Approximable minimization problems and optimal solutions on random inputs

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Abstract

In this paper we extend recent work about logical criteria for approximation properties of optimization problems. We focus on the relationship between logical expressibility and expected asymptotic growth of optimal solutions on random inputs. This further develops a probabilistic approach due to Behrendt, Compton and Grädel showing that expected optimal solutions for any problem in the class MAX Σ_1 grows essentially like a polynomial. We show that there is a similar result for MIN $F^+\Pi_1$, a syntactic class of minimization problems which provides a logical criterion for approximability. As a consequence, we show that some important problems do not belong to MIN $F^+\Pi_1$.

1 Introduction

Many NP decision problems are derived from optimization problems by putting a bound on the cost of a desired solution. The NP-completeness of the corresponding decision problem implies that the underlying optimization problem is not solvable in polynomial time, unless P = NP. However, it still may admit efficient *approximation algorithms* which find solutions whose relative error compared to an optimal solution is bounded by a constant. Although many results about approximation properties of optimization problems are known, it would be interesting to have general "structural" reasons why certain problems are approximable and others are not.

Papadimitriou and Yannakakis [12] discovered that the approximation properties of optimization problems are related, at least in some important cases, to their representation by logical formulae. Motivated by Fagin's characterization of NP in terms of existential second-order logic [6], they introduced the classes MAX SNP and MAX NP and showed that every problem in these classes is approximable in polynomial time up to a constant factor. Kolaitis and Thakur

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[7] investigated the logical definability of optimization problems in a systematic way. They introduced the classes MAX Σ_n (resp. MAX Π_n) consisting of all maximization problems $\mathcal{Q}$ whose input instances are finite structures $\mathfrak{A}$ of a fixed vocabulary σ, such that the cost of an optimal solution of $\mathcal{Q}$ on input $\mathfrak{A}$ is definable by an expression

$$\mathrm{opt}_{\mathcal{Q}}(\mathfrak{A}) = \min_{\bar{S}} |\{\bar{x} : \mathfrak{A} \models \psi(\bar{x}, \bar{S})\}|$$

where $\psi(\bar{x}, \bar{S})$ is a Σ_n-formula (resp. Π_n-formula) and where $\bar{S}$ are predicate variables not contained in

σ. Recall that Σ_n (resp. Π_n) are the classes of first-order formulae in prenex normal form with n alternating blocks of quantifiers beginning with $\exists$ (resp. $\forall$). The class MAX FO consists of the maximization problems that are definable in the same way by arbitrary first-order formulae.

Kolaitis and Thakur proved that these classes collapse to a strict hierarchy of four levels.

$$\text{MAX } \Sigma_0 \subsetneq \text{MAX } \Sigma_1 \subsetneq \text{MAX } \Sigma_2 = \text{MAX } \Pi_1 \subsetneq \text{MAX } \Pi_2 = \text{MAX FO}$$

and that all maximization problems with polynomially bounded cost function fall into this hierarchy. The classes MAX SNP and MAX NP are the closures of MAX Σ_0 and MAX Σ_1 under appropriate approximation-preserving reductions. Thus, the class MAX Σ_1 provides a sufficient *syntactic criterion for approximability*.

For minimization problems, the definability classes MIN Σ_n and MIN Π_n are defined analogously. Surprisingly the situation for minimization is not dual to maximization. The minimization classes collapse to a strict hierarchy of only two levels:

$$\text{MIN } \Sigma_0 = \text{MIN } \Sigma_1 \subsetneq \text{MIN } \Sigma_2 = \text{MIN } \Pi_1 = \text{MIN FO}$$

Moreover, even the class at the bottom level, MIN Σ_0, contains problems that are not approximable if $P \neq NP$. Thus, in the case of minimization the quantifier structure of the formulae does not suffice to establish a syntactic criterion for approximability.

However, by restricting also the occurrences of the new relation symbols, Kolaitis and Thakur could isolate a different syntactic class, called MIN $F^+\Pi_1$, all whose problems are approximable up to a constant factor:

Definition 1.1 For every $k \in \mathbb{N}$, the class MIN $F^+\Pi_1(k)$ consists of all minimization problems whose optimum can be expressed as

$$\mathrm{opt}_{\mathcal{Q}}(\mathfrak{A}) = \min_{S}\{|S| : (\mathfrak{A}, S) \models \forall \bar{x} \psi(\bar{x}, S)\},$$

where S is a single predicate, $\psi(\bar{x}, S)$ is a quantifier-free formula in conjunctive normal form (CNF), in which all occurrences of S are positive and S occurs at most k times in each clause. Moreover, let MIN $F^+\Pi_1 = \bigcup_k$ MIN $F^+\Pi_1(k)$.

MIN $F^+\Pi_1$ is a proper subclass of MIN Σ_0 (see in [8] for details). A canonical example for a problem in MIN $F^+\Pi_1$ is MIN VERTEX COVER: Given a graph, it asks for a smallest vertex cover, i.e. a set C of vertices such that every edge contains at least one vertex of C. Indeed, this problem is definable by the expression

$$\mathrm{opt}_{\mathrm{MIN\ VC}}(G) = \min_S\{|S| : (G,S) \models \forall x \forall y (Exy \rightarrow (Sx \vee Sy))\}.$$

Also the generalization of MIN VERTEX COVER to k-hypergraphs is expressible in MIN $F^+\Pi_1$, for any $k \in \mathbb{N}$. It is known that there exist efficient approximation algorithms for these problems. Moreover, every problem in MIN $F^+\Pi_1$ is reducible to some MIN k-HYPERVERTEX COVER problem by an approximation preserving reduction. As a consequence, the class MIN $F^+\Pi_1$ provides a logical criterion for approximability.

Behrendt, Compton and Grädel introduced in [2] a probabilistic approach that characterizes the growth of expected optimal solution sizes for problems that belong to MAX Σ_1. The method is related to the theory of asymptotic probabilities for logically defined properties and to the probabilistic method from combinatorics. The main result states that for every problem in MAX Σ_1, the expected value of an optimal solution on a random input grows like $\Theta(p(n))$, where $p(n)$ is a polynomial and n the cardinality of the input. This remains true for an extended class MAX Σ_1^{FP} which allows the formulae to contain predicates definable in fixed point logic (see [2] for details). Recently, this approach was extended to definability classes of counting functions [5].

In this paper we will show that a similar probabilistic criterion holds for the class MIN $F^+\Pi_1$. As an application we will show that MIN DOMINATING SET does not belong to MIN $F^+\Pi_1$ because almost surely the size of a minimal dominating set grows asymptotically like $\log n$. It is also known (see in [1] and [3]) that the chromatic number of a random graph grows asymptotically like $n/2\log n$. Therefore MIN GRAPH COLOURING does not belong to MIN $F^+\Pi_1$ either.

2 Preliminaries

We begin by presenting the necessary material on optimization problems.

Definition 2.1 An *NP optimization problem* is a tuple $\mathcal{Q} = (\mathcal{I}_\mathcal{Q}, \mathcal{F}_\mathcal{Q}, f_\mathcal{Q}, \mathrm{opt})$ such that

- $\mathcal{I}_\mathcal{Q}$ is the set of *input instances*.
- $\mathcal{F}_\mathcal{Q}(I)$ is the set of *feasible solutions* for input I. Here 'feasible' means that the size of the elements $S \in \mathcal{F}_\mathcal{Q}(I)$ is polynomially bounded in the size of I and that the set $\{(I,S) : S \in \mathcal{F}_\mathcal{Q}(I)\}$ is recognizable in polynomial time.
- $f_\mathcal{Q} : \{(I,S) : S \in \mathcal{F}_\mathcal{Q}(I)\} \rightarrow \mathbb{N}$ is a polynomial-time computable function, called the *cost function*.

- $\mathrm{opt}_\mathcal{Q}$ is one of the two functions defined below with $\mathcal{I}_\mathcal{Q}$ as domain and positive integers as values:

  $$\mathrm{opt}_\mathcal{Q}(I) = \min_S f_\mathcal{Q}(I,S) \quad \text{or} \quad \mathrm{opt}_\mathcal{Q}(I) = \max_S f_\mathcal{Q}(I,S).$$

  In the first case we say $\mathcal{Q}$ is a *minimization problem* and in the second case we say $\mathcal{Q}$ is a *maximization problem.*

For every NP optimization problem $\mathcal{Q}$, the following decision problem is in NP: Given an instance I of $\mathcal{Q}$ and a natural number k, is there a solution $S \in \mathcal{F}_\mathcal{Q}(I)$ such that $f_\mathcal{Q}(I,S) \leq k$ when opt $=$ min, (or $f_\mathcal{Q}(I,S) \geq k$, when opt $=$ max).

Example: MIN DOMINATING SET. The set of input instances $\mathcal{I}_\mathcal{Q}$ is the set of all graphs $G = (V,E)$. The set $\mathcal{F}_\mathcal{Q}(G)$ consists of all dominating sets of G, i.e., of the subsets D of V such that every vertex of $V - D$ has a neighbour in D. The cost function $f_\mathcal{Q}$ gives the cardinality of D and the optimum is the minimum of $f_\mathcal{Q}$ over all dominating sets.

Definition 2.2 We say that an NP optimization problem $\mathcal{Q}$ is *constant-approximable* if there exist a constant c and a polynomial time algorithm that produces a feasible solution S for a given instance I of $\mathcal{Q}$ such that

$$c \geq \begin{cases} \frac{f_\mathcal{Q}(I,S)}{\mathrm{opt}_\mathcal{Q}(I)} & \text{if } \mathcal{Q} \text{ is a minimization problem} \\ \\ \frac{\mathrm{opt}_\mathcal{Q}(I)}{f_\mathcal{Q}(I,S)} & \text{if } \mathcal{Q} \text{ is a maximization problem} \end{cases}$$

A weaker notion of approximability used in the literature requires that only the cost $opt_\mathcal{Q}(I)$ of an optimal solution be approximated; thus for MIN DOMINATING SET it would only be required that the algorithm approximates the domination number $\gamma(G)$, not that it actually finds a nearly minimal dominating set.

The class MIN $F^+\Pi_1$ as given by Definition 1.1 provides a sufficient logical criterion for the approximability in this weaker sense:

Theorem 2.3 *If $\mathcal{Q} \in$ MIN $F^+\Pi_1$ then there is a polynomial-time algorithm approximating $\mathrm{opt}_\mathcal{Q}$ up to a constant factor.*

It is possible to modify the definition of expressibility classes like MAX Σ_1 and MIN $F^+\Pi_1$ to ensure that all problems in these classes are indeed constant-approximable also in the stronger sense as given by Definition 2.2. When the stronger notion of approximability is used, then a more restrictive notion for the representation of an optimization problem by a logical formula is required. Unfortunately this problem is swept under the carpet by most authors. We refer to [2, 11] for a discussion of this issue and for precise definitions.

As usual, for functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f = \Theta(g)$, if there are constants $c, d > 0$ such that $cg(n) \leq f(n) \leq dg(n)$ for sufficiently large n, and we write $f \sim g$ to indicate that $\lim_{n\to\infty} f(n)/g(n) = 1$.

3 Atomic types and asymptotic probabilities

In this section we give some definitions and results from the theory of asymptotic probabilities.

Definition 3.1 Let σ be a finite relational vocabulary and $x_1, \ldots, x_k$ be a sequence of distinct variables. A maximally consistent set t of σ-atoms and negated σ-atoms (including equalities and inequalities) in $x_1, \ldots, x_k$ is called an *atomic σ-type in $x_1 \ldots, x_k$*. Since such a set is always finite, we can form in first-order logic the conjunction of the formulae in t; by abuse of notation, we denote this conjunction by $t(x_1, \ldots, x_k)$. On every σ-structure $\mathfrak{A}$, the type t defines the set of realizations $t^{\mathfrak{A}} = \{\bar{u} \in A^k : \mathfrak{A} \models t(\bar{u})\}$.

A $\varnothing$-type is called an equality type e. So e is a maximally consistent set of equalities $x_i = x_j$ and inequalities $x_i \neq x_j$, where $1 \leq i < j \leq k$, which defines on every structure $\mathfrak{A}$ the set $e^{\mathfrak{A}} = \{\bar{u} \in A^k : \mathfrak{A} \models e(\bar{u})\}$.

Atomic types play an important rôle in considerations about asymptotic probabilities of logically defined properties. Given a σ-sentence ψ of some logic L and a sequence $(\Omega(n))_{n\in\mathbb{N}}$ of probability spaces, where $\Omega(n)$ contains σ-structures with universe $\{0, \ldots, n-1\}$, we denote by $\mu_n(\psi)$ the probability that $\mathfrak{A} \models \psi$ for a randomly chosen $\mathfrak{A} \in \Omega(n)$. An interesting question is, whether the limit $\mu(\psi) := \lim_{n\to\infty} \mu_n(\psi)$ exists and to determine the possible values of $\mu(\psi)$ as ψ ranges over the sentences in L.

For the uniform distribution (i.e. $\Omega(n)$ assigns to all σ-structures over $\{0, \ldots, n-1\}$ the same probability), there is a 0-1 law for many important logics, such as first-order logic, fixpoint logic, infinitary logic $L^{\omega}_{\infty\omega}$ and for some fragments of existential second-order logic. This means that for every sentence in these logics, the asymptotic probability $\mu(\psi)$ does always exist and is either 0 or 1. We refer to [4] for a survey on asymptotic probabilities.

Most of the 0-1 laws proved so far for logical languages rely on the so-called extension axioms. Suppose we have atomic types $s(x_1, \ldots, x_k)$ and $t(x_1, \ldots, x_k, x_{k+1})$ such that $s \subset t$. Then, in a random structure $\mathfrak{A}$, we can almost surely extend every realization $\bar{u} \in s^{\mathfrak{A}}$ to a realization $(\bar{u}, u_{k+1}) \in t^{\mathfrak{A}}$.

Kolaitis and Vardi [10] established a 0-1 law for the infinitary logic $L^{\omega}_{\infty\omega}$. In fact they proved a stronger result: Every formula in $L^{\omega}_{\infty\omega}$ is almost surely equivalent to a (finite) disjunction of atomic types.

Theorem 3.2 *For every formula $\psi(\bar{x})$ in the infinitary logic $L^{\omega}_{\infty\omega}$ there exist atomic types $t_i(\bar{x})$ such that*

$$\mathfrak{A} \models \forall\bar{x}(\psi(\bar{x}) \leftrightarrow \bigvee_i t_i(\bar{x}))$$

for all but an exponentially decreasing fraction of structures $\mathfrak{A}$ when the size of $\mathfrak{A}$ tends to infinity.

We will use this theorem to perform a probabilistic analysis of the expressive power of the class MIN $F^+\Pi_1$. In addition we will need some results about the expected size of atomic types on random structures.

For every equality type e there is a polynomial $p_e(n)$ such that $|e^{\mathfrak{A}}| = p_e(n)$ for each $\mathfrak{A}$ of cardinality n. Every σ-type t defines an equality type e such that $e \subseteq t$ and for every σ-structure $\mathfrak{A}$, $t^{\mathfrak{A}} \subseteq e^{\mathfrak{A}}$.

For each $n \in \mathbb{N}$, let $\Omega(n)$ be the probability space of all σ-structures over universe $\{0, \dots, n-1\}$ with uniform probability distribution, and let T be a random variable on each $\Omega(n)$ defined by $T(\mathfrak{A}) = |t^{\mathfrak{A}}|$ where t is an atomic type. The expectation of T is denoted by $E(T)$. The following observation is obvious.

Lemma 3.3 *Let t be an atomic σ-type and e be the equality type defined by t. There exists a constant c_t such that almost surely*

$$E(T) = c_t p_e(n).$$

Now, let t be a type of tuples $\bar{x}_1, \dots, \bar{x}_k$. Note that we can decompose the formula $t(\bar{x}_1, \dots, \bar{x}_k)$ in the following way:

$$t(\bar{x}_1, \dots, \bar{x}_k) = t_1(\bar{x}_1) \wedge \dots \wedge t_k(\bar{x}_k) \wedge \varphi_e(\bar{x}_1, \dots, \bar{x}_k) \wedge \beta(\bar{x}_1, \dots, \bar{x}_k)$$

such that t_i is the type of $\bar{x}_i$, such that φ_e contains only equalities $y = z$ and inequalities $y \neq z$ where y and z are components of two different tuples among $\bar{x}_1, \dots, \bar{x}_k$, and where β contains only atoms and negated atoms (no equalities and inequalities) $R\bar{y}$ where $\bar{y}$ intersects at least two tuples among $\bar{x}_1, \dots, \bar{x}_k$.

We call two realizations $\bar{u} = \bar{u}_1, \dots, \bar{u}_k$ and $\bar{v} = \bar{v}_1, \dots, \bar{v}_k$ of t in $\mathfrak{A}$ *disjoint* if $\bar{u}_i \neq \bar{v}_j$ for all i, j. This does not exclude that some component of $\bar{u}_i$ appears also in $\bar{v}_j$.

Lemma 3.4 *Let t be an atomic type in $\bar{x}_1, \dots, \bar{x}_k$ with the decomposition*

$$t(\bar{x}_1, \dots, \bar{x}_k) = t_1(\bar{x}_1) \wedge \dots \wedge t_k(\bar{x}_k) \wedge \varphi_e(\bar{x}_1, \dots, \bar{x}_k) \wedge \beta(\bar{x}_1, \dots, \bar{x}_k)$$

as above. Let $p_i(n)$ be the size of the equality type e_i that corresponds to t_i and let $q(n) = \min_i p_i(n)$. Then there exists a constant $d > 0$ such that for a random structure $\mathfrak{A} \in \Omega(n)$, there exist almost surely at least $dq(n))$ pairwise disjoint realizations $\bar{u} = (\bar{u}_1, \dots, \bar{u}_k)$ in $t^{\mathfrak{A}}$.

Proof. For simplicity of notation, we assume that $q(n) = p_1(n) \leq \dots \leq p_k(n)$.

We extend all $\bar{u}_1 \in e_1^{\mathfrak{A}}$ in some arbitrary order to a tuple $\bar{u} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k)$ such that

- $\bar{u}$ is of type e, where e is the equality type that corresponds to t.
- the jth component $\bar{u}_j$ of $\bar{u}$ is different from the jth components of all previously chosen tuples $\bar{u}$. This is possible because $p_i(n) \leq p_j(n)$ for $i < j$.

Let U be the set of the tuples $\bar{u}$ obtained in this way. Clearly $|U| = |e_1^{\mathfrak{A}}| = q(n)$. Almost surely at least a constant fraction $cq(n)$ of these tuples belong to $t^{\mathfrak{A}}$. Moreover, each $\bar{u} \in U$ has a common component with at most $k(k-1)$ other tuples of U. So almost surely we have at least $cq(n)/k(k-1)$ disjoint tuples in $t^{\mathfrak{A}}$. ■

4 A probabilistic criterion for MIN $F^+\Pi_1$

In this section we establish a probabilistic (necessary) criterion for membership of an optimization problem $\mathcal{Q}$ in MIN $F^+\Pi_1$.

First we need a normal form for first-order formulae that represent problems $\mathcal{Q}$ in the class MIN $F^+\Pi_1$.

Lemma 4.1 *Let $\mathcal{Q}$ be a problem in* MIN $F^+\Pi_1(k)$. *Then there exists an existential first-order formula $\psi(\bar{y}_1,\ldots,\bar{y}_k)$ such that for all input structures $\mathfrak{A}$*

$$opt_{\mathcal{Q}}(\mathfrak{A}) = \min_S\{|S| : (\mathfrak{A},S) \models \forall\bar{y}_1\ldots\forall\bar{y}_k[\psi(\bar{y}_1,\ldots,\bar{y}_k) \rightarrow \bigvee_{j=1}^{k} S\bar{y}_j]\},$$

and $\bar{y}_j$ is of length m.

PROOF. According to Definition 1.1 we can express the optimum of $\mathcal{Q}$ for all input structures $\mathfrak{A}$ as:

$$\mathrm{opt}_{\mathcal{Q}}(\mathfrak{A}) = \min_S\{|S| : (\mathfrak{A},S) \models \forall\bar{y}\hat{\psi}(\bar{y},S)\},$$

where S is of arity m, $\bar{y} = (y_1,\ldots,y_l)$ and $\hat{\psi}(\bar{y},S)$ is a quantifier-free CNF formula in which all occurrences of S are positive, and S occurs at most k times in each clause. So there exist formulae $\psi_i(\bar{y},S)$ such that $\hat{\psi}(\bar{y},S) \equiv \bigwedge_i \psi_i(\bar{y},S)$ and $\psi_i(\bar{y},S) \equiv \neg\varphi_i(\bar{y}) \vee \bigvee_{j=1}^{t_i} S\bar{y}_{i_j}$, $t_i \leq k$, where φ_i is the negation of the disjunction of the formulas where S does not appear and $\bar{y}_{i_j}$ is of length m. Without loss of generality we can assume that S occurs exactly k times in each clause. We can then make the following transformations.

$$\begin{aligned}
\forall\bar{y}\hat{\psi}(\bar{y},S) &\equiv \forall\bar{y}[\bigwedge_i \psi_i(\bar{y},S)]\\
&\equiv \forall\bar{y}[\bigwedge_i(\varphi_i(\bar{y}) \rightarrow \bigvee_{j=1}^{k} S\bar{y}_{i_j})]\\
&\equiv \forall\bar{z}_1\ldots\forall\bar{z}_k\forall\bar{y}[\bigwedge_i(\varphi_i(\bar{y}) \wedge \bar{z}_1 = \bar{y}_{i_1} \wedge\ldots\wedge \bar{z}_k = \bar{y}_{i_k} \rightarrow \bigvee_{j=1}^{k} S\bar{z}_j)]\\
&\equiv \forall\bar{z}_1\ldots\forall\bar{z}_k\forall\bar{y}[(\bigvee_i(\varphi_i(\bar{y}) \wedge \bigwedge_{r=1}^{k} \bar{z}_r = \bar{y}_{i_r})) \rightarrow \bigvee_{j=1}^{k} S\bar{z}_j]\\
&\equiv \forall\bar{z}_1\ldots\forall\bar{z}_k[(\exists\bar{y}(\bigvee_i(\varphi_i(\bar{y}) \wedge \bigwedge_{r=1}^{k} \bar{z}_r = \bar{y}_{i_r}))) \rightarrow \bigvee_{j=1}^{k} S\bar{z}_j]
\end{aligned}$$

■

Let $\mathcal{Q}$ be an optimization problem whose input instances are finite structures over a fixed vocabulary σ. We consider the behaviour of $\mathrm{opt}_{\mathcal{Q}}(\mathfrak{A})$ on a randomly chosen σ-structure $\mathfrak{A}$. Again, let $\Omega(n)$ be the probability space of all σ-structures over universe $\{0, \ldots, n-1\}$ with uniform probability distribution. So $\mathrm{opt}_\mathcal{Q}$ is a random variable on each $\Omega(n)$ whose expected value is denoted by $E(\mathrm{opt}_\mathcal{Q})$.

Theorem 4.2 (Probabilistic criterion for Min $F^+\Pi_1$**)** *Let $\mathcal{Q}$ be a problem in* Min $F^+\Pi_1$. *Then there exists a polynomial $p(n)$ and a constant $\varepsilon > 0$ such that almost surely*

$$\varepsilon p(n) \leq opt_\mathcal{Q}(\mathfrak{A}) \leq p(n)$$

for a randomly chosen σ-structure $\mathfrak{A}$.

Proof. For every $\mathcal{Q} \in$ Min $F^+\Pi_1$ there exists a $k \in \mathbb{N}$ such that $\mathcal{Q} \in$ Min $F^+\Pi_1(k)$. By Lemma 4.1 we can express the optimum of $\mathcal{Q}$ as:

$$\mathrm{opt}_\mathcal{Q}(\mathfrak{A}) = \min_S\{|S| : (\mathfrak{A}, S) \models \forall\bar{y}_1 \ldots \forall\bar{y}_k[\psi(\bar{y}_1, \ldots, \bar{y}_k) \rightarrow \bigvee_{j=1}^{k} S\bar{y}_j]\}$$

where $\bar{y}_j$ is of length m. By Lemma 3.2 there exist atomic types $t_i(\bar{x}_1, \ldots, \bar{x}_k)$ such that

$$\mathfrak{A} \models \forall\bar{x}_1 \cdots \forall\bar{x}_k(\psi(\bar{x}_1, \ldots, \bar{x}_k) \leftrightarrow \bigvee_i t_i(\bar{x}_1, \ldots, \bar{x}_k))$$

for all but an exponentially decreasing fraction of structures $\mathfrak{A}$.

We decompose each atomic type t_i as in the previous section and write

$$t_i(\bar{x}_1, \ldots, \bar{x}_k) = t_{i,1}(\bar{x}_1) \wedge \ldots t_{i,k}(\bar{x}_k) \wedge \varphi_i(\bar{x}_1, \ldots, \bar{x}_k) \wedge \beta_i(\bar{x}_1, \ldots, \bar{x}_k).$$

The equality type associated with $t_{i,j}(\bar{x}_j)$ is denoted $e_{i,j}$ and its size is given by the polynomial $p_{i,j}(n)$. For all i there exists a $j(i)$ such that $p_{i,j}(n) = \min_\ell p_{i,\ell}(n)$, i.e., $e_{i,j(i)}$ is an equality type of minimal size among the $e_{i,\ell}$.

Let $q_i(n) = p_{i,j(i)}(n)$ and set $S_0(\mathfrak{A}) = \bigcup_i t_{i,j(i)}^{\mathfrak{A}}$. It is clear that S_0 is a feasible solution for $\mathcal{Q}$ on $\mathfrak{A}$, and

$$|S_0(\mathfrak{A})| \leq \sum_i q_i(n) = O(\max_i q_i(n)).$$

On the other side, Lemma 3.4 tells us that almost surely, there exist $cq_i(n)$ disjoint tuples $\bar{u}_1, \ldots, \bar{u}_k$ such that $\mathfrak{A} \models t_i(\bar{u}_1, \ldots, \bar{u}_k)$. For each of these tuples a feasible solution $S(\mathfrak{A})$ must contain at least one component. Since the tuples are disjoint, it follows that $\mathrm{opt}_\mathcal{Q}(\mathfrak{A}) \geq cq_i(n)$. ■

A closer look at the proof of the theorem reveals that the statement fails only on a exponentially decreasing fracture of structures, and we can infer:

Corollary 4.3 *Let Q be a problem in* MIN $F^+\Pi_1$. *Then there exists a polynomial $p(n)$ such that*

$$E(opt_Q) = \Theta(p(n)).$$

Proposition 4.4 MIN DOMINATING SET *is not in* MIN $F^+\Pi_1$.

PROOF. As usual in graph theory, we denote the size of a minimal dominating set of a graph G by $\gamma(G)$ (the domination number). We prove that for a random graph G with n nodes and edge probability $1/2$, almost surely

$$\gamma(G) \sim \log n.$$

This fact together with Theorem 4.2 implies the statement of the proposition. Let $X_r(G)$ be the number of different dominating sets of cardinality r. Thus

$$\gamma(G) = \min\{r \in \mathbb{N} : X_r(G) \geq 1\}.$$

We estimate the random variable X_r for $r = r(n)$.

First, let D be an arbitrary set of $r(n) = \log n$ vertices. The probability that a fixed vertex of $V - D$ has no edge into D is $2^{-\log n} = 1/n$. Hence with probability $1 - n^{-1}$ there is an edge into D, and since we have $n - \log n$ vertices in $V - D$ the probability that D is a dominating set is $d(n) = (1 - n^{-1})^{n - \log n}$. Since $\lim_{n\to\infty} d(n) = e^{-1}$, we infer that almost surely $\gamma(G) \leq \log n$.

Now, let $r(n) = (1-\varepsilon)\log n$ for any $\varepsilon > 0$. There are $\binom{n}{r(n)}$ different choices for D in V. Define the indicator random variable X^D by

$$X_r^D(G) := \begin{cases} 1 & : \quad D \text{ is a dominating set of } G \\ 0 & : \quad \text{otherwise} \end{cases}$$

Obviously, $X_r = \sum_D X_r^D$. By linearity of expectation it follows that

$$\begin{aligned} E(X_r) &= \sum_D E(X_r^D) = \binom{n}{(1-\varepsilon)\log n}(1 - n^{-1+\varepsilon})^{n-(1-\varepsilon)\log n} \\ &= \binom{n}{(1-\varepsilon)\log n}(1 - n^{-1+\varepsilon})^{n^{1-\varepsilon}n^{-1+\varepsilon}(n-(1-\varepsilon)\log n)} \end{aligned}$$

Recall that $\lim_{n\to\infty}(1 - n^{-s})^{n^s} = e^{-1}$ and that $\binom{n}{m} \leq n^m$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} E(X_r) &\leq \lim_{n\to\infty} n^{(1-\varepsilon)\log n} e^{-n^{-1+\varepsilon}(n-(1-\varepsilon)\log n)} \leq \\ &\leq \lim_{n\to\infty} e^{O(\log^2 n) - n^{\varepsilon}} = 0. \end{aligned}$$

By Markov's inequality, $P[X_r \geq 1] \leq E(X_r)$, so almost surely a random graph with n vertices does not contain a dominating set of cardinality $(1-\varepsilon)\log n$.

Thus, $\gamma(G) \sim \log n$ almost surely. ∎

Proposition 4.5 MIN GRAPH COLOURING *is not in* MIN $F^+\Pi_1$.

PROOF. By [1] almost surely the chromatic number $\chi(G)$ of a graph G satisfies $\chi(G) \sim n/2\log n$. ∎

5 Closures under infinitary reductions

As in [2] we will show that the criterion remains true for extensions of MIN $F^+\Pi_1$ which use more powerful logics than first-order logic.

Definition 5.1 Let L and K be classes of logical formulae. A formula ψ over vocabulary σ is in L^K if it can be obtained from a formula $\varphi \in L$ of vocabulary $\sigma \cup \{P_1, \dots, P_k\}$ and formulae $\alpha_1(\bar{z}_1), \dots, \alpha_k(\bar{z}_k) \in K$ of vocabulary σ by substituting $\alpha_i(\bar{x})$ for all occurrences of atoms $P_i(\bar{x})$ in φ, i.e.,

$$\psi = \varphi(\alpha_1/P_1, \dots, \alpha_k/P_k).$$

Theorem 5.2 (Extended probabilistic criterion) *Let $K \subseteq L^\omega_{\infty\omega}$ and let $\mathcal{Q}$ be a problem in* MIN $F^+\Pi_1^K$. *Then there exists a polynomial $p(n)$ such that almost surely $opt_{\mathcal{Q}}(\mathfrak{A}) = \Theta(p(n))$ for a randomly chosen σ-structure $\mathfrak{A}$.*

The proof follows from Lemma 3.2 which implies that in a probabilistic sense, the predicates from K can be eliminated. Thus, for every formula ψ in $F^+\Pi_1^K$, there is a formula ψ' in $F^+\Pi_1$ such that ψ is equivalent to ψ' almost surely.

In particular the probabilistic criterion holds for MIN $F^+\Pi_1^{\mathrm{FP}}$ because fixed point logic is contained in $L^\omega_{\infty\omega}$. Note that fixed point logic reduction is a polynomial-time reduction and that in general polynomial-time reductions do not preserve asymptotic probabilities.

Corollary 5.3 MIN GRAPH COLOURING *and* MIN DOMINATING SET *are not in the class* MIN $F^+\Pi_1^{\mathrm{FP}}$.

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