0-1 Laws for Recursive Structures*

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Abstract

We discuss resource-bounded measures on the class of recursive structures and prove that with respect to such measures a random recursive structure is almost surely isomorphic to the unique countable model of the extension axioms.

1 Introduction

An interesting branch of model theory is the study of 0-1 laws. A class of structures on which a probability measure has been defined has a 0-1 law for a particular logic if the class of models of every sentence in this logic has measure 0 or 1.

In finite model theory one studies 0-1 laws for asymptotic probabilities. Fix a finite relational vocabulary τ and let $\mu_n(\psi)$ denote the fraction of the τ-structures with universe $\{0,\dots,n-1\}$ that satisfy ψ. The 0-1 law for first order logic, proved independently by Fagin [2] and Glebskii et al. [4], states that for first-order sentences the limit $\mu(\psi) = \lim_{n\to\infty} \mu_n(\psi)$ always exists and is either 0 or 1. Gaifman [3] considered countable structures with universe ω and finite relational vocabulary. He proved that with respect to the Lebesgue measure every property of countable structures that is closed under isomorphisms has probability either 0 or 1. The proof is based on the fact that the set of *extension axioms* of a finite relational vocabulary σ has a unique countable model, called the *countable random σ-structure* $\mathcal{R}$, and that the isomorphism class of $\mathcal{R}$ has measure one.

Hirst and Harel [6] examined recursive structures. They studied infinite sequences of recursive structures on which the extension axioms have asymptotic probability 1 and proved that first-order logic satisfies the 0-1 law on such sequences. We present here a different approach to 0-1 laws for recursive structures. We use *resource-bounded measure theory* (introduced by Lutz) to define a meaningful measure on classes of recursive structures and prove 0-1 laws with respect to this measure.

A non-standard, but convenient way to describe the Lebesgue measure on $\{0,1\}^\omega$ is in terms of betting strategies or, equivalently, *martingales*. A player repeatedly bets part of her current capital on the successive bits of an unknown string $\chi \in \{0,1\}^\omega$. If she correctly predicts the next bit she wins twice the money she has bet, otherwise

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she gets nothing. The player *succeeds* on χ if she becomes infinitely rich. Clearly, she tries to find a betting strategy that is always useful. It is known that a class $\mathcal{C} \subseteq \{0,1\}^\omega$ has Lebesgue measure 0 iff there exists a strategy to succeed on every string of $\mathcal{C}$. By imposing restrictions on the martingales describing the strategies one can define *resource-bounded* measures on small subsets of $\{0,1\}^\omega$. For instance, by restricting the player to recursive martingales a measure on the set of recursive strings is obtained. For an extensive study of martingales we refer to [9].

Since recursive σ-structures can be identified with recursive strings in $\{0,1\}^\omega$ we obtain a meaningful measure for recursive structures. We show that with respect to this measure, the isomorphism class of the random σ-structure $\mathcal{R}$ has measure one also within the class of recursive structures. We prove this by exhibiting a recursive betting strategy which is successful on all recursive structures on which at least one extension axiom fails. As a consequence, every logic has a 0-1 law on the class of recursive structures.

2 Preliminaries

Let σ be a finite vocabulary of relation symbols.

Definition 2.1. A *countable* σ-structure is a σ-structure with universe $A = \omega$. We denote the class of countable σ-structures by $\mathtt{str}_\omega(\sigma)$. A countable σ-structure $\mathfrak{A}$ is *recursive* if every relation of $\mathfrak{A}$ is recursive. We write $\mathtt{recstr}(\sigma)$ for the class of recursive σ-structures.

For $\mathfrak{A} \in \mathtt{str}_\omega(\sigma)$ and $n \in \mathbb{N}$, we denote by $\mathfrak{A}|_n$ the substructure of $\mathfrak{A}$ with universe $\{0, \dots, n-1\}$.

For every fixed vocabulary σ we can identify countable σ-structures with infinite binary sequences. Here is one possible identification for the case where σ consists of just one binary relation symbol E. A generalization to arbitrary finite vocabularies is straightforward.

Describe a countable structure $\mathfrak{A} = (\omega, E)$ by an infinite binary table $(a_{i,j})_{i,j\in\omega}$ where $a_{i,j} = 1$ if $(i,j) \in E$ and $a_{i,j} = 0$ otherwise. Any bijection $f : \omega \times \omega \to \omega$ will now produce an encoding $\chi_\mathfrak{A} = c_0 c_1 \cdots$ of $\mathfrak{A}$ by setting $c_{f(i,j)} = a_{i,j}$. For our purpose it is convenient to enumerate $\omega \times \omega$ in such a way that the substructures $\mathfrak{A}|_n$ are represented by initial portions of $\chi_\mathfrak{A}$. We therefore identify a structure $\mathfrak{A}$ described by the table

$$\begin{array}{|ccccc} \hline a_{0,0} & a_{0,1} & \cdots & a_{0,k} & \cdots \\ a_{1,0} & a_{1,1} & \cdots & a_{1,k} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ a_{k,0} & a_{k,1} & \cdots & a_{k,k} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{array}$$

with the sequence $\chi_\mathfrak{A} = a_{0,0}a_{0,1}a_{1,1}a_{1,0}\cdots a_{0,k}\cdots a_{k,k}\cdots a_{k,0}\cdots$.

We call $\chi_\mathfrak{A}$ the *characteristic sequence* of $\mathfrak{A}$. It is clear that $\mathfrak{A}$ is a recursive structure if and only if $\chi_\mathfrak{A}$ is a recursive sequence. Thus we can identify the class of recursive σ-structures with the class $\mathtt{REC}$ of recursive sequences in $\{0,1\}^\omega$.

An interesting subclass of $\mathtt{str}_\omega(\sigma)$ is $\mathtt{pstr}(\sigma)$. It consists of all σ-structures $\mathfrak{A}$ that are polynomial-time computable in the sense that there exists a polynomial $p(n)$ such that each substructure $\mathfrak{A}|_n$ can be computed in time $p(n)$. We call these structures *polynomial structures*. We can identify $\mathtt{pstr}(\sigma)$ with the class $\mathtt{POL}$ of *polynomial sequences* in $\{0,1\}^\omega$, i.e., the sequences whose initial substrings of length n can be computed in time $p(n)$ where $p(n)$ is a polynomial.

For each fixed vocabulary σ we thus have the following identifications:

$$\begin{aligned} \mathtt{str}_\omega(\sigma) &\simeq \{0,1\}^\omega \\ \mathtt{recstr}(\sigma) &\simeq \mathtt{REC} \\ \mathtt{pstr}(\sigma) &\simeq \mathtt{POL} \end{aligned}$$

3 A measure on the class of recursive structures

The *Lebesgue probability distribution* on $\{0,1\}^\omega$ can be defined via the random experiment where an infinite binary sequence is chosen by independent tosses of a fair coin for each symbol of the sequence. For any class of sequences $\mathcal{C} \subseteq \{0,1\}^\omega$, we denote by $\mu_L(\mathcal{C})$ the probability that the sequence x is in $\mathcal{C}$ if x is randomly chosen according to the Lebesgue probability distribution. It is well-known that there is a class $\mathcal{C}$ for which $\mu_L(\mathcal{C})$ is not defined.

Lutz [7] introduced in the *resource-bounded measure theory* to investigate problems in structural complexity theory. Our exposition follows Mayordomo's PhD thesis [8].

Let us consider the following single player game. Given an unknown infinite binary sequence χ and a starting capital $c_0 \in \mathbb{R}^+$, the player bets a part of her money on the successive bits of χ in the following way.

Before the n^{th} bet the player knows the first n bits of χ and has capital c_n. She bets an amount $b_n < c_n$ either that the $(n+1)^{\text{st}}$ bit of χ is 0 or 1. If she wins, her capital is increased by b_n so that her new capital is $c_{n+1} = c_n + b_n$; otherwise she looses the stake and is left with capital $c_{n+1} = c_n - b_n$. Of course the player always has the option to pass on the n^{th} bet which simply means that $b_n = 0$.

The player *succeeds* if she gets infinitely rich.

A *strategy* for this game is a function $g : \{0,1\}^* \to \{0,1\} \times [0,\infty)$ that tells the player how to bet according to the information she has. Let w_n be the initial string of χ of length n, then $g(w_n) = (u,v)$ says that she should bet the amount $b_n = v$ on the $(n+1)^{\text{st}}$ bit of χ being u.

According to this strategy-function there is a *capital-function* $d_g : \{0,1\}^* \to [0,\infty)$ which computes the player's capital after initial string w_n if she uses the strategy g, i.e. $d_g(w_n) = c_n$. We can compute d_g recursively from g (and vice versa):

$$d_g(wu) = \begin{cases} d_g(w) + v & \text{if } g(w) = (u,v) \\ d_g(w) - v & \text{if } g(w) = (1-u,v). \end{cases}$$

Functions that behave like d_g are called *martingales*.

Definition 3.1. A *martingale* is a function $d : \{0,1\}^* \to [0,\infty)$ such that

$$d(w) = \frac{d(w0) + d(w1)}{2}$$

for all $w \in \{0,1\}^*$. A martingale d is *successful* for an infinite sequence χ iff $\limsup_{n\to\infty} d(w_n) = \infty$. For each martingale d, we write $S^\infty[d]$ for the set of sequences for which d is successful:

$$S^\infty[d] := \{\chi : \limsup_{n\to\infty} d(w_n) = \infty\}.$$

Classes with Lebesgue measure 0 and 1 can be described by martingales as follows.

- A class $\mathcal{C}$ has *Lebesgue measure* 0 ($\mu_L(\mathcal{C}) = 0$) iff there is a martingale d such that $\mathcal{C} \subseteq S^\infty[d]$.
- A class $\mathcal{C}$ has *Lebesgue measure* 1 ($\mu_L(\mathcal{C}) = 1$) iff its complement $\overline{\mathcal{C}}$ has Lebesgue measure 0.

This is just a reformulation of classical definitions of the Lebesgue measure (compare [8, 9]).

Since the class of recursive sequences is countable it has Lebesgue measure 0. To define a meaningful measure within this class, Lutz restricted the admissible martingales by so-called *measure resource bounds.* We consider two measure resource bounds Γ, namely $\Gamma = \mathtt{rec}$, the class of all recursive functions and $\Gamma = \mathtt{p}$, the class of all polynomial-time computable functions. Γ-measures are defined by appropriate restrictions to the martingale.

Definition 3.2.
- A class $\mathcal{C}$ has Γ-*measure 0* iff there exists a martingale $d \in \Gamma$ such that $\mathcal{C} \subseteq S^\infty[d]$.
- A class $\mathcal{C}$ has Γ-*measure 1* iff its complement $\overline{\mathcal{C}}$ has Γ-*measure 0.*

Lutz proved that REC does not have rec-measure 0 and that POL does not have p-measure 0:

Theorem 3.3 (Lutz).
- *For every recursive martingale d there exists a recursive sequence χ such that d is not successful for χ.*
- *For every polynomial-time computable martingale d there exists a polynomial sequence χ such that d is not successful for χ.*

It turns out that $\Gamma = \mathtt{rec}$ (respectively p) is the largest measure resource bound such that REC (respectively POL) does not have Γ-measure 0. Meaningful measures on REC and POL are obtained by restricting the rec-measure to REC and the pol-measure to POL.

Definition 3.4.
- A class $\mathcal{C}$ has *measure* 0 in REC ($\mu_{\mathtt{rec}}(\mathcal{C}) = 0$) iff $\mathcal{C} \cap \mathtt{REC}$ has rec-measure 0.
- A class $\mathcal{C}$ has *measure* 1 in REC ($\mu_{\mathtt{rec}}(\mathcal{C}) = 1$) iff $\overline{\mathcal{C}}$ has *measure* 0 in REC.
- A class $\mathcal{C}$ has *measure* 0 in POL ($\mu_{\mathtt{p}}(\mathcal{C}) = 0$) iff $\mathcal{C} \cap \mathtt{POL}$ has p-measure 0.
- A class $\mathcal{C}$ has *measure* 1 in POL ($\mu_{\mathtt{p}}(\mathcal{C}) = 1$) iff $\overline{\mathcal{C}}$ has *measure* 0 in POL.

We investigate the notion of probabilities of properties on countable, recursive and polynomial structures using resource bounded measures.

Definition 3.5. Let σ be a fixed vocabulary and let P be a property of countable σ-structures. We define the probability that P is true on recursive σ-structures as the recursive measure of the set of characteristic sequences of recursive σ-structures with property P within REC, provided the measure exists (and similarly for polynomial structures).

$$\mu_L(P) = \mu_L(\{\chi_{\mathfrak{A}} : \mathfrak{A} \in \mathtt{str}_\omega(\sigma) \text{ and } \mathfrak{A} \text{ has property } P\})$$

$$\mu_{\mathtt{rec}}(P) = \mu_{\mathtt{rec}}(\{\chi_{\mathfrak{A}} : \mathfrak{A} \in \mathtt{recstr}(\sigma) \text{ and } \mathfrak{A} \text{ has property } P\})$$

$$\mu_{\mathtt{p}}(P) = \mu_{\mathtt{p}}(\{\chi_{\mathfrak{A}} : \mathfrak{A} \in \mathtt{pstr}(\sigma) \text{ and } \mathfrak{A} \text{ has property } P\}).$$

Let L be a logic and σ a vocabulary. We say that the *0-1 law* holds for L on countable, recursive respectively polynomial σ-structures iff every property P expressible in L has probability 0 or 1 on $\mathtt{str}_\omega(\sigma)$, $\mathtt{recstr}(\sigma)$ respectively $\mathtt{pstr}(\sigma)$.

4 Atomic types and extension axioms

Most of the 0-1 laws for finite structures are based on the fact that the *extension axioms* have asymptotic probability 1. We refer to [1] for a survey on asymptotic probabilities.

Definition 4.1. Let σ be a finite relational vocabulary. A maximally consistent set t of σ-atoms and negated σ-atoms (including equalities and inequalities) in the variables $x_1, \ldots, x_m$ is called an *atomic σ-type in $x_1 \ldots, x_m$*. Since such a set is always finite, we can form in first-order logic the conjunction of the formulae in t; by slight abuse of notation, we denote this conjunction by $t(x_1, \ldots, x_m)$. On every σ-structure $\mathfrak{A}$, the atomic type t defines the set of realizations $t^{\mathfrak{A}} = \{\bar{u} \in A^k : \mathfrak{A} \models t[\bar{u}]\}$.

Let $s(x_1, \ldots, x_m)$ and $t(x_1, \ldots, x_{m+1})$ be atomic types of distinct elements (i.e. containing the inequalities $x_i \neq x_j$ for all $i < j \leqslant m$). We say that $t(x_1, \ldots, x_{m+1})$ is an extension of $s(x_1, \ldots, x_m)$ if $s \subseteq t$, which means that for every structure $\mathfrak{A}$ and elements $u_1, \ldots, u_m, u_{m+1}$,

$$\mathfrak{A} \models t[u_1, \ldots, u_m, u_{m+1}] \Longrightarrow \mathfrak{A} \models s[u_1, \ldots, u_m].$$

Definition 4.2. For every $m, s(x_1, \ldots, x_m)$ and $t(x_1, \ldots, x_{m+1})$, where t is an extension of s, we define the *m-extension axiom*

$$\rho_{s,t} = \forall x_1 \ldots \forall x_m (s(x_1, \ldots, x_m) \to \exists x_{m+1} t(x_1, \ldots, x_{m+1})).$$

Let T be the set of all extension axioms. Using a back and forth argument Gaifman [3] showed that T is an ω-categorical theory, i.e. that T has up to isomorphism exactly one countable model $\mathcal{R}$, called the *countable random structure over σ*. As a consequence, T is complete.

The completeness of T and the fact that every extension axiom has asymptotic probability 1 imply the 0-1 law for first-order logic on finite structures. In fact, a first-order sentence ψ has asymptotic probability 1 if and only if $T \models \psi$, i.e. if and only if ψ holds in $\mathcal{R}$, the countable random structure.

We next collect some simple numerical facts on atomic types and extension axioms. Let $\sigma = \{R_1, \dots , R_k\}$, where each relation R_i is of arity a_i, and let m be a natural number. Define $a = \max a_i$.

(i) Each σ-type t in m variables $x_1, \dots , x_m$ contains $\sum_{i=1}^{k} m^{a_i} \leqslant k \cdot m^a$ different σ-atoms.

(ii) There are $2^{\sum m^{a_i}} \leqslant 2^{k \cdot m^a}$ different σ-types in $x_1, \dots , x_m$.

(iii) The number of m-extension axioms is $2^{\sum (m+1)^{a_i}} \leqslant 2^{k \cdot (m+1)^a}$.

(iv) Let $\rho_{s,t}$ be an m-extension axiom and let the tuple $\bar{u} = (u_1, \dots , u_m)$ be of type s. To check for an arbitrary z whether $(u_1, \dots , u_m, z)$ is of type t needs $\sum_{i=1}^{k}((m + 1)^{a_i} - m^{a_i}) \leqslant k((m + 1)^a - m^a)$ evaluations of atoms.

5 0-1 laws

Gaifman [3] showed that for fixed relational vocabulary σ the isomorphism class of the random σ-structure $\mathcal{R}$ has Lebesgue measure 1 within the class of all countable σ-structures. We prove an analogous result for the class of recursive σ-structures with respect to the `rec`-measure.

Theorem 5.1. *The isomorphism class of $\mathcal{R}$ has recursive measure 1 within* $\mathtt{recstr}(\sigma)$.

Proof. It suffices to show that the conjunction of all extension axioms has recursive measure 1, i.e $\mu_{\mathtt{rec}}(\bigwedge \rho_{s,t}) = 1$. We present a recursive winning strategy, that is successful on all recursive structures that satisfy $\bigvee \neg\rho_{s,t}$. The negation of an m-extension axiom is a formula of the form

$$\neg\rho_{s,t} = \exists\bar{x}(s(\bar{x}) \wedge \forall x_{m+1} \neg t(\bar{x}, x_{m+1})).$$

The intuitive idea of the strategy is to search systematically for a pair $(\rho_{s,t}, \bar{u})$ such that $\mathfrak{A} \models s[\bar{u}]$ but $\mathfrak{A} \models \neg t[\bar{u}, v]$ for all $v \in \omega$. Once such a pair is found the player will always win in every remaining round of the game.

For simplicity of notation we assume that $\sigma = \{E\}$, where E is a binary relation symbol. It is straightforward to extend the proof to arbitrary finite vocabularies.

We play on the infinite sequence $a_{0,0}a_{0,1}a_{1,1}a_{1,0} \dots a_{0,k} \dots a_{k,k} \dots a_{k,0} \dots$ with starting capital c_0. For each n, round n is the portion of the game on the substring $a_{0,n} \dots a_{n,n} \dots a_{n,0}$, i.e. the substring that defines the extension of $\mathfrak{A}|_n$ to $\mathfrak{A}|_{n+1}$.

$$
\begin{array}{|cccc:c:c}
\hline
 & & & & a_{0,n} & \cdots \\
 & \mathfrak{A}|_n & & & a_{1,n} & \cdots \\
 & & & & \vdots & \\
 & & & & a_{n-1,n} & \cdots \\
\hdashline
a_{n,0} & a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} & \cdots \\
\hdashline
\vdots & \vdots & & \vdots & \vdots & \ddots
\end{array}
$$

Let c_n be the capital before round n.

Round 0. The player passes: $b(\varepsilon) = (0, 0)$ and $c_1 = c_0$.

Round n. After playing round $n-1$ the player knows the substructure $\mathfrak{A}|_n$. There are only finitely many extension axioms that do not hold in $\mathfrak{A}|_n$, because all m-extension axioms with $m > n$ are trivially satisfied in $\mathfrak{A}|_n$. For each m-extension axiom $\rho_{s,t}$ let

$$\varphi_{s,t}(\bar{x}) := s(\bar{x}) \wedge \forall x_{m+1} \neg t(\bar{x}, x_{m+1})$$

(expressing that $\rho_{s,t}$ 'fails at $\bar{x}$'). For each tuple $\bar{u} = (u_1, \dots, u_m)$ of distinct elements of $\{0, \dots, n-1\}$ with atomic type s and for every atomic type $t \supseteq s$ of $m+1$ elements, let

$$f(n, t, \bar{u}) := \begin{cases} n - \max(u_1, \dots, u_m) & \text{if } \mathfrak{A} \models \varphi_{s,t}[\bar{u}] \\ 0 & \text{if } \mathfrak{A} \models \neg\varphi_{s,t}[\bar{u}]. \end{cases}$$

Note that, if $\mathfrak{A}|_n \models \varphi_{s,t}[\bar{u}]$ and $\max(u_1, \dots, u_m) < i \leqslant n$ then also $\mathfrak{A}|_i \models \varphi_{s,t}[\bar{u}]$. Therefore $f(n, t, \bar{u})$ describes the number of $i \leqslant n$ with $\mathfrak{A}|_i \models \varphi_{s,t}[\bar{u}]$, i.e. the number of rounds $i \leqslant n$ in which $\bar{u}$ already has witnessed that $\mathfrak{A}|_i \models \neg\rho_{s,t}$.

To describe the strategy of the player in round n we distinguish two cases: If $\mathfrak{A}|_n \models \rho_{s,t}$ for all extension axioms then the player passes in round n and the capital remains the same, $c_{n+1} = c_n$.

Otherwise, compute a pair $(\rho_{s,t}, \bar{u})$ for which $f(n, t, \bar{u})$ is maximal and let $u_{m+1} = n$. The extension $\mathfrak{A}|_{n+1}$ of $\mathfrak{A}|_n$ is described by the bits $a_{0,n} \cdots a_{n,n} a_{n,n-1} \cdots a_{n,0}$ of $\chi_{\mathfrak{A}}$. To determine whether

$$\mathfrak{A}|_{n+1} \models t[\bar{u}, u_{m+1}],$$

exactly the $2m+1$ bits of $Z := \{a_{n,u_i}, a_{u_i,n} : i = 1, \dots, m\} \cup \{a_{n,n}\}$ are relevant and there is precisely one assignment $\alpha : Z \to \{0, 1\}$ such that $\mathfrak{A}|_{n+1} \models t[\bar{u}, u_{m+1}]$.

The idea of the strategy is that the player bets nothing on the bits outside of Z and bets on the outcome $1 - \alpha(z)$ for every $z \in Z$, until the first bet has been successful (after that the player passes for the rest of round n). The capital that the player risks on each bet is calculated in such a way to guarantee that she has won something if one bet has been successful, and that she has half the money left even if she looses on all $2m + 1$ bets.

Let $z_1, \dots, z_{2m+1}$ be the bits of Z in the order in which they are precessed (this order is determined by the order on $u_1, \dots, u_m$). Further, let

$$\varepsilon(n) = \frac{c_n}{2(2^{2m+1}-1)}.$$

Having lost her bets on $z_1, \dots, z_i$, she bets for z_{i+1} the capital $2^i\varepsilon(n)$ on the outcome $1-\alpha(z_{i+1})$. If she wins, she passes for the rest of round n and terminates the round with capital

$$c_{n+1} = c_n - \sum_{j=1}^{i} 2^{j-1}\varepsilon(n) + 2^i\varepsilon(n) = c_n + \varepsilon(n).$$

If she looses on all $2m+1$ bets, her remaining capital is

$$c_{n+1} = c_n - \sum_{i=1}^{2m+1} 2^{i-1}\varepsilon(n) = c_n - (2^{2m+1}-1)\varepsilon(n) = c_n/2.$$

Clearly the player never runs out of money. If $\mathfrak{A}$ satisfies $\bigvee \neg\rho_{s,t}$, then there is a pair $(\rho_{s,t}, \bar{u})$ such that $\mathfrak{A} \models \varphi_{s,t}[\bar{u}]$ that is detected after finitely many rounds (say in round n_0). In each of the following rounds, the player will use this pair (since it continues to be the pair that has survived the most rounds) and thus wins in each of the following rounds. For all $n \geqslant n_0$ we have

$$c_n = c_{n_0} + \sum_{i=n_0}^{n-1} \varepsilon(i)$$

which tends to infinity with growing n.

Clearly the strategy is recursive. ∎

Time complexity of this strategy: It is easy to see that the most time-consuming part of the strategy is to find the pair $(\rho_{s,t}, \bar{u})$ for which $f(n,t,\bar{u})$ is maximal. Let σ be an arbitrary finite relational signature as in Section 4. There are at most

$$\sum_{m=1}^{n-1} \frac{n!}{(n-m)!} \cdot 2^{(k(m+1)^a)} \text{ possible pairs } (\rho_{s,t}, \bar{u}).$$

To check that $\bar{u}$ is of type s requires at most $k \cdot m^a$ evaluations of atoms. We then have to check for all $x_{m+1} \neq u_i$ whether $\bar{u}, x_{m+1}$ is of type t which requires at most $(n-m)k((m+1)^a - m^a)$ evaluations. In the worst case the player therefore needs

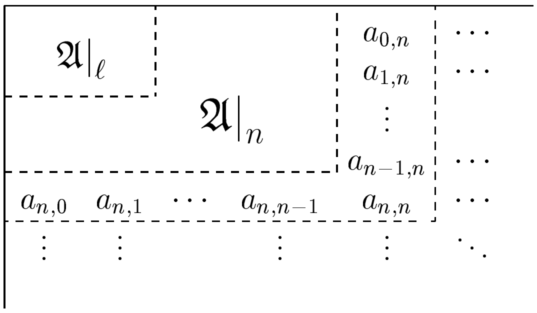
$$\sum_{m=1}^{n-1} \frac{n!}{(n-m)!} \cdot 2^{(k(m+1)^a)} k(m^a + (n-m)((m+1)^a - m^a))$$

basic steps. This is bounded by 2^{n^r} for some natural number r.

Using this estimation we can transform the strategy into a polynomial-time strategy and obtain a 0-1 law for polynomial structures.

Theorem 5.2. *The class of all polynomial structures that are isomorphic to $\mathcal{R}$ has measure 1 in* $\mathtt{pstr}(\sigma)$.

Proof. The main idea of the strategy is that we only consider a substructure $\mathfrak{A}|_\ell$ of $\mathfrak{A}|_n$ which is sufficiently small such that finding the pair $(\rho_{s,t}, \bar{u})$ with maximal value $f(\ell, t, \bar{u})$ can be done in time polynomial in n.



Let $\ell = \log^{1/r} n$ where r is the natural number from the estimation above. The computation of $\varepsilon(n)$ is then also polynomial in n because $m \leqslant \ell$ and we can compute c_n in polynomial-time in n. The rest of the strategy is polynomial-time anyway. It is clear that if $\mathfrak{A}$ satisfies $\neg \bigwedge \rho_{s,t}$ we will also find with this strategy after finitely many rounds the pair $(\rho_{s,t}, \bar{u})$ such that

$$\mathfrak{A} \models \varphi_{s,t}[\bar{u}].$$

This proves the theorem. ∎

As a direct consequence of the completeness of T and Theorems 5.1 and 5.2, we obtain the following general result.

Theorem 5.3 (Transfer Theorem). *For every isomorphism closed property P of countable σ-structures (e.g. for every property expressible in a logic) we have:*

$$\mathcal{R} \textit{ has property } P \iff \mu_L(P) = 1 \iff \mu_{\mathtt{rec}}(P) = 1 \iff \mu_{\mathtt{p}}(P) = 1$$

6 Properties of the countable random graph

We conclude this paper with a collection of some properties of the countable random (undirected) graph, the so-called *Rado graph.* These facts are folklore, but (at least some of them) are hard to find in the literature.

First of all, we present two explicit descriptions of the Rado graph. The first is in terms of *hereditarily finite* sets. Recall that that a set is hereditarily finite if it is a finite set of finite sets of finite sets of .... Thus HF, the set of hereditarily finite sets is the smallest collection of sets such that $\varnothing \in \mathrm{HF}$, and for all $k \in \omega$ and $a_1, \dots, a_k \in \mathrm{HF}$, also $\{a_1, \dots, a_k\} \in \mathrm{HF}$. Now let $\mathcal{R}$ be the graph (HF, E) where $E = \{(a, b) \in \mathrm{HF} \times \mathrm{HF} : a \in b \vee b \in a\}$. It is easy to verify that $\mathcal{R} \models T$, so $\mathcal{R}$ is indeed the Rado graph. Another description is $\mathcal{R} = (\omega, E)$ where $E = \{(i, j) : p_i | j \vee p_j | i\}$ where p_i denotes the ith prime number.

Obviously the Rado graph possesses all properties that are derivable from the extension axioms. For instance the Rado graph has *diameter* 2. This follows immediately from the the axioms

$$\forall x \exists y \neg Exy,$$
$$\forall x \forall y((\neg Exy \wedge x \neq y) \to \exists z(Exz \wedge Eyz)).$$

As a consequence, the Rado graph is *connected*.

Further, the extension axioms imply that every finite graph is isomorphic to an induced subgraph of the Rado graph. Therefore the Rado graph does *not* have any property with a finite forbidden subgraph. For instance, since a graph is planar if and only if it does not have K_5 and $K_{3,3}$ as a minor it is clear that the Rado graph is not planar.

An interesting property of countable graphs is the existence of a *Hamilton path*. There is more than one possibility to define this notion for countable graphs. Here we say that G is Hamiltonian if there exists a bijective homomorphism from $(\omega, succ)$, the natural numbers with successor, to G, i.e. if there exists a one-way infinite path through G that hits every node precisely once. Harel proved that the problem whether a given recursive graph is Hamiltonian is highly undecidable, namely Σ_1^1-complete [5]. But the Rado graph is Hamiltonian. To see this consider a random graph $G = (\omega, E)$ where $(i, i+1) \in E$ for all $i \in \omega$ and all other pairs are connected by an edge independently with probability 1/2. It is clear that almost surely G is a model of the extension axioms and therefore isomorphic to the Rado graph. Since G is Hamiltonian by definition, so is the Rado graph. Thus, recursive random graphs are almost surely Hamiltonian.

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