrak{G}$ be a free Abelian

subgroup of maximal rank. Then $\text{rank}(\mathfrak{G}) = \text{rank}(\mathfrak{H})$ and there is a subsemigroup $\mathfrak{T}$ of $\mathfrak{S}$ such that $\iota(T)$ generates $\mathfrak{H}$. In order to show that $\text{rank}(\mathfrak{H})$ is bounded by $f(c)$ it suffices to show that the rank of $\langle X \rangle$ is bounded by $f(c)$ for every finite subset X of $\mathfrak{H}$. So let X be a finite subset of $\mathfrak{H}$. Then there is a finite subset $U \subseteq T$ with $X \subseteq \langle \iota(U) \rangle$. By Lemma 12 there is a finite set V with $U \subseteq V \subseteq T$ such that $|V + V| \leq c|V|$. Hence $\text{rank}(\langle X \rangle) \leq \text{rank}(\langle \iota(V) \rangle) \leq f\left(\frac{|V+V|}{|V|}\right) \leq f(c)$. ◄

▶ **Corollary 16.** *The following classes are not uniformly ω-automatic:*
1. *The class $\{(\mathbb{Z}^n, +) \mid n \in N\}$ for any infinite set $N \subseteq \mathbb{N} \setminus \{0\}$. In particular the class of all free abelian groups is not uniformly ω-automatic.*
2. *The class $\{(\mathbb{N}^n, +) \mid n \in N\}$ for any infinite set $N \subseteq \mathbb{N} \setminus \{0\}$.*

4.3 Boolean Algebras

The previous results demonstrate that ω-automatic presentations with advice capture a much greater variety of structures than ordinary ω-automatic presentations. Moreover, uniformly ω-automatic classes can be surprisingly rich. On the other side we have also seen that (classes of) structures that are presentable in this way are still subject to certain restrictions. Therefore one might ask if there are also examples where we do not gain anything from the possibility to access an advice string or where the only uniformly ω-automatic classes are the trivial ones. In this section we will show that both is the case for the class of countable Boolean algebras. For the following we need the fact that every countable Boolean algebra is isomorphic to the interval algebra $\mathfrak{B}_{\mathfrak{L}}$ of a linear order $\mathfrak{L}$. The interval algebra of a linear order $\mathfrak{L}$ is the set algebra generated by the half-open intervals $\{[x, y) \mid x, y \in \mathfrak{L}, x < y\}$. A Boolean algebra $\mathfrak{B}$ is called *super-atomic* if for every element $b \in B \setminus \{\emptyset\}$ there is an atom $a \in B$ with $a \subseteq b$. Any super-atomic countable Boolean algebra is isomorphic to $\mathfrak{B}_\alpha$ for an ordinal α. For an introduction to the theory of Boolean algebras we refer to [9]. The automatic Boolean algebras have been fully classified in [15]: A countably infinite Boolean algebra is automatic if, and only if, it is isomorphic to $\mathfrak{B}_{\omega n}$ for some $n \geq 1$.

▶ **Theorem 17.** *A countably infinite Boolean algebra is ω-automatic with advice if, and only if, it is isomorphic to $\mathfrak{B}_{\omega n}$ for some natural number $n \geq 1$.*

Proof. Let $\mathfrak{B}$ be a countably infinite Boolean algebra that is not isomorphic to $\mathfrak{B}_{\omega n}$ for all $n \geq 1$. Suppose $\mathfrak{B}$ is ω-automatic with advice. Then $\mathfrak{B}$ has an injective ω-automatic presentation $\mathfrak{b}[\alpha]$ over finite words. We can add the length-lexicographical order to $\mathfrak{b}$ and obtain a presentation of $\mathfrak{B}^{\leq_{\text{llex}}} := (\mathfrak{B}, \leq_{\text{llex}})$ where $\leq_{\text{llex}}$ constitutes a linear order of order type ω on $\mathfrak{B}$. We show that one can construct from $\mathfrak{B}^{\leq_{\text{llex}}}$ an ω-automatic presentation with advice α of the ordinal ω^ω, which has already been shown not to be ω-automatic with advice in [13]. Note that ω^ω is isomorphic to the set $\{(n_0, \ldots, n_k) \in \mathbb{N}^* \mid n_k \neq 0\}$ with the length-lexicographical ordering on $\mathbb{N}^*$.

We construct an advice regular set P of pairwise disjoint elements of $\mathfrak{B}$ such that there are infinitely many $b \in \mathfrak{B}$ with $b \subseteq p$ for every $p \in P$. For the interval algebra of $(\mathbb{Q}, <)$, which is, up to isomorphism, the only countable atomless Boolean algebra, such a set P would be for example $P = \{[n, n+1) \mid n \in \mathbb{N}\}$. For $\mathfrak{B}_\alpha$ with $\alpha \geq \omega^2$ an example would be $P = \{[\omega n, \omega(n+1)) \mid n \in \mathbb{N}\}$. Therefore we can conclude that $\mathfrak{B}$ contains such a set because if $\mathfrak{B}$ contains an element $c \in B \setminus \{\emptyset\}$ such that there is no atom a with $a \subseteq c$, then the Boolean subalgebra $\{x \cap c \mid x \in B\}$ is an atomless countable Boolean algebra. As the example above shows, there is an infinite set P in $\mathfrak{B}$ with the described property. If $\mathfrak{B}$

contains no such element, then $\mathfrak{B}$ is super-atomic and therefore isomorphic to $\mathfrak{B}_\alpha$ for an ordinal $\alpha \geq \omega^2$, in which case there is also such a P.

It remains to show how to obtain such a set P that is ω-regular with advice α. Consider the formula $\psi_P(x, y) \coloneqq ((x = y \lor x \cap y = \emptyset) \land \exists^\infty z(z \subseteq x) \land \exists^\infty z(z \subseteq y))$. Then $\mathfrak{B} \models \exists^{2\text{-ram}} xy\psi_P(x, y)$ and an application of Lemma 7 to the relation defined by $\psi_P(x, y)$ yields an infinite set P with $\mathfrak{B} \models \psi_P(a, b)$ for all $a, b \in P$ that is ω-regular with advice α and has the required property. Let $p(i)$ denote the i-th element of P in the well-order $\leq_{\text{llex}}$ and for every $p \in P$ let $a_p(i)$ be the i-th element below p with respect to $\leq_{\text{llex}}$ (here we have to omit the empty set in the enumeration of the elements below p). We encode a sequence $(n_0, n_1, \ldots, n_k)$ as the element $\bigcup_{0 \leq i \leq k} a_{p(i)}(n_i)$. Thus for example the sequence $(2, 0, 1)$ is encoded by the element $a_{p(0)}(2) \cup a_{p(1)}(0) \cup a_{p(2)}(1)$. Note that the i-th component of the sequence encoded by an element m can be retrieved from m as $p(i) \cap m$ and the length of a sequence is determined by the greatest i such that $p(i) \cap m \neq \emptyset$. It is now a simple exercise to construct a first-order interpretation of ω^ω. ◄

The next step is to ask which classes of countable Boolean algebras are uniformly ω-automatic. Trivially this is the case if the class is a finite collection of Boolean algebras of the form $\mathfrak{B}_{\omega n}$. We show that these are indeed the only examples. In the following we find it easier to work with a slightly different view on the structure of a Boolean algebra $\mathfrak{B}_{\omega n}$. Note that $\mathfrak{B}_\omega$ is isomorphic to the set algebra of all finite and co-finite subsets of ω, that is on $\mathcal{P}_{\text{fc}}(\omega) = \{X \subseteq \omega \mid X \text{ or } \omega \setminus X \text{ is finite}\}$, and that $\mathfrak{B}_{\omega n} \cong \mathfrak{B}_\omega^n$ for all $n \geq 1$.

▶ **Lemma 18.** *There exists an* FOC-*interpretation $\mathcal{I}$ such that the following holds: If $\mathfrak{A} \cong (\mathfrak{B}_\omega^n, P, \preceq)$, i.e. $\mathfrak{A}$ is isomorphic to the n-fold product of the Boolean algebra $\mathfrak{B}_\omega$, expanded by the unary relation $P = \{(\omega, \emptyset, \ldots, \emptyset), (\emptyset, \omega, \emptyset, \ldots, \emptyset), \ldots, (\emptyset, \ldots, \emptyset, \omega)\}$ and a linear order $\preceq$ of order-type ω, then $\mathcal{I}(\mathfrak{A}) \cong (\mathbb{N}^n, +)$.*

Proof. It is not hard to see that $(\mathbb{N}, +)$ is FOC-interpretable in $(\mathfrak{B}_\omega, P, \preceq)$. Without loss of generality we can assume that $\preceq$ behaves on the atoms like the natural linear order, that is $\{0\} \prec \{1\} \prec \cdots$. The idea is to identify the finite sets in $\mathfrak{B}_\omega$ with the binary expansions of natural numbers. Accordingly, a finite set X presents the number $n(X) = \sum_{i \in X} 2^i$. The domain formula has to express that x is a finite set. This can be done by the formula $\delta'(x) \coloneqq \neg\exists^\infty y(y \subseteq x)$. It is routine to construct a formula $\varphi'_+(x, y, z)$ such that for all finite sets $X, Y, Z \subseteq \omega$ it holds that $(\mathfrak{B}_\omega, P, \preceq) \models \varphi_+(X, Y, Z)$ if, and only if, $n(X) + n(Y) = n(Z)$. We describe how to transform this interpretation into an interpretation $\mathcal{I}$ that interprets $(\mathbb{N}^n, +)$ in $(\mathfrak{B}_\omega^n, P, \preceq)$ for all $n \geq 1$. The idea for the general case is to use the predicate P to perform addition in every component separately. An element of $(m_1, \ldots, m_n)$ is now encoded by the tuple $(X_1, \ldots, X_n)$ of finite sets with $n(X_i) = m_i$ for all $1 \leq i \leq n$. The correctness of the addition now has to be checked for every component separately. Accordingly we define $\delta(x) \equiv \forall p \in P : \delta''(x \cap p, p)$ and $\varphi_+(x, y, z) \equiv \forall p \in P : \varphi''_+(x \cap p, y \cap p, z \cap p, p)$ where $\varphi''_\mathbb{N}$ and δ'' are obtained from δ' and φ'_+ by restricting all quantifications to elements below p. ◄

Next we show that the predicate P is definable in $(\mathfrak{B}_\omega^n, \preceq)$.

▶ **Lemma 19.** *Let $(X_{ij})_{1 \leq i,j \leq n}$ be a collection of finite subsets of ω such that for all $i \in \{1, \ldots, n\}$:*
▪ *$X_{ki} \cap X_{\ell i} = \emptyset$ for all $1 \leq k < \ell \leq n$ with $k, \ell \neq i$ and*
▪ *$X_{ii} = \bigcup_{j \neq i} X_{ji}$.*
Then there is an automorphism of $\mathfrak{B}_\omega^n$ which maps the n-tuple

$$((X_{i1}, \ldots, X_{i(i-1)}, \omega \setminus X_{ii}, X_{i(i+1)}, \ldots, X_{in}))_{1 \leq i \leq n}$$

of elements of $\mathcal{P}_{fc}(\omega)^n$ to the tuple $((\omega, \emptyset, \ldots, \emptyset), (\emptyset, \omega, \emptyset, \ldots, \emptyset), \ldots, (\emptyset, \ldots, \emptyset, \omega))$.

▶ **Lemma 20.** *There is an FOC-interpretation $\mathcal{J}$ such that the following is true: If a structure $\mathfrak{A}$ is isomorphic to $(\mathfrak{B}_\omega^n, \leq)$, where $\leq$ is a linear order on $\mathcal{P}_{fc}(\omega)^n$ of order type ω then the structure $\mathcal{J}(\mathfrak{A})$ is isomorphic to a structure $(\mathfrak{B}_\omega^n, P, \preceq)$, where P and $\preceq$ are as in Lemma 18.*

Proof. It suffices to show that a set of the form $\{(X_{i1}, \ldots, X_{i(i-1)}, \omega \setminus X_{ii}, X_{i(i+1)}, \ldots, X_{in}) \mid 1 \leq i \leq n\}$, where $(X_{ij})_{1 \leq i,j \leq n}$ are as described in Lemma 19, is definable by an FOC-formula.

First, we define the elements of $\mathfrak{B}_\omega^n$ that are finite in all but exactly one component. This is done by the following formula $\mathrm{Comp}(x) := \exists^\infty z(z \subseteq x) \wedge \neg \exists y(y \subseteq x \wedge \exists^\infty z(z \subseteq y) \wedge \exists^\infty z(z \subseteq x \setminus y))$, which states that x is infinite and there is no infinite subset y of x such that $x \setminus y$ is also infinite. This ensures that $x = (X_1, \ldots, X_{i-1}, \omega \setminus X_i, X_{i+1}, \ldots, X_n)$ for some finite sets $X_1, \ldots, X_n \subseteq \omega$ and some $i \leq n$.

Next we employ the linear order to preselect n such elements which are infinite in pairwise different components. $\mathrm{Sel}(x) := \mathrm{Comp}(x) \wedge \forall y((\mathrm{Comp}(y) \wedge y < x) \rightarrow \neg \exists^\infty z(z \subseteq x \cap y))$. The elements of Sel are not yet of the type that we need. First, the elements of Sel might have finite intersections and second there might be finitely many atoms that are not below any element of Sel. Therefore we need to modify the elements of Sel so that they are disjoint and every atom is below one of these elements. This, however, can easily be achieved by a first-order formula. ◀

▶ **Corollary 21.** *Let $\mathcal{C}$ be a class of countably infinite Boolean algebras. Then the following three conditions are equivalent:*
1. *$\mathcal{C}$ is uniformly automatic.*
2. *$\mathcal{C}$ is uniformly ω-automatic.*
3. *$\mathcal{C} \cong \{\mathfrak{B}_{\omega n} \mid n \in N\}$ for some finite set $N \subseteq \mathbb{N} \setminus \{0\}$.*

Proof. Clearly (1) implies (2) and (3) implies (1) because every Boolean algebra of the form $\mathfrak{B}_{\omega n}$ is automatic and every finite class of automatic structures is uniformly automatic.

It remains to prove that (2) implies (3). Let $\mathcal{C}$ be a uniformly ω-automatic class of countably infinite Boolean algebras. Then every $\mathfrak{B} \in \mathcal{C}$ is ω-automatic with advice and hence isomorphic to some $\mathfrak{B}_{\omega n}$. Consequently $\mathcal{C} \cong \{\mathfrak{B}_{\omega n} \mid n \in N\}$ for some set $N \subseteq \mathbb{N} \setminus \{0\}$. Suppose that N is infinite. By Theorem 6 there is an injective uniformly ω-automatic presentation $\mathfrak{c}$ of $\mathcal{C}$ over finite words. We can now expand $\mathfrak{c}$ by the length lexicographical order and use the interpretations $\mathcal{I}$ and $\mathcal{J}$ from Lemma 18 and Lemma 20 to obtain a uniformly ω-automatic presentation of the class $\{(\mathbb{N}^n, +) \mid n \in N\}$, contradicting Corollary 16. ◀

4.4 Monoids, Groups, and Integral Domains

The ideas and methods presented in the previous sections are powerful enough to provide several other non-automaticity results. In particular it can be shown that:

- The free semigroup with two generators is not a substructure of any countable structure that is ω-automatic with advice.
- $(\mathbb{N}, \cdot)$ is not a substructure of any countable structure that is ω-automatic with advice.
- No infinite integral domain has an injective ω-automatic presentation with advice.
- The field of reals is not ω-automatic with advice.

Due to space constraints we have to abstain from presenting the proofs here. Instead we refer the interested reader to the upcoming full version of this paper.

5 Closure Properties and Counting Problems

It is easy to see that the class of automatic structures is closed under the standard composition operators such as disjoint union and direct product and that the same constructions still work in the presence of an advice. However, when we consider uniformly automatic classes then the closure under such compositions is a much stronger notion because compositions of arbitrary width must be presented uniformly. It is therefore not surprising that the situation becomes more diverse in this setting.

▶ **Definition 22.** Let $\mathcal{C}$ be a class of τ-structures. Then $\mathcal{C}^{\times}$ denotes the closure of $\mathcal{C}$ under direct products and, in case that τ is relational, $\mathcal{C}^{\uplus}$ denotes the closure of $\mathcal{C}$ under disjoint unions. That is $\mathcal{C}^{\times} = \{\mathfrak{A}_1 \times \cdots \times \mathfrak{A}_n \mid n \geq 1, \mathfrak{A}_1, \ldots, \mathfrak{A}_n \in \mathcal{C}\}$ and $\mathcal{C}^{\uplus} = \{\mathfrak{A}_1 \uplus \cdots \uplus \mathfrak{A}_n \mid n \geq 1, \mathfrak{A}_1, \ldots, \mathfrak{A}_n \in \mathcal{C}\}$.

It is not hard to see that uniformly (ω-)tree-automatic classes behave very well under the two closure operators that we defined above.

▶ **Lemma 23.** *Let $\mathcal{C}$ be a uniformly (ω-)tree-automatic class of structures. From a given (ω-)tree-automatic presentation $(P, \mathfrak{c})$ of $\mathcal{C}$ one can effectively construct (ω-)tree-automatic presentations $(P^{\times}, \mathfrak{c}^{\times})$ of $\mathcal{C}^{\times}$, and $(P^{\uplus}, \mathfrak{c}^{\uplus})$ of $\mathcal{C}^{\uplus}$. Moreover, the regularity of the advice set is preserved.*

For uniformly (ω-)automatic classes we do not enjoy the same closure properties as in the tree case.

▶ **Corollary 24.** *There is a regularly automatic class $\mathcal{C}$ such that the closure under direct products $\mathcal{C}^{\times}$ is not uniformly ω-automatic.*

Proof. The free Abelian groups of finite rank are up to isomorphism the finite direct products of the automatic structure $(\mathbb{Z}, +)$. Hence by Corollary 16, $\{(\mathbb{Z}, +)\}^{\times}$ is not uniformly ω-automatic. ◀

However, if we restrict ourselves to classes of finite structures then uniformly automatic classes are equally well behaved.

▶ **Lemma 25.** *Let $\mathcal{C}$ be a uniformly automatic class of* finite *structures. From a given automatic presentation $(P, \mathfrak{c})$ of $\mathcal{C}$ one can effectively construct a uniformly automatic presentation $(P^{\times}, \mathfrak{c}^{\times})$ of $\mathcal{C}^{\times}$ and $(P^{\uplus}, \mathfrak{c}^{\uplus})$ of $\mathcal{C}^{\uplus}$. Moreover, regularity of the advice set is preserved.*

Product closures play an important role in the structure theory of Abelian groups, since classification theorems in that domain often take the form: $\mathfrak{G}$ is a group with property Prop, if, and only if, $\mathfrak{G}$ is isomorphic to a direct sum of groups from a class $\mathcal{C}_{\text{Prop}}$; where $\mathcal{C}_{\text{Prop}}$ is a class of prime groups with property Prop, in the sense that they have no non-trivial decomposition into groups with property Prop and that $\mathcal{C}_{\text{Prop}}^{\times} = \mathcal{C}$.

The *divisible Abelian groups* are an example for a class that has a classification theorem of this form. An Abelian group $\mathfrak{G}$ is called *divisible*, if for every $g \in G$ and $n \in \mathbb{N}$ there is a $h \in G$ with $n \cdot h = g$. The prime groups in the classification theorem for divisible Abelian groups are $(\mathbb{Q}, +)$ and the Prüfer p-groups $\mathbb{Z}(p^{\infty})$, which are isomorphic to $\mathbb{Q}_{\chi_p}/\mathbb{Z}$ where $\chi_p = (\chi_p)_{q \in \mathbb{P}}$ is the sequence with $(\chi_p)_p = \infty$ and $(\chi_p)_q = 0$ for $p \neq q$. Let us use in general the notation $\mathbb{Z}(n^{\infty})$ for the subgroup of $\mathbb{Q}/\mathbb{Z}$ that is generated by $\{\frac{1}{n^k} \mid k \in \mathbb{N}\}$. Note that $\{\mathbb{Z}(p^{\infty}) \mid p \in \mathbb{P}\}^{\times} = \{\mathbb{Z}(n^{\infty}) \mid n \geq 2\}$ thus the countable divisible Abelian groups are the product closure of the class $\mathcal{C}_{\text{Div}} = \{(\mathbb{Q}, +)\} \cup \{\mathbb{Z}(n^{\infty}) \mid n \geq 2\}$.

Note that from the regularly ω-automatic presentation of $(\mathbb{Q}_c, +, <, \mathbb{Z})$ in Theorem 11 we can obtain a regularly automatic presentation of the subgroups of $\mathbb{Q}/\mathbb{Z}$ via a first-order interpretation. This is a parameterised automatic presentation of $\mathcal{C}_{\mathrm{Div}}$ with parameter set $P = \{\mathrm{bin}(2)\#\mathrm{bin}(3)\ldots\} \cup \{(\mathrm{bin}(n)\#)^\omega : n \geq 2\}$. Lemma 23 yields an ω-tree automatic presentation of $\mathcal{C}_{\mathrm{Div}}^\times$ with parameter set $P^\times$, which has a decidable MSO-theory.

▶ **Corollary 26.** *The class of countable divisible Abelian groups is strongly ω-tree-automatic.*

Even in cases when a class of structures does not admit a regular presentation up to isomorphism (possibly already due to cardinality reasons), a regular presentation up to elementary equivalence may still be possible, and we would thereby still get a decision procedure for the first-order theory of the class. This is in fact also possible for the class of all Abelian groups itself, whereby we can reprove Szmielew's decidability result for the first-order theory of Abelian groups [21]. We rely here on the fact that every Abelian group is elementary equivalent to a Szmielew group and every Szmielew group is isomorphic to a countably infinite direct sum of subgroups of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ [11]. From this fact and Lemma 23 we obtain the following corollary.

▶ **Corollary 27.** *There is a regularly ω-tree-automatic presentation of the class of all Abelian groups up to elementary equivalence.*

At last, let us mention an application that arises from the semantic shift in the transition between automatic presentations with advice and uniformly automatic presentations. In model theory one is often interested in the number of isomorphism types of structures from a given class that satsify a given formula. It turns out that counting problems of this kind can also be handled in our framework if the presentation of the class fulfills the following uniqueness condition: An [ω-](tree-)automatic presentation $\mathfrak{c}$ of a class $\mathcal{C}$ has the **unique representation property** if for any $\mathfrak{A} \in \mathcal{C}$ there is exactly one parameter α such that $\mathcal{S}(\mathfrak{c}[\alpha]) \cong \mathfrak{A}$.

▶ **Lemma 28.** *If a class $\mathcal{C}$ has a regularly (ω-)automatic presentation $\mathfrak{c}$ with the unique representation property then there is a uniform decision procedure to determine, for given $\varphi \in$ FO and $k, m \in \mathbb{N}$, whether the number of pairwise non-isomorphic models of φ in $\mathcal{C}$ is a finite number n with $n \equiv k \mod m$, whether it is at most countably infinite, or whether it is uncountable.*

Proof. We can construct from $\mathfrak{c}$ an automatic presentation of $\mathfrak{A}_\mathcal{C} = ((\biguplus_{\alpha \in P} \mathcal{S}(\mathfrak{c}[\alpha])) \cup \{\mathcal{S}(\mathfrak{c}[\alpha]) \mid \alpha \in P\}, \sim)$, where $\mathcal{S}(\mathfrak{c}[\alpha]) \sim a$ holds if $a \in \mathcal{S}(\mathfrak{c}[\alpha])$. Let ψ be the formula obtained from φ by relativising all quantifiers Qx by $a \sim x$. Because of the unique representation property the decision problems reduce to checking whether $\mathfrak{A}_\mathcal{C}$ satisfies $\exists^{(k,m)} a\psi$, $\exists^{\leq \aleph_0} a\psi$, or $\exists^{>\aleph_0} a\psi$, respectively. ◀

Unfortunately even in the simplest conceivable case, that is regularly automatic classes of finite sets, the unique representation property is undecidable.

▶ **Theorem 29.** *The problem to decide whether a regularly automatic presentation has the unique representation property is Π_1^0-complete for the empty signature and for signatures with only monadic predicates (even for classes of finite structures), and Π_1^1-hard for any signature with at least one predicate of arity at least two.*

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A   Proofs Omitted from Section 3

▶ **Theorem 6.** *A class $\mathcal{C}$ of countable structures has a parameterised ω-automatic presentation $\mathfrak{c}$ with parameter set P, if, and only if, it has an injective parameterised ω-automatic presentation over finite words $\mathfrak{c}'$ with the same parameter set P. Moreover $\mathfrak{c}'$ can be effectively constructed from $\mathfrak{c}$.*

Proof. Let $\mathfrak{c} = (\mathcal{A}, \mathcal{A}_\approx, (\mathcal{A}_R)_{R \in \tau})$ be the given presentation with advice alphabet Γ. We slightly modify the presentation in such a way that $\mathfrak{d}[\alpha]$ is a well-defined structure for every $\alpha \in \Gamma^\omega$. For this matter replace $\mathcal{A}_R$ by an automaton $\mathcal{A}'_R$ with $L(\mathcal{A}'_R) = \{\overline{a} \otimes \alpha \in L(\mathcal{A}_R) | \overline{a} \in L(\mathcal{A}[\alpha])^r\}$ which ensures that $\mathcal{A}'_R[\alpha]$ recognises a relation on the domain $L(\mathcal{A}[\alpha])$ for every $\alpha \in \Gamma^\omega$. To ensure that $\approx_\alpha$ is always a congruence relation, we replace $\mathcal{A}_\approx$ by an automaton $\mathcal{A}'_\approx$ so that $\mathcal{A}'_\approx[\alpha]$ recognises $L(\mathcal{A}_\approx[\alpha])$ if this is a congruence relation, and the equality relation otherwise. Since the property that $\approx_\alpha$ is a congruence relation is first-order definable, it is a property that can be checked by an automaton, i.e. such an $\mathcal{A}'_\approx$ can indeed be constructed. Define now the structure $\mathcal{S}_\approx(\mathfrak{c})$ as the disjoint union of all $\mathcal{S}_\approx(\mathfrak{c}[\alpha])$ for $\alpha \in \Gamma^\omega$. Then $(\mathcal{A}, \mathcal{A}'_\approx, (\mathcal{A}'_R)_{R \in \tau})$ is an ω-automatic presentation of $\mathcal{S}_\approx(\mathfrak{c})$. In the following we makes use of [12, Proposition 3.1] which gives a criterion for countability of sets that are defined by a formula $\varphi(x, \overline{z})$ with parameters $\overline{z}$ in an ω-automatic structure. We apply the criterion to the formula $\varphi(x, z) := x \sim z$ and the $\tau \uplus \{\sim\}$-structure that we get from $\mathcal{S}_\approx(\mathfrak{c})$ by enlarging the universe with the elements α for every $\alpha \in \Gamma^\omega$ and defining the relation $\sim$ on the enlarged universe via $w \sim \alpha :\Leftrightarrow \alpha \in \Gamma^\omega$ and $w \in \mathcal{S}_\approx(\mathfrak{c}[\alpha])$. It is easy to see that $\mathfrak{A}_\mathfrak{c}$ is still an ω-automatic structure. Since $\varphi(-, \alpha)$ defines a countable set in $\mathfrak{c}$ for every $\alpha \in P$ the proposition thus implies that that there is a constant c computable from $\mathfrak{c}$ such that the formula

$$\psi(z, x_1, \ldots, x_c) := \bigwedge_{1 \le i \le c} x_i \sim z \wedge \forall x (x \sim z \to \exists y (\bigvee_{1 \le i \le c} y \sim_e x_i \wedge y \approx x))$$

defines an ω-automatic relation $R_\psi \subseteq \Gamma^\omega \times (\Sigma^\omega)^c$ in $\mathfrak{A}_\mathfrak{c}$ with $\alpha R_\psi \neq \emptyset$ for every $\alpha \in P$. By the uniformisation theorem for ω-automatic relations [6], there is an ω-automatic function $f_{R_\psi} : R_\psi(\Sigma^\omega)^c \to (\Sigma^\omega)^c$ with $f_{R_\psi}(\alpha) \in \alpha R_\psi$ for all $\alpha \in R_\psi(\Sigma^\omega)^c$.

We are now prepared to construct $\mathfrak{d}'$. Intuitively we are going to use f_{R_ψ} to pick $x_1, \ldots, x_c$ from the original presentation such that every element has a $\approx$-representative in $L := L(\mathcal{A}[\alpha]) \cap \bigcup_{1 \le i \le c} [x_i]_{\sim_e}$. Then for every $y \in L$ we just cut y from the point where it coincides with some x_i and annotate the resulting string with the respective end-class. More

formally we first expand the alphabet Σ by new symbols $\{1, \ldots, c\}$. The domain automaton is constructed from the formula:

$$\varphi_A(\alpha, x) := \exists y \in \Sigma^*, i \in \{1, \ldots, c\}(x = yi\square^\omega \wedge (y(f_{R_\psi}(\alpha)_i[|y|, \infty)) \in L(\mathcal{A}[\alpha])).$$

Similarly for $S \in \{\approx\} \cup \tau$ we construct an automaton by the formula

$$\varphi_S(\alpha, y_1 i_1 \square^\omega, \ldots, y_k i_k \square^\omega) := S(y_1(f_{R_\psi}(\alpha)_{i_1}[|y_1|, \infty)), \ldots, y_k(f_{R_\psi}(\alpha)_{i_k}[|y_k|, \infty))).$$

The corresponding relations can easily be recognised by Muller automata. The presentation can be made injective by taking the length-lexicographic smallest representative of any $\approx$-class. ◀

B   Proofs Omitted from Section 4

▶ **Lemma 19.** *Let $(X_{ij})_{1 \le i, j \le n}$ be a collection of finite subsets of ω such that for all $i \in \{1, \ldots, n\}$:*
- *$X_{ki} \cap X_{\ell i} = \emptyset$ for all $1 \le k < \ell \le n$ with $k, \ell \ne i$ and*
- *$X_{ii} = \bigcup_{j \ne i} X_{ji}$.*

Then there is an automorphism of $\mathfrak{B}_\omega^n$ which maps the n-tuple

$$((X_{i1}, \ldots, X_{i(i-1)}, \omega \setminus X_{ii}, X_{i(i+1)}, \ldots, X_{in}))_{1 \le i \le n}$$

of elements of $\mathcal{P}_{fc}(\omega)^n$ to the tuple $((\omega, \emptyset, \ldots, \emptyset), (\emptyset, \omega, \emptyset, \ldots, \emptyset), \ldots, (\emptyset, \ldots, \emptyset, \omega))$.

Proof. The automorphism is constructed as follows: For all $1 \le i \le n$ fix a bijection π_i between the atoms below $(X_{i1}, \ldots, X_{i(i-1)}, \omega \setminus X_{ii}, X_{i(i+1)}, \ldots, X_{in})$ and the atoms below $(\emptyset, \ldots, \emptyset, \underbrace{\omega}_{\text{position } i}, \emptyset, \ldots, \emptyset)$. Because of the properties of $(X_{ij})_{1 \le i, j \le n}$, every atom appears in the domain and the range of exactly one π_i. Thus, we can combine $\pi_1, \ldots, \pi_n$ to a permutation $\pi = \bigcup_{1 \le i \le n} \pi_i$ on the atoms of $\mathfrak{B}_\omega^n$. We lift π to a permutation ρ on $\mathcal{P}(\omega)^n$ by

$$\rho((X_1, \ldots, X_n)) = \bigcup_{a \text{ atom below } (X_1, \ldots, X_n)} \pi(a).$$

Then ρ is an automorphism on $(\mathcal{P}(\omega), \cup, \cap, \omega, \emptyset)^n$ because ρ is derived from a permutation of the atoms. Every such permutation is an automorphism because $(\mathcal{P}(\omega), \cup, \cap, \omega, \emptyset)^n$ is isomorphic to $(\mathcal{P}(\omega), \cup, \cap, \omega, \emptyset)$ via the automorphism

$$(X_1, \ldots, X_n) \mapsto \bigcup_{1 \le i \le n} \{na + (i-1) \mid a \in X_i\}.$$

Further, the restriction of ρ to $\mathcal{P}_{\text{fc}}(\omega)^n$ is a permutation on $\mathcal{P}_{\text{fc}}(\omega)^n$, hence ρ is an automorphism on $\mathfrak{B}_\omega^n \subseteq (\mathcal{P}(\omega), \cup, \cap, \omega, \emptyset)^n$. ◀

C   Proofs Omitted from Section 5

▶ **Lemma 23.** *Let $\mathcal{C}$ be a uniformly $(\omega\text{-})$tree-automatic class of structures. From a given $(\omega\text{-})$tree-automatic presentation $(P, \mathfrak{c})$ of $\mathcal{C}$ one can effectively construct $(\omega\text{-})$tree-automatic presentations $(P^\times, \mathfrak{c}^\times)$ of $\mathcal{C}^\times$, and $(P^\uplus, \mathfrak{c}^\uplus)$ of $\mathcal{C}^\uplus$. Moreover, the regularity of the advice set is preserved.*
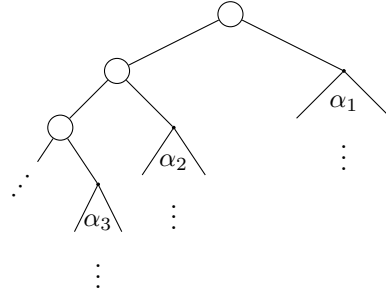
■ **Figure 1** The Parameters for the class $\mathcal{C}^\times$

Proof. Construction of $(P^\times, \mathfrak{c}^\times)$: Suppose $\mathcal{C}$ is presented by the uniform (ω-)tree-automatic presentation $\mathfrak{c}$ over the advice set P. As the construction is rather straightforward we only give the parameter set for the presentation and the idea for the encoding. The parameter set consists of all trees where the right child of every node in the left-most branch induces a subtree which is in P. This is depicted in Figure 1 . Such an advice presents the structure $\mathcal{S}(\mathfrak{c}[\alpha_1]) \times \mathcal{S}(\mathfrak{c}[\alpha_2]) \times \cdots \times \mathcal{S}(\mathfrak{c}[\alpha_n])$. Let $t_1, \ldots, t_n$ be elements of $\mathcal{S}(\mathfrak{c}[\alpha_1]), \ldots, \mathcal{S}(\mathfrak{c}[\alpha_n])$, respectively. Then the element $(t_1, \ldots, t_n)$ is put together in the same way as the advices.

Construction of $(P^\uplus, \mathfrak{c}^\uplus)$: We use the same advice set as in (1). The elements are encoded by trees where all except one node $0^n 1$ of the form $\{0\}^* 1$ are leafs labelled with a new dummy symbol and $0^n 1$ induces a subtree t such that $t \in L(\mathcal{A}[\alpha_n])$ ◀

▶ **Lemma 24.** *Let $\mathcal{C}$ be a regularly automatic class of finite structures. Then there is a presentation $\mathfrak{c} = (\mathcal{A}, (\mathcal{A}_R)_{R \in \tau})$ with regular parameter set P such that $|w| = |\alpha|$ for all $w \in L(\mathcal{A}[\alpha])$.*

Proof. Let $\mathfrak{c}' = (\mathcal{A}', (\mathcal{A}'_R)_{R \in \tau})$ be an injective uniform automatic presentation of $\mathcal{C}$ where the regular parameter set P' is recognised by $\mathcal{A}'_p$. Since $\mathfrak{c}$ is injective and $\mathcal{C}$ is a class of finite structures, $L(\mathcal{A}'[\alpha])$ is finite for all $\alpha \in L(\mathcal{A}'_p)$. Extend the alphabet of the presentation by a new padding symbol $\#$ and consider the injective mapping $\pi : L(\mathcal{A}'_p) \to L(\mathcal{A}'_p)\#^*$ given by $\alpha \mapsto \alpha\#^k$ with $k = \max\{0, \max\{|w| - |\alpha| \mid \alpha \otimes w \in L(\mathcal{A}'_p)\}\}$. The language $P := \pi(L(\mathcal{A}'_p))$ is regular and we can effectively construct a corresponding automaton $\mathcal{A}_p$ from $\mathcal{A}'_p$ and $\mathcal{A}'$. Similarly we can construct automata for the languages

$$L_A := \{\alpha\#^n \otimes w\#^k \mid \alpha \otimes w \in L(\mathcal{A}') \wedge \alpha\#^n \in P \wedge |\alpha\#^n| = |w\#^k|\} \text{ and}$$
$$L_R := \{\alpha\#^n \otimes w_1\#^{k_1} \otimes \ldots \otimes w_r\#^{k_r} \mid$$
$$\alpha \otimes w_1 \otimes \ldots \otimes w_r \in L(\mathcal{A}'_R) \wedge \alpha\#^n \in P \wedge |\alpha\#^n| = |w_i\#^{k_i}|\}.$$

Together these automata form the presentation we are looking for.   ◀

▶ **Lemma 25.** *Let $\mathcal{C}$ be a uniformly automatic class of* finite *structures. From a given automatic presentation $(P, \mathfrak{c})$ of $\mathcal{C}$ one can effectively construct a uniformly automatic presentation $(P^\times, \mathfrak{c}^\times)$ of $\mathcal{C}^\times$ and $(P^\uplus, \mathfrak{c}^\uplus)$ of $\mathcal{C}^\uplus$. Moreover, regularity of the advice set is preserved.*

Proof. Let $\mathfrak{c} = (\mathcal{A}, (\mathcal{A}_R)_{R \in \tau}, \pi)$ be an automatic presentation of $\mathcal{C}$ over the advice set P. Construction of $(P^\times, \mathfrak{c}^\times)$: By Lemma 24, we might assume that for all $\alpha \in P$ and all $w \in L(\mathcal{A}[\alpha])$ we have $|\alpha| = |w|$. As parameter set for $\mathcal{C}^\times$ we can now take $(P\#)^* P$, where $\alpha_1 \# \ldots \# \alpha_n$ is an advice for $\mathcal{S}(\mathfrak{c}[\alpha_1]) \times \cdots \times \mathcal{S}(\mathfrak{c}[\alpha_n])$. The construction of a uniform presentation $\mathfrak{c}^\times$ of $\mathcal{C}^\times$ from $\mathfrak{c}$ is straight forward. On reading $\alpha_1 \# \ldots \# \alpha_n$ as parameter,

the automaton $\mathcal{A}^\times$ should accept exactly the words $w_1 \# \ldots \# w_n$ with $w_i \in L(\mathcal{A}[\alpha_i])$ for all $i \in \{1, \ldots, n\}$. This can obviously be done by an automaton since all words in $L(\mathcal{A}[\alpha_i])$ have the same length as α_i. The same holds for the presentation of the relations $R \in \tau$.

Construction of $(P^\times, \mathfrak{c}^\times)$: We construct a presentation over the advice set $(P\#)^* P$, where $\alpha_1 \# \cdots \# \alpha_n$ should be an advice for $\biguplus_{1 \le i \le n} \mathcal{S}(\mathfrak{c}[\alpha_i])$. For an advice $\alpha_1 \# \cdots \# \alpha_n$ we encode the elements of $\mathcal{S}(\alpha_i)$, $1 \le i \le n$, by the language $L_i = \#^{|\alpha_1 \# \cdots \# \alpha_{i-1} \#|} L(\mathcal{A}[\alpha])$. Intuitively we shift the encodings of the elements in the copy of the i-th summand so that it matches again with the beginning of the i-th advice. Obviously one can construct a parameterised automaton with $L(\mathcal{A}[\alpha]) = \bigcup_{1 \le i \le n} L_i$ for all $\alpha = \alpha_1 \# \cdots \# \alpha_n \in (P\#)^* P$. It is an easy exercise to construct the rest of the presentation.   ◀

▶ **Theorem 29.** *The problem to decide whether a regularly automatic presentation has the unique representation property is* Π^0_1*-complete for the empty signature and for signatures with only monadic predicates (even for classes of finite structures), and* Π^1_1*-hard for any signature with at least one predicate of arity at least two.*

Proof. Π^1_1-hardness for signatures with binary relations follows directly from the the fact that the isomorphism problem for automatic structures is Σ^1_1-complete [15]. Obviously, given automatic presentations $\mathfrak{d}_0, \mathfrak{d}_1$, one can construct a parameterised presentation $\mathfrak{c}$ over the parameters $\{0, 1\}$ such that $\mathcal{S}(\mathfrak{c}[0]) = \mathcal{S}(\mathfrak{d}_0)$ and $\mathcal{S}(\mathfrak{c}[1]) = \mathcal{S}(\mathfrak{d}_1)$. Then $\mathfrak{c}$ has the unique representation property if, and only if, $\mathcal{S}(\mathfrak{d}_0) \not\cong \mathcal{S}(\mathfrak{d}_1)$.

In order to establish Π^0_1-completeness for the empty signature we adopt a technique used by Kuske et al. in [18] to show that the ismomorphism problem for automatic equivalence relations is Π^0_1-complete. More precisely we use encodings of polynomials by automata to reduce Hilbert's 10th problem to the uniqueness problem. The problem can be formulated as follows: given polynomials $p, q \in \mathbb{N}[x_1, \ldots, x_k]$ decide whether $p(\bar{a}) = q(\bar{a})$ for some $\bar{a} \in \mathbb{N}^k$. In [18] it is shown that for every polynomial p with nonnegative coefficients one can construct an automaton $\mathcal{A}_p$ such that on input $1^{n_1} \otimes \ldots \otimes 1^{n_k}$ the automaton $\mathcal{A}_p$ has exactly $p(n_1, \ldots, n_k)$ accepting runs [18, Lemma 2]. Given an automaton $\mathcal{A}_p$ we can construct the following automatic presentation $\mathfrak{c}_p$ of the class $\{(\{0, \ldots, m-1\}) \mid \exists \bar{n}(p(\bar{n}) = m)\}$. The parameter language of $\mathfrak{c}$ is $\{1^{n_1} \otimes \ldots \otimes 1^{n_k} \mid \bar{n} \in \mathbb{N}^k\}$. For a parameter α the domain language is $\{w \in Q_p^* \mid w$ is an accepting run of $\mathcal{A}_p$ on $\alpha\}$, which is uniformly automatic since an automaton can check while reading $\alpha \otimes w$ if w is an accepting run of $\mathcal{A}_p$ on α. To complete the proof note that we can construct injective polynomials C_k for any arity k. For $p, q \in \mathbb{N}[x_1, \ldots, x_n]$ define $p' := C_{k+1}(x_1, \ldots, x_k, p(x_1, \ldots, x_k))$ and $q' := C_{k+1}(x_1, \ldots, x_k, q(x_1, \ldots, x_k))$. Then p' and q' are both injective and $p'(\bar{a}) = q'(\bar{b})$ holds if, and only if, $\bar{a} = \bar{b}$ and $p(\bar{a}) = q(\bar{b})$. Now let $\mathfrak{c}$ be the parameter disjoint union of $\mathfrak{c}_{p'}$ and $\mathfrak{c}_{q'}$. By the aforementioned properties of p', q', $\mathfrak{c}$ has the unique representation property if, and only if. $p(\bar{a}) \neq q(\bar{a})$ for all $\bar{a} \in \mathbb{N}^k$. This establishes the hardness for Π^0_1. Further, the isomorphism problem is decidable for automatic structures with purely monadic signatures. Hence the uniqueness problem is in Π^0_1 since we can just enumerate all pairs of distinct parameters (α, β) from the regular parameter set and check if $\mathcal{S}(\mathfrak{c}[\alpha]) \cong \mathcal{S}(\mathfrak{c}[\beta])$.   ◀