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Partition Width

Structures of Bounded Partition Width

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Preface

The investigation of monadic second-order logic started around 1960 with the work of Büchi, Elgot, McNaughton, and Rabin on the monadic theories of the natural numbers and the infinite binary tree. This research revealed a close connection between MSO and automata theory leading to automata-based decision procedures for a wide range of monadic theories. A wealth of applications of these results ensued in the fields of modal logic and automatic verification. Not only did many decidability results directly follow from the decidability of the MSO-theory of the binary tree, but also the automata-theoretic techniques employed by Büchi and Rabin could be adopted to obtain efficient decision procedures for weaker logics.

As far as further development on monadic second-order logic itself is concerned, Shelah gave new proofs of their results by purely model theoretic means and, together with Gurevich, they investigated the monadic theory of linear orders. In another article Baldwin and Shelah computed Hanf and Löwenheim numbers for monadic theories. Finally, a stronger version of Rabin's theorem was given by Muchnik.

In a separate line of research which developed out of the study of graph grammars, Engelfriet, Seese, and Courcelle investigated the monadic second-order theory of certain classes of graphs and the relationship between these theories and well-known complexity measures from graph theory.

Let us summarise the work on monadic second-order logic done so far. There have been three main lines of research.

(a) *The investigation of specific structures.*

- The natural numbers with successor (ω, suc) and expansions by certain unary predicates [11, 36, 14].
- The binary tree $(2^{<\omega}, \mathrm{suc}_0, \mathrm{suc}_1)$ [61].
- The real line $(\mathbb{R}, <)$ and other linear orderings [70, 43, 45].
- Grids [53].

(b) *Constructions on structures that preserve monadic properties.*

- Interpretations [60, 24].
- The composition method [70].
- Unravellings of transition systems [26].
- Substitutions [31].
- Inverse rational substitutions [15].
- The construction of Muchnik [69, 78].

(c) *Algebraic and model theoretic research.*

- The computation of Hanf and Löwenheim numbers [2, 1].
- Algebraic classification of graphs with decidable GSO-theory [68].
- The monadic theory of sparse graphs [28].
- Definability and axiomatisability for certain classes of graphs [20, 22].

In the present thesis we will concentrate on this last topic by investigating the question of which monadic theories are simple. In particular, we are looking for theories that are decidable or at least simple enough that we are able to derive structure theorems.

We propose to draw the line between simple and complicated theories by defining that a structure has a simple monadic theory if and only if it can be interpreted in some (possibly infinite) coloured tree.

The class of structures obtained this way generalises the class of graphs of bounded clique width which was originally defined by Courcelle, Engelfriet, and Rozenberg [30] via graph grammars. Later on Courcelle [24] proved that every class of finite graphs of bounded clique width can be interpreted in a suitable class of finite trees. In the same vein, we will introduce terms denoting arbitrary relational structures and we show that a structure can be denoted by a term if and only if it is interpretable in some (possibly infinite) tree. Furthermore, we obtain an equivalent characterisation via hierarchical decompositions of the structure which can be used to define a complexity measure, called *partition width*, which provides our generalisation of the notion of clique width.

The intuitive idea that structures interpretable in a tree have a simple monadic theory is supported by several model theoretic results we obtain for this class. Finiteness of partition width is preserved by elementary embeddings and we will prove a compactness theorem for structures of finite partition width. Furthermore, no such structure has the independence property or, equivalently, infinite VC-dimension, that is, in no structure of finite partition width it is possible to encode, in a first-order way, all subsets of some infinite set by single elements.

After having obtained a class of simple structures the obvious next question is whether this characterisation is precise. That is, we would like to prove that all other structures have a complicated monadic theory. We conjecture that every structure of infinite partition width contains arbitrarily large finite MSO-definable grids. This would imply that the full second-order theory of the class of finite sets can be interpreted in the monadic second-order theory of every structure

of infinite partition width. In particular, every such structure would have an undecidable MSO-theory. Therefore, a proof of this conjecture would settle the conjecture of Seese [68] which states that every graph with decidable MSO-theory has finite clique width.

We try to obtain an answer to this question by developing a theory of connectedness based on cuts and separations that is symmetric with regard to edges and non-edges. After sufficient preparations of this kind we are able to translate the core of the original proof of Robertson and Seymour's Excluded Grid Theorem from tree width to partition width. Despite these encouraging results, both, a full analogue of the Excluded Grid Theorem and the conjecture itself remain open.

In the second part of the thesis we turn to the investigation of subclasses consisting of structures with decidable monadic theory that, furthermore, admit a finite representation. Mainly, we will consider the class of structures that can be interpreted in the complete binary tree without additional unary predicates. We will study algebraic properties of these structures including a characterisation of all linear orders contained in this class. We will also show that every such structure can be finitely axiomatised in guarded second-order logic with cardinality quantifiers.

The organisation of this thesis is as follows. We start in Chapter 1 by giving a survey on several variants of monadic second-order logic. We will present operations on structures that preserve monadic properties and, after introducing the required automata-theoretic concepts, we prove an extension of the theorem of Muchnik which is one of the strongest decidability results in logic known today.

After these logical prerequisites we give an introduction to the theory of graph grammars in Chapter 2. We present several classes of graphs defined by such grammars, define the notion of clique width, and compare the clique width and the tree width of a given graph.

In Chapter 3 we start to develop a model theory for monadic second-order logic by generalising the concept of clique width from countable graphs to relational structures of arbitrary cardinality. We study how this new measure, which we call *partition width*, behaves under certain operations on structures, and we give an existential condition for large partition width. As far as model theoretic questions are concerned, we show that a variant of partition width is invariant under elementary extensions and compactness, and we prove that no structure of finite partition width has the independence property.

The main open problem in the field of clique width is Seese's conjecture which states that a graph with decidable MSO-theory has finite clique width. In Chapter 4 we make some progress in this direction by developing a theory of cuts and connectedness suitable for dealing with clique width. This allows us to transfer the main part

of the proof of Robertson and Seymour's Excluded Grid Theorem from tree width to partition width. Nevertheless, both a full analogue of the Excluded Grid Theorem and Seese's conjecture remain open.

In the last part of the thesis we investigate structures of finite partition width that can be encoded by a finite amount of information. In Chapter 5 we define a hierarchy of classes of such structures and present several algebraic characterisations of the class of tree-interpretable structures, the lowest class in this hierarchy. In particular, we will study paths in tree-interpretable graphs and we derive a characterisation of all tree-interpretable linear orders.

The final chapter is devoted to the proof that every tree-interpretable structure is finitely $\mathrm{GSO}(\exists^{\kappa})$-axiomatisable. We show that the cardinality quantifiers are really needed and we present some simple applications to the automorphism group of a tree-interpretable structure.

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1 First- and Second-Order Logic

After fixing our notation we will present some variants of monadic second-order logic and give an overview over their properties. We will investigate operations on structures and their effect on monadic theories, and we will present decidability results for monadic theories based on these operations and on automata-theoretic techniques.

1.1 Notation and conventions

Let us recall some basic definitions and fix our notation. Let $[n] := \{0, \dots, n-1\}$. We tacitly identify tuples $\bar{a} = a_0 \dots a_{n-1} \in A^n$ with functions $[n] \to A$ and frequently we write $\bar{a}$ for the set $\{a_0, \dots, a_{n-1}\}$. This allows us to write $\bar{a} \subseteq \bar{b}$ or $\bar{a} = \bar{b}|_I$ for $I \subseteq [n]$. The length of a sequence $\bar{a} \in A^\alpha$ is $|\bar{a}| = \alpha$. The complement of a set X is denoted by $\overline{X}$. Recall that the α-fold iterated exponentiation $\beth_\alpha(\kappa)$ is defined by

$\beth_\alpha(\kappa)$

$$\beth_0(\kappa) = \kappa \quad \text{and} \quad \beth_\alpha(\kappa) = \sup\left\{ 2^{\beth_\beta(\kappa)} \;\middle|\; \beta < \alpha \right\}.$$

We will use this notation also for finite κ.

For the most part, we will only consider relational structures $\mathfrak{M} = (M, R_0, R_1, \dots)$. The set of relation symbols $\{R_0, R_1, \dots\}$ is called the *signature* of $\mathfrak{M}$. When speaking of the arity of a structure or a signature we mean the supremum of the arities of its relations. A *transition system* is a structure of arity at most 2. We reserve the term (directed) *graph* for is a transition systems $\mathfrak{G} = (V, E)$ with a single edge relation. If E is irreflexive and symmetric then we call $\mathfrak{G}$ *undirected*.

Logic. MSO, monadic second-order logic, is the extension of first-order logic FO by quantification over sets. In places where the exact definition matters – say when considering the quantifier rank of a formula – we will use a variant without first-order variables where the atomic formulae are of the form $Y = Z$, $Y \subseteq Z$, and $RX_0 \dots X_{n-1}$, for set variables X_i, Y, Z and relations R. Using slightly nonstandard

semantics we say that an atom of the form $R\bar{X}$ holds if there are elements $a_i \in X_i$ such that $\bar{a} \in R$. Note that we do not require the X_i to be singletons. Obviously, each MSO-formula can be brought into this form.

FO_k, MSO_k — By FO_k and MSO_k we denote the fragments of the respective logic that consists of those formulae with quantifier rank at most k.

$\mathrm{Th}_{\mathfrak{L}}(\mathfrak{M})$ — $\mathrm{Th}_{\mathfrak{L}}(\mathfrak{M})$ is the $\mathfrak{L}$-theory of a structure $\mathfrak{M}$. We denote the fact that $\mathfrak{N}$ is an $\mathfrak{L}$-elementary extension of $\mathfrak{M}$ by $\mathfrak{M} \preceq_{\mathfrak{L}} \mathfrak{N}$, and we set $\mathfrak{M} \equiv_{\mathfrak{L}} \mathfrak{N}$ if $\mathrm{Th}_{\mathfrak{L}}(\mathfrak{M}) = \mathrm{Th}_{\mathfrak{L}}(\mathfrak{N})$.

$\mathfrak{M} \preceq_{\mathfrak{L}} \mathfrak{N}$

$\mathfrak{M} \equiv_{\mathfrak{L}} \mathfrak{N}$

$\mathfrak{L}(\exists^\omega)$ — If $\mathfrak{L}$ is a logic containing MSO then we denote by $\mathfrak{L}(\exists^\omega)$ the extension of $\mathfrak{L}$ by the predicate $|X| \geq \aleph_0$, and the logic $\mathfrak{L} + \mathrm{C}$ extends $\mathfrak{L}$ by predicates of the form $|X| \equiv k \pmod m$ for all $k, m < \omega$. We adopt the convention that $|X| \equiv k \pmod m$ is false for infinite sets X. Hence $\mathfrak{L} + \mathrm{C}$ subsumes $\mathfrak{L}(\exists^\omega)$. Finally, $\mathfrak{L}(\exists^\kappa)$ is the extension of $\mathfrak{L}$ by first-order quantifiers $\exists^\lambda$ meaning "there exist at least λ many", for all cardinals λ.

$\mathfrak{L} + \mathrm{C}$

$\mathfrak{L}(\exists^\kappa)$

φ^X — We denote the relativisation of a formula φ to a set X by φ^X. A formula $\varphi(\bar{x})$ where each free variable is first-order defines on a given structure $\mathfrak{M}$ the relation $\varphi^{\mathfrak{M}} := \{ \bar{a} \mid \mathfrak{M} \vDash \varphi(\bar{a}) \}$.

$\varphi^{\mathfrak{M}}$

$\kappa^{<\alpha}$ — Trees. Let κ be a cardinal and α an ordinal. By $\kappa^{<\alpha}$ we denote the set of all functions $\beta \to \kappa$ for $\beta < \alpha$. The empty sequence is denoted by ε. For $x, y \in \kappa^{<\alpha}$, we write $x \preceq y$ if x is a prefix of y, and the longest common prefix of x and y is denoted by $x \sqcap y$. If $x = yz$ then $y^{-1}x := z$. By x/k we denote the prefix of x of length $|x| - k$.

ε

$x \preceq y$

$x \sqcap y$

$y^{-1}x$

x/k

Let $\leq_{\mathrm{lex}}$ be the lexicographic order, and $\leq_{\mathrm{ll}}$ the length-lexicographic one, that is,

$x \leq_{\mathrm{lex}} y$

$x \leq_{\mathrm{ll}} y$

$$x \leq_{\mathrm{ll}} y \quad :\text{iff} \quad |x| < |y|, \quad \text{or} \quad |x| = |y| \text{ and } x \leq_{\mathrm{lex}} y.$$

tree — A (directed) *tree* is a partial order $(T, \preceq)$ whose universe $T \subseteq \kappa^{<\alpha}$ is closed under prefixes. Sometimes we also add the successor functions $\mathrm{suc}_c(x) := xc$ for $c < \kappa$. *Labelled trees* are either represented as structures $(T, \preceq, (P_i)_{i \in \Lambda})$ with additional unary predicates P_i for each label $i \in \Lambda$, or as functions $t : T \to \Lambda$.

labelled tree

undirected tree — An *undirected tree* is an undirected graph that is acyclic and connected. An undirected tree is called *ternary* if all non-leaves have degree 3.

ternary

term — Let Υ be a signature. A *Υ-term* is a labelled tree $T \to \Upsilon$ such that the number of successors of a node $w \in T$ equals the arity of its label. The tree T is called the *domain* of the term. Note that terms may be infinite.

domain

$\kappa \to (\mu)^\lambda_\chi$ — Ramsey's theorem. Let κ, λ, μ, and χ be cardinals. We denote by $\kappa \to (\mu)^\lambda_\chi$ the fact that, given any colouring of the set

$$\wp_\lambda(\kappa) := \{ X \subseteq \kappa \mid |X| = \lambda \}$$

using χ colours, there exists a subset $Z \subseteq \kappa$ of size μ such that all sets $X \in \wp_\lambda(Z)$ have the same colour. In order to avoid clumsy descriptions we further define

$R(m)^d_p$

$$R(m)^d_p := \min\left\{ n \mid n \to (m)^d_p \right\}.$$

Recall that Ramsey's theorem states that $R(m)^d_p < \aleph_0$ for finite values of m, d, and p.

1.2 Guarded second-order logic and tree width

Guarded logics were introduced as generalisations of modal logics. In the course of this programme Grädel, Hirsch, and Otto [41] defined *guarded second-order logic* which, on sparse structures, i.e., structures whose relations contain few tuples, can be considered as a generalisation of monadic second-order logic. Restricted to graphs this logic coincides with the logic MS_2, a variant of monadic second-order logic defined by Courcelle where one can also quantify over sets of edges.

Definition 1.2.1. Let $\mathfrak{M}$ be a relational structure.

(a) A tuple $\bar{a} \subseteq M$ is *guarded* if there exists a relation R of $\mathfrak{M}$ and some tuple $\bar{c} \in R$ such that $\bar{a} \subseteq \bar{c}$. The relation R may be the equality predicate =. If we want to specify the relation witnessing guardedness we say that the tuple $\bar{a}$ is guarded *by* R. *guarded tuple*

(b) A relation $S \subseteq M^n$ is *guarded* if every tuple $\bar{a} \in S$ is guarded. *guarded relation*

Note that every singleton $a \in M$ is guarded by = and, consequently, so is every unary relation $A \subseteq M$.

Definition 1.2.2. *Guarded second-order logic*, GSO, has the same syntax as full second-order logic, but semantically all second-order quantifiers are restricted to range only over guarded relations. *guarded second-order logic* GSO

Remark. For finite signatures, there exists a formula $\gamma(\bar{x})$ that states that the tuple $\bar{x}$ is guarded. Therefore, by requiring that every second-order quantifier is of the form

$$\big(\exists R.\forall \bar{x}(R\bar{x} \to \gamma(\bar{x}))\big) \quad \text{or} \quad \big(\forall R.\forall \bar{x}(R\bar{x} \to \gamma(\bar{x}))\big),$$

we can replace the semantic restriction in the definition of GSO by a purely syntactic one.

Lemma 1.2.3. MSO $\subseteq$ GSO.

Proof. Since every unary relation is guarded it follows that we can use unrestricted set quantifiers in GSO. □

Example. For graphs guarded quantification amounts to quantification over sets of vertices and sets of edges. Hence, we can, for instance, express that a graph $\mathfrak{G} = (V, E)$ contains a Hamiltonian cycle by the GSO-sentence

$$\exists H\big[\forall x\forall y(Hxy \to Exy) \wedge \forall x\exists^{=1} yHxy \wedge \forall x\exists^{=1} yHyx \\ \wedge \forall X\big((\exists xXx \wedge \forall x\forall y(Xx \wedge Hxy \to Xy)) \to \forall xXx\big)\big]$$

To every relational structure we can associate two transition systems in a natural way.

Definition 1.2.4. Let $\mathfrak{M} = (M, R_0, R_1, \dots)$ be a structure.

Gaifman graph $\mathcal{G}(\mathfrak{M})$

(a) The *Gaifman graph* of $\mathfrak{M}$ is the graph $\mathcal{G}(\mathfrak{M}) = (M, E)$ whose edge relation E consists of all guarded pairs of $\mathfrak{M}$.

incidence structure $\mathfrak{M}^{\mathcal{I}}$

(b) The *incidence structure* of $\mathfrak{M}$ is the structure

$$\mathfrak{M}^{\mathcal{I}} := (V, (\pi_i)_i, R_0, R_1, \dots)$$

where

- the universe V consists of all guarded tuples of $\mathfrak{M}$ (note that, in particular, $M \subseteq V$),
- $\pi_i : V \to M \subseteq V$ is the projection to the i-th coordinate, and
- the relations $R_i \subseteq V$ are considered as unary predicates.

Note that every guarded tuple in $\mathcal{G}(\mathfrak{M})$ is also guarded in $\mathfrak{M}$. Therefore, we can translate GSO-formulae over $\mathcal{G}(\mathfrak{M})$ to formulae over $\mathfrak{M}$.

Similarly, each guarded relation $S \subseteq M^n$ in $\mathfrak{M}$ corresponds to a set $S \subseteq V$ in $\mathfrak{M}^{\mathcal{I}}$ and, conversely, every set $S \subseteq V$ can be written as a union $S = \bigcup_i S_i$ of guarded relations $S_i \subseteq M^{n_i}$. Using this correspondence we can translate GSO-formulae over $\mathfrak{M}$ to MSO-formulae over $\mathfrak{M}^{\mathcal{I}}$ and vice versa. In that way, GSO can indeed be considered as a variant of MSO.

Lemma 1.2.5. *Let $\mathfrak{M}$ be a structure of finite signature.*

(a) *For every formula $\varphi(\bar{x}, \bar{Z}) \in$ GSO, there exists a* GSO*-formula $\varphi^{\mathcal{G}}(\bar{x}, \bar{Z})$ such that*

$$\mathcal{G}(\mathfrak{M}) \models \varphi(\bar{a}, \bar{S}) \quad \text{iff} \quad \mathfrak{M} \models \varphi^{\mathcal{G}}(\bar{a}, \bar{S})$$

for all tuples $\bar{a} \subseteq M$ and all guarded relations $S_i \subseteq M^{n_i}$.

(b) *For every formula $\varphi(\bar{x}, \bar{Z}) \in$ GSO, there exists an* MSO*-formula $\varphi^{\mathcal{I}}(\bar{x}, \bar{Z})$ (where each Z_i is considered as set variable) such that*

$$\mathfrak{M} \models \varphi(\bar{a}, \bar{S}) \quad \text{iff} \quad \mathfrak{M}^{\mathcal{I}} \models \varphi^{\mathcal{I}}(\bar{a}, \bar{S})$$

for all tuples $\bar{a} \subseteq M$ and all guarded relations $S_i \subseteq M^{n_i}$.

Conversely, every MSO*-formula over $\mathfrak{M}^{\mathcal{I}}$ can be translated into a corresponding* GSO*-formula over $\mathfrak{M}$.*

There exists a strong link between guarded second-order logic and the notion of tree width introduced by Robertson and Seymour [63].

Definition 1.2.6. Let $\mathfrak{M}$ be a structure.

(a) A *tree decomposition* of $\mathfrak{M}$ is a family $(F_v)_{v \in T}$ of subsets $F_v \subseteq M$ indexed by an undirected tree T that satisfies the following conditions: *tree decomposition*

- $\bigcup_{v \in T} F_v = M$.
- For every $a \in M$, the set $\{ v \in T \mid a \in F_v \}$ is connected.
- For every guarded tuple $\bar{a} \subseteq M$, there exists a node $v \in T$ such that $\bar{a} \subseteq F_v$.

(b) The *width* of a tree decomposition $(F_v)_{v \in T}$ is the number *width*

$$\operatorname{twd}(F_v)_v := \sup \{ |F_v| - 1 \mid v \in T \}.$$

(The -1 has historical reasons. Its only effect is making notation slightly more complex.) The *tree width* of $\mathfrak{M}$ is the minimal width of a tree decomposition of $\mathfrak{M}$. *tree width* $\operatorname{twd} \mathfrak{M}$

Lemma 1.2.7. *For every structure $\mathfrak{M}$, we have* $\operatorname{twd} \mathfrak{M} = \operatorname{twd} \mathcal{G}(\mathfrak{M})$.

Proof. Every tree decomposition of $\mathfrak{M}$ is also a tree decomposition of $\mathcal{G}(\mathfrak{M})$. Conversely, one can show that, if $(F_v)_v$ is a tree decomposition of $\mathcal{G}(\mathfrak{M})$, then, for every clique $X \subseteq M$ in the Gaifman graph, there exists a component F_v with $X \subseteq F_v$ (see, e.g., Diestel [33], Lemma 12.3.5). Hence, $(F_v)_v$ is also a tree decomposition of $\mathfrak{M}$. □

One important characterisation of tree width is the Excluded Grid Theorem of Robertson and Seymour [64]. The following improvement is by Robertson, Seymour, and Thomas [66].

Theorem 1.2.8 (Excluded Grid Theorem)**.** *Let $\mathfrak{M}$ be a structure of tree width* $\operatorname{twd} \mathfrak{M} > 20^{2n^5}$*. Then $\mathcal{G}(\mathfrak{M})$ contains an $n \times n$ grid as minor.*

It follows that the GSO-theory of every structure of infinite tree width is undecidable.

Theorem 1.2.9 (Seese [68])**.** *If $\mathfrak{M}$ is a structure of infinite tree width then its* GSO*-theory is undecidable.*

Proof. Let $\mathfrak{G} = (V, E) := \mathcal{G}(\mathfrak{M})$ be the Gaifman graph of $\mathfrak{M}$. Every minor $\mathfrak{H}$ of $\mathfrak{G}$ can be encoded by three relations:

- the subset $V_0 \subseteq V$ of those vertices that are not deleted,
- the subset $E_0 \subseteq E$ of those edges which are not deleted, and
- the subset $E_1 \subseteq E_0$ of those edges that are contracted.

Given V_0, E_0, and E_1, every sentence $\varphi \in \mathrm{MSO}$ over $\mathfrak{H}$ can be translated into an MSO-formula $\psi(V_0, E_0, E_1)$ over $\mathfrak{G}$ such that

$$\mathfrak{H} \models \varphi \quad \text{iff} \quad \mathfrak{G} \models \psi(V_0, E_0, E_1)\,.$$

Hence, the MSO-theory of the class $\mathcal{M}$ of minors of $\mathcal{G}(\mathfrak{M})$ can be interpreted into the GSO-theory of $\mathfrak{M}$. By the Excluded Grid Theorem, $\mathcal{M}$ contains the class $\mathcal{K}$ of all finite grids. Since $\mathcal{K}$ is finitely MSO-axiomatisable, it follows that $\mathrm{Th}_{\mathrm{MSO}}(\mathcal{K})$ is interpretable in $\mathrm{Th}_{\mathrm{GSO}}(\mathfrak{M})$. A simple encoding of domino problems shows that the former is undecidable (see Seese [67]), and the result follows. □

uniformly k-sparse

Definition 1.2.10. A structure $\mathfrak{M}$ is *uniformly k-sparse* if

$$\big|R|_X\big| \leq k|X|\,,$$

for every set $X \subseteq M$ and all relations R.

Lemma 1.2.11. *Planar transition systems are uniformly 7-sparse.*

Proof. Consider a finite substructure $\mathfrak{A}$ with n vertices and fix some edge relation E_λ. The $\{E_\lambda\}$-reduct (A, E_λ) of $\mathfrak{A}$ is a directed graph whose underlying undirected graph has at most $3n - 6$ edges (see, e.g., Diestel [33], Corollary 4.2.7, or Bollobás [9], Theorem I. 16). Taking possible self-loops into account (of which there are at most n) it follows that (A, E_λ) has at most $7n - 12$ edges. □

Lemma 1.2.12. *Let $\mathfrak{M}$ be a transition system with m binary relations. If* $\operatorname{twd}\mathfrak{M} \leq k$ *then $\mathfrak{M}$ is uniformly $(k^2 + k)$-sparse.*

Proof. Let $\mathfrak{M}_0 \subseteq \mathfrak{M}$ and let $(F_v)_{v \in T}$ be a tree decomposition of $\mathfrak{M}_0$ of width at most k such that $F_u \not\subseteq F_v$ for all nodes $u \neq v$. If $|M_0| \leq k$ then there is nothing to prove. Otherwise, fix an arbitrary node $r \in T$ and consider T as directed tree with root r. For every directed edge (u, v) in this tree we can fix some element $a \in F_v \setminus F_u$. That way we obtain an injective function mapping the edges of T into M_0. It follows that $|T| \leq |M_0| + 1$. Let $E := \bigcup_{v \in T} F_v \times F_v$. Then E is of size $|E| \leq k^2\big(|M_0| + 1\big)$ and every edge relation E_λ of $\mathfrak{M}_0$ is contained in E. Consequently,

$$|E_\lambda| \leq k^2\big(|M_0| + 1\big) \leq k^2\left(|M_0| + \frac{|M_0|}{k}\right) = (k^2 + k)|M_0|\,. \qquad \square$$

Courcelle [28] has shown that GSO collapses to MSO on uniformly k-sparse graphs.

Theorem 1.2.13. *Let $k < \omega$. For every sentence $\varphi \in$ GSO there exists a sentence $\psi \in$ MSO such that*

$$\mathfrak{M} \models \varphi \quad \text{iff} \quad \mathfrak{M} \models \psi$$

for every uniformly k-sparse transition system $\mathfrak{M}$.

Corollary 1.2.14. *If the* GSO*-theory of a transition system $\mathfrak{M}$ is decidable then* GSO *collapses to* MSO *on $\mathfrak{M}$.*

Proof. By Theorem 1.2.9, the decidability of $\mathrm{Th}_{\mathrm{GSO}}(\mathfrak{M})$ implies that $k := \operatorname{twd} \mathfrak{M} < \aleph_0$. Hence, by Lemma 1.2.12, $\mathfrak{M}$ is uniformly $(k^2 + k)$-sparse, and the result follows from the preceding theorem. □

1.3 Functors

One approach to investigate the theory of a given structure $\mathfrak{M}$ consists in showing that $\mathfrak{M}$ can be obtained from structures with known theories by operations that are compatible with the logic under consideration. In the present section we will introduce several operations which allow us to compute the $\mathfrak{L}_0$-theory of $\mathfrak{M}$ from the $\mathfrak{L}_1$-theory of another structure for certain logics $\mathfrak{L}_0$ and $\mathfrak{L}_1$.

Definition 1.3.1. Let $\mathfrak{L}_0$ and $\mathfrak{L}_1$ be logics. An $(\mathfrak{L}_0, \mathfrak{L}_1)$*-functor* is an operation $\mathcal{F}$ on structures with the following properties: $(\mathfrak{L}_0, \mathfrak{L}_1)$*-functor*

(1) There exists an effective function mapping each $\mathfrak{L}_0$-sentence φ to a sentence $\varphi^{\mathcal{F}} \in \mathfrak{L}_1$ such that

$$\mathcal{F}(\mathfrak{M}) \models \varphi \quad \text{iff} \quad \mathfrak{M} \models \varphi^{\mathcal{F}} \qquad \text{for every structure } \mathfrak{M}.$$

(2) $\mathcal{F}$ preserves elementary embeddings, i.e., $\mathfrak{M} \preceq_{\mathfrak{L}_1} \mathfrak{N}$ implies $\mathcal{F}(\mathfrak{M}) \preceq_{\mathfrak{L}_0} \mathcal{F}(\mathfrak{N})$.

If $\mathfrak{L}_0 = \mathfrak{L}_1$ we will simply call $\mathcal{F}$ an $\mathfrak{L}_0$-functor.

Remark. If $\mathcal{F}$ is an $\mathfrak{L}$-functor and $\mathfrak{M} \equiv_{\mathfrak{L}} \mathfrak{N}$, then $\mathcal{F}(\mathfrak{M}) \equiv_{\mathfrak{L}} \mathcal{F}(\mathfrak{N})$.

Example. The following operations on structures fit into the framework of functors.

(1) *Reducts.* If $\mathfrak{M}$ is a τ-structure and $\tau_0 \subseteq \tau$ then, for any reasonable logic $\mathfrak{L}$, the τ_0-reduct $\mathcal{F}(\mathfrak{M}) := \mathfrak{M}|_{\tau_0}$ is an $\mathfrak{L}$-functor with $\varphi^{\mathcal{F}} := \varphi$.

(2) *Definable expansions.* A sequence $\psi_i(\bar{x})$, $i \in I$, of MSO-formulae induces the MSO-functor $\mathcal{F}(\mathfrak{M}) := (\mathfrak{M}, (\psi_i^{\mathfrak{M}})_i)$. We obtain $\varphi^{\mathcal{F}}$ by replacing each new relation by the formula ψ_i defining it.

(3) *Definable substructures.* For every $\psi(x) \in \mathrm{MSO}$, we can define the MSO-functor $\mathcal{F}(\mathfrak{M}) := \mathfrak{M}|_{\psi^{\mathfrak{M}}}$ mapping a structure $\mathfrak{M}$ to the substructure defined by ψ. We can compute $\varphi^{\mathcal{F}}$ by relativising every quantifier to ψ.

(4) *Factorisation by definable equivalence relations.* Each formula $\psi(x, y) \in \mathrm{MSO}$ defining an equivalence relation induces an MSO-functor $\mathcal{F}(\mathfrak{M}) := \mathfrak{M}/\psi^{\mathfrak{M}}$ where $\varphi^{\mathcal{F}}$ is obtained from φ by replacing every equation $t = t'$ by the formula $\psi(t, t')$ and by restricting set quantifiers to sets closed under the equivalence relation defined by ψ.

(5) We have seen in the previous section that the operation which associates to a structure its Gaifman graph is a GSO-functor, and the mapping to the incidence structure is a (MSO, GSO)-functor.

(6) If all formulae in (2) – (4) are first-order then these operations are FO-functors.

(7) The operations (1) – (3) are also (MSO + C)-functors.

1.3.1 Interpretations

We combine the operations (1) – (4) of the previous example into a single one.

Definition 1.3.2. Let $\mathfrak{L}$ be a logic, and τ and σ relational signatures.

$\mathfrak{L}$-interpretation

(a) A (one-dimensional) $\mathfrak{L}$-*interpretation* is a sequence

$$\mathcal{I} = \big\langle \delta(x),\ \varepsilon(x, y),\ (\varphi_R(\bar{x}))_{R \in \tau} \big\rangle$$

of $\mathfrak{L}$-formulae of signature σ. Given a σ-structure $\mathfrak{M}$ it defines the τ-structure

$$\mathcal{I}(\mathfrak{M}) := \big(\delta^{\mathfrak{M}},\ (\varphi_R^{\mathfrak{M}})_{R \in \tau}\big)\big/\varepsilon^{\mathfrak{M}}.$$

To make this expression well-defined we require that $\varepsilon^{\mathfrak{M}}$ is a congruence of the structure $(\delta^{\mathfrak{M}},\ (\varphi_R^{\mathfrak{M}})_R)$.

$\mathcal{I} : \mathfrak{N} \leq_{\mathfrak{L}} \mathfrak{M}$

(b) A structure $\mathfrak{N}$ is $\mathfrak{L}$-*interpretable* in $\mathfrak{M}$ if there exists an $\mathfrak{L}$-interpretation $\mathcal{I}$ such that $\mathfrak{N} \cong \mathcal{I}(\mathfrak{M})$. In this case we write $\mathcal{I} : \mathfrak{N} \leq_{\mathfrak{L}} \mathfrak{M}$. The epimorphism $(\delta^{\mathfrak{M}},\ (\varphi_R^{\mathfrak{M}})_R) \to \mathfrak{N}$ induced by the isomorphism $\mathcal{I}(\mathfrak{M}) \cong \mathfrak{N}$ is called *coordinate map* and also denoted by $\mathcal{I}$.

coordinate map

injective

(c) An interpretation $\mathcal{I}$ is *injective* if the coordinate map is injective, i.e., if $\varepsilon(x, y) \equiv x = y$.

Since every composition of $\mathfrak{L}$-functors is again an $\mathfrak{L}$-functor, it follows that MSO-interpretations are MSO-functors and FO-interpretations FO-functors. We even have the stronger result that the function $\varphi \mapsto \varphi^{\mathcal{F}}$ can be extended to formulae with free variables.

Interpretation Lemma. *If* $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{MSO}} \mathfrak{B}$ *then*

$$\mathfrak{A} \models \varphi(\mathcal{I}(\bar{b})) \quad \text{iff} \quad \mathfrak{B} \models \varphi^{\mathcal{I}}(\bar{b}) \quad \textit{for all } \varphi \in \mathrm{MSO} \textit{ and } \bar{b} \subseteq \delta^{\mathfrak{B}}.$$

In particular, MSO*-interpretations are* MSO*-functors. The same holds, if we replace* MSO *with* FO*. Finally, if* $\mathcal{I}$ *is injective then we can also translate* (MSO + C)*-formulae.*

We illustrate interpretations by considering structures interpretable in a tree $(\alpha^{<\beta}, \preceq, \bar{P})$. These will play an important role in Chapter 3.

Lemma 1.3.3. *Let* α *and* β *be ordinals.*

(a) $(\alpha^{<\beta}, \preceq) \leq_{\mathrm{FO}} (2^{<\alpha\beta}, \preceq, P)$ *for some unary predicate* P*.*

(b) $(\alpha^{<\beta}, \preceq, (\mathrm{suc}_i)_{i<\alpha}, \bar{P}) \leq_{\mathrm{FO}_1} (\alpha^{<\beta}, \preceq, \bar{P}, \bar{Q})$ *for suitable unary predicates* Q_i*,* $i < \alpha$*.*

Proof. (a) Let $h : \alpha^{<\beta} \to 2^{<\alpha\beta}$ be the function which replaces in $\bar{\gamma} \in \alpha^{<\beta}$ each γ_i by the sequence $1^{\gamma_i}0$. Then for $x, y \in \alpha^{<\beta}$ we have

$$x \preceq y \quad \text{iff} \quad h(x) \preceq h(y)\,.$$

Setting $P := \operatorname{rng} h$ we obtain the desired interpretation.

(b) Let $Q_i := \{\, wi \mid w \in \alpha^{<\beta} \,\}$ for $i < \alpha$. Then

$$\mathrm{suc}_i(x, y) \quad \text{iff} \quad x \prec y \wedge P_i y \wedge \neg\exists z(x \prec z \prec y)\,. \qquad \square$$

1.3.2 Products

In general, products only preserve first-order theories. To obtain MSO-functors we consider the special case of products by a fixed finite structure.

Definition 1.3.4. Let $\mathfrak{M}$ be a σ-structure and $\mathfrak{N}$ a τ-structure. The *product* of $\mathfrak{M}$ and $\mathfrak{N}$ is the structure

product $\mathfrak{M} \times \mathfrak{N}$

$$\mathfrak{M} \times \mathfrak{N} := \big(M \times N, (R^{\mathrm{o}})_{R\in\sigma}, (R^{1})_{R\in\tau}, =_{\mathrm{o}}, =_{1}\big)$$

with relations

$$R^{\mathrm{o}} := \big\{\, \big(\langle a_{\mathrm{o}}, b_{\mathrm{o}}\rangle, \ldots, \langle a_r, b_r\rangle\big) \;\big|\; (a_{\mathrm{o}}, \ldots, a_r) \in R^{\mathfrak{M}} \,\big\}\,,$$
$$R^{1} := \big\{\, \big(\langle a_{\mathrm{o}}, b_{\mathrm{o}}\rangle, \ldots, \langle a_r, b_r\rangle\big) \;\big|\; (b_{\mathrm{o}}, \ldots, b_r) \in R^{\mathfrak{N}} \,\big\}\,,$$
$$\langle a, b\rangle =_{\mathrm{o}} \langle a', b'\rangle \quad :\text{iff} \quad a = a'\,,$$
$$\langle a, b\rangle =_{1} \langle a', b'\rangle \quad :\text{iff} \quad b = b'\,.$$

$[m] \times \mathfrak{N}$

Mostly, we will consider the special case that $\mathfrak{M} = [m]$ is a finite set. By $m \times \mathfrak{N}$ we will denote the product $\mathfrak{M} \times \mathfrak{N}$ where $\mathfrak{M} = ([m], 0, \dots, m-1)$ is the set $[m]$ enriched by constants for every element.

If $\mathfrak{M}$ is a finite structure then the operation $\mathcal{F}(\mathfrak{N}) := \mathfrak{M} \times \mathfrak{N}$ is an MSO-functor.

Lemma 1.3.5. *Let $\mathfrak{M}$ be a finite structure and $\mathfrak{N}$ an arbitrary one. For every formula $\varphi(X_0, \dots, X_{n-1}) \in \mathrm{MSO}$ an all sets $A_0, \dots, A_{n-1} \subseteq M$ there exists a formula $\varphi_{\bar{A}}(\bar{X}) \in \mathrm{MSO}$ such that*

$$\mathfrak{M} \times \mathfrak{N} \models \varphi(B_0, \dots, B_{n-1})$$

iff $$\mathfrak{N} \models \varphi_{\pi_0(B_0) \dots \pi_0(B_{n-1})}(\pi_1(B_0), \dots, \pi_1(B_{n-1}))\,,$$

where $\pi_0 : M \times N \to M$ and $\pi_1 : M \times N \to N$ are the projections on the respective coordinate.

Proof. We construct $\varphi_{\bar{a}}(\bar{x})$ by induction on φ. For atomic formulae we have

$$\begin{aligned}
(R^0\bar{X})_{\bar{A}} &:= \begin{cases} \text{true} & \text{if } A_0 \times \dots \times A_{n-1} \cap R^{\mathfrak{M}} \neq \varnothing\,, \\ \text{false} & \text{otherwise,} \end{cases} \\
(R^1\bar{X})_{\bar{A}} &:= R\bar{X}\,, \\
(X =_0 Y)_{\bar{A}} &:= \begin{cases} \text{true} & \text{if } A_0 \cap A_1 \neq \varnothing\,, \\ \text{false} & \text{otherwise,} \end{cases} \\
(X =_1 Y)_{\bar{A}} &:= X \cap Y \neq \varnothing\,.
\end{aligned}$$

Boolean combinations remain unchanged

$$\begin{aligned}
(\varphi \wedge \psi)_{\bar{A}} &:= \varphi_{\bar{A}} \wedge \psi_{\bar{A}}\,, \\
(\neg\varphi)_{\bar{A}} &:= \neg\varphi_{\bar{A}}\,,
\end{aligned}$$

and for quantifiers we have

$$\begin{aligned}
(\exists Y\varphi)_{\bar{A}} &:= \exists Y \bigvee_{P \subseteq M} \varphi_{\bar{A}P}\,, \\
(\forall Y\varphi)_{\bar{A}} &:= \forall Y \bigwedge_{P \subseteq M} \varphi_{\bar{A}P}\,.
\end{aligned}$$

□

Interpretations in trees are closed under products with finite structures.

Lemma 1.3.6. *Let $\mathcal{I} : \mathfrak{N} \leq_{\mathrm{MSO}} (2^{<\omega}, \mathrm{suc}_0, \mathrm{suc}_1, \bar{R})$ for arbitrary relations $\bar{R}$. If $\mathfrak{M}$ is a finite σ-structure then*

$$\mathfrak{M} \times \mathfrak{N} \leq_{\mathrm{MSO}} (2^{<\omega}, \mathrm{suc}_0, \mathrm{suc}_1, \bar{R})\,.$$

Proof. Let $\mathcal{I} = \langle \delta(x),\ \varepsilon(x,y),\ (\varphi_R(\bar{x}))_{R\in\tau}\rangle$. W.l.o.g. we assume that $M = [n]$. We code the element $(k, w) \in [n] \times 2^{<\omega}$ by the word $w10^k$. Let $\psi_k(x,y)$ be the formula stating that $x = y10^k$. We obtain an interpretation

$$\mathcal{I}' : \left\langle \delta',\ \varepsilon',\ (\varphi'_{R^0})_{R\in\sigma},\ (\varphi'_{R^1})_{R\in\tau},\ \varphi'_{=_0},\ \varphi'_{=_1}\right\rangle$$

of $\mathfrak{M}\times\mathfrak{N}$ in $(2^{<\omega}, \mathrm{suc}_0, \mathrm{suc}_1)$ by defining

$$\begin{aligned}
\delta'(x) &:= \exists y\Big(\delta(y) \wedge \bigvee_{k<n} \psi_k(x,y)\Big),\\
\varepsilon'(x,y) &:= \exists u\exists v\Big(\varepsilon(u,v) \wedge \bigvee_{k<n} (\psi_k(x,u)\wedge\psi_k(y,v))\Big),\\
\varphi'_{R^0}(\bar{x}) &:= \exists \bar{y} \bigvee_{\bar{k}\in R^{\mathfrak{M}}} \bigwedge_i \psi_{k_i}(x_i,y_i),\\
\varphi'_{R^1}(\bar{x}) &:= \exists \bar{y}\Big(\varphi_R(\bar{y}) \wedge \bigvee_{\bar{k}\subseteq[n]} \bigwedge_i \psi_{k_i}(x_i,y_i)\Big),\\
\varphi'_{=_0}(x,y) &:= \exists u\exists v \bigvee_{k<n} (\psi_k(x,u)\wedge\psi_k(y,v)),\\
\varphi'_{=_1}(x,y) &:= \exists u\exists v\Big(\varepsilon(u,v) \wedge \bigvee_{k,l<n} (\psi_k(x,u)\wedge\psi_l(y,v))\Big).
\end{aligned}$$

□

1.3.3 Iterations and unravellings

The next operation, introduced by Muchnik, arranges disjoint copies of a structure in a tree-like fashion.

Definition 1.3.7. Let $\mathfrak{M} = (M, R_0, \dots)$ be a τ-structure. The *iteration* of $\mathfrak{M}$ is the structure $\mathfrak{M}^* := (M^{<\omega}, \mathrm{suc}, \mathrm{cl}, R_0^*, \dots)$ of signature $\tau^* := \tau \cup \{\mathrm{suc}, \mathrm{cl}\}$ where $\mathfrak{M}^*$

$$\begin{aligned}
\mathrm{suc} &:= \{\,(w, wa) \mid w \in M^{<\omega},\ a \in M\,\},\\
\mathrm{cl} &:= \{\,waa \mid w \in M^{<\omega},\ a \in M\,\},\\
R_i^* &:= \{\,(wa_0, \dots, wa_r) \mid w \in M^{<\omega},\ \bar{a} \in R_i\,\}.
\end{aligned}$$

The fact that iterations are MSO-functors is one of the strongest decidability results for monadic second-order logic currently known. We will defer the proof until Section 1.6 since it requires techniques we have not yet introduced.

Theorem 1.3.8 (Muchnik)**.** *For every sentence $\varphi \in$ MSO one can effectively construct a sentence $\varphi^* \in$ MSO such that*

$$\mathfrak{M} \vDash \varphi^* \quad \text{iff} \quad \mathfrak{M}^* \vDash \varphi \qquad \textit{for all structures } \mathfrak{M}.$$

Corollary 1.3.9. *If $\mathfrak{M} \equiv_{\mathrm{MSO}} \mathfrak{N}$ then $\mathfrak{M}^* \equiv_{\mathrm{MSO}} \mathfrak{N}^*$.*

We give three examples of decidability results that immediately follow from this theorem. The first one is Rabin's tree theorem.

Theorem 1.3.10 (Rabin). *The monadic second-order theory of the binary tree* $\mathfrak{T}_2 := (2^{<\omega}, \mathrm{suc}_0, \mathrm{suc}_1)$ *is decidable.*

$\mathfrak{T}_2$

Proof. Let $\mathfrak{M} := ([2], P_0, P_1)$ be the structure with binary universe and predicates $P_c := \{c\}$. Its iteration is the tree

$$\mathfrak{M}^* = (2^{<\omega}, \mathrm{suc}, \mathrm{cl}, P_0^*, P_1^*)$$

into which we can interpret $\mathfrak{T}_2$ by defining

$$\mathrm{suc}_c(x, y) := \mathrm{suc}(x, y) \wedge P_c^* y\,.$$

Since the MSO-theory of the finite structure $\mathfrak{M}$ is decidable so is the one of $\mathfrak{T}_2$. □

A generalisation is the theorem of Le Tourneau [77] and Shelah [70] stating that the theory of all trees is decidable.

Theorem 1.3.11. *The monadic second-order theory of the class* $\mathcal{T}$ *of all trees* $(T, \leq)$ *with* $T \subseteq \kappa^{<\omega}$*, for some cardinal* κ*, is decidable.*

Proof. Let $\varphi \in \mathrm{MSO}$. We have $\varphi \in \mathrm{Th}_{\mathrm{MSO}}(\mathcal{T})$ iff

$$(\kappa^{<\omega}, \leq) \models \forall X(\text{“}X \text{ is prefix-closed”} \to \varphi^X) \qquad \text{for all } \kappa\,.$$

Hence, it is sufficient to show that the MSO-theory of the class of all trees of the form $(\kappa^{<\omega}, \leq)$ is decidable. Actually, by the same reasoning we only need to consider such trees with $\kappa \geq \aleph_0$.

We claim that $(\kappa^{<\omega}, \leq) \equiv_{\mathrm{MSO}} (\aleph_0^{<\omega}, \leq)$ for infinite κ. Let $\mathfrak{M}_\kappa$ be the structure of size κ with empty signature. By an Ehrenfeucht-Fraïssé game we can show that $\mathfrak{M}_\kappa \equiv_{\mathrm{MSO}} \mathfrak{M}_{\aleph_0}$ for all $\kappa \geq \aleph_0$. It follows that $\mathfrak{M}_\kappa^* \equiv_{\mathrm{MSO}} \mathfrak{M}_{\aleph_0}^*$. Since $\mathcal{I} : (\kappa^{<\omega}, \leq) \leq_{\mathrm{MSO}} \mathfrak{M}_\kappa^*$ for an interpretation $\mathcal{I}$ that does not depend on κ we have $(\kappa^{<\omega}, \leq) \equiv_{\mathrm{MSO}} (\aleph_0^{<\omega}, \leq)$ as desired.

We have seen that $\varphi \in \mathrm{Th}_{\mathrm{MSO}}(\mathcal{T})$ iff

$$(\aleph_0^{<\omega}, \leq) \models \forall X(\text{“}X \text{ is prefix-closed”} \to \varphi^X)\,.$$

By Muchnik's theorem, the decidability of $\mathrm{Th}_{\mathrm{MSO}}(\mathfrak{M}_{\aleph_0})$ implies the one of $\mathrm{Th}_{\mathrm{MSO}}(\aleph_0^{<\omega}, \leq)$, . □

By contrast, the theory of trees $\kappa^{<\alpha}$ with $\alpha > \omega$ is undecidable.

Theorem 1.3.12. *The monadic second-order theory of any class* $\mathcal{K}$ *containing some tree* $(\kappa^{<\alpha}, \leq)$ *with* $\kappa > 1$ *and* $\alpha > \omega$ *is undecidable.*

Proof. Shelah [70] has shown that the first-order theory of arithmetic is interpretable in the monadic theory of the real line $(\mathbb{R}, \leq)$. Since $(\mathbb{R}, \leq) \leq_{\mathrm{MSO}} (2^{<\omega+1}, \preceq, P_0, P_1)$ where $P_c := \{ wc \mid w \in 2^{<\omega} \}$ it follows that $\mathrm{Th}_{\mathrm{MSO}}(2^{<\omega+1}, \preceq, P_0, P_1)$ is undecidable. Let $\psi(X, Y_0, Y_1) \in \mathrm{MSO}$ be a formula stating that the substructure induced by X when expanded by the unary predicates Y_0 and Y_1 is isomorphic to the tree $(2^{<\omega+1}, \preceq, P_0, P_1)$. It follows that, for $\varphi \in \mathrm{MSO}$,

$$(2^{<\omega+1}, \preceq) \models \varphi(P_0, P_1)$$

$$\text{iff} \quad \forall X \forall Y_0 \forall Y_1 (\psi(X, Y_0, Y_1) \to \varphi^X(Y_0, Y_1)) \in \mathrm{Th}_{\mathrm{MSO}}(\mathcal{K}). \qquad \square$$

The last applications concerns unravellings of transition systems.

Definition 1.3.13. Let $\mathfrak{M} = (V, (E_\lambda)_\lambda, \bar{P})$ be a transition system. Its *unravelling* is the forest *unravelling*

$$\mathcal{U}(\mathfrak{M}) := (V^{\mathrm{u}}, (E_\lambda^{\mathrm{u}})_\lambda, \bar{P}^{\mathrm{u}})$$

whose universe consists of all finite paths through $\mathfrak{M}$. The relations are

$$E_\lambda^{\mathrm{u}} := \{ (wa, wab) \mid wab \in V^{\mathrm{u}},\ (a, b) \in E_\lambda \},$$
$$P_i^{\mathrm{u}} := \{ wa \in V^{\mathrm{u}} \mid a \in P_i \}.$$

Theorem 1.3.14. *The operation of unravelling is an* MSO*-functor.*

Proof. We construct an MSO-interpretation $\mathcal{I} : \mathcal{U}(\mathfrak{M}) \leq_{\mathrm{MSO}} \mathfrak{M}^*$. First, we define a formula $\psi_\lambda(x, y)$ stating that $x = wa$ and $y = wab$ for some edge $(a, b) \in E_\lambda$. From wa we can define the element $z = waa$ with the help of the clone relation and then state that $(waa, wab) \in E_\lambda^*$.

$$\psi_\lambda(x, y) := \exists z (\mathrm{suc}(x, z) \wedge \mathrm{cl}(z) \wedge E_\lambda^* z y).$$

The reflexive and transitive closure $\preceq$ of the relation suc is also MSO-definable. The set of all paths through $\mathfrak{M}$ is given by the formula

$$\delta(x) := \forall y \forall z \Big(\mathrm{suc}(y, z) \wedge z \preceq x \to \bigvee\nolimits_\lambda \psi_\lambda(y, z) \Big),$$

and the relations of $\mathcal{U}(\mathfrak{M})$ can be defined by

$$\varphi_{E_\lambda}(x, y) := \delta(y) \wedge \psi_\lambda(x, y),$$
$$\varphi_{P_i}(x) := \delta(x) \wedge P_i^* x. \qquad \square$$

The next lemma shows that iterations and interpretations commute.

Lemma 1.3.15. *If $\mathcal{I} : \mathfrak{M} \leq_{\mathrm{MSO}} \mathfrak{N}$ is an injective interpretation then there exists an injective interpretation $\mathcal{I}^* : \mathfrak{M}^* \leq_{\mathrm{MSO}} \mathfrak{N}^*$.*

Proof. Let $\mathcal{I} = \langle \delta(x), (\varphi_R(\bar{x}))_R \rangle$. To construct

$$\mathcal{I}^* = \langle \delta^*(x),\ \varphi^*_{\mathrm{suc}}(x,y),\ \varphi^*_{\mathrm{cl}}(x),\ (\varphi^*_R(\bar{x}))_R \rangle$$

we set

$$\begin{aligned}
\delta^*(x) &:= \forall y \forall z (\mathrm{suc}(y,z) \wedge z \preceq x \to \delta^y(z)),\\
\varphi^*_{\mathrm{suc}}(x,y) &:= \mathrm{suc}(x,y),\\
\varphi^*_{\mathrm{cl}}(x) &:= \mathrm{cl}(x),\\
\varphi^*_R(\bar{x}) &:= \exists y \Big(\bigwedge_i \mathrm{suc}(y,x_i) \wedge \varphi^y_R(\bar{x}) \Big),
\end{aligned}$$

where $\preceq$ is the reflexive and transitive closure of suc in $\mathfrak{N}^*$ and ψ^y denotes the relativisation of ψ to the set $\{ z \mid \mathrm{suc}(y,z) \}$. □

1.3.4 Generalised sums

All operations presented so far construct one structure from another one. Next we will describe a quite general and versatile way to compose a structure out of many different parts. Transferring results of Feferman and Vaught [39] from products and first-order theories to unions and MSO-theories, Shelah [70] introduced the *composition method* in order to prove in a uniform way all of the decidability results for monadic second-order known at that time. For a readable overview see [75, 44].

A *generalised sum* of a sequence $(\mathfrak{M}_i)_{i \in I}$ of structures consists of their disjoint union where we add an equivalence relation $\sim$ whose classes are the components M_i. Furthermore, the index set I may be an arbitrary structure $\mathfrak{I}$ whose relations are also added.

Definition 1.3.16. Let $\mathfrak{I}$ be a σ-structure and $(\mathfrak{M}_i)_{i \in I}$ a sequence of τ-structures indexed by elements of $\mathfrak{I}$.

generalised sum $\bigcup_{i \in \mathfrak{I}} \mathfrak{M}_i$

The *generalised sum* of $(\mathfrak{M}_i)_{i \in I}$ is the structure

$$\bigcup_{i \in \mathfrak{I}} \mathfrak{M}_i = \big(N, \sim, (R^0)_{R \in \sigma}, (R^1)_{R \in \tau} \big)$$

with universe $N := \bigcup \{ (i,a) \mid i \in I,\ a \in M_i \}$ and relations

$$\begin{aligned}
(i,a) \sim (j,b) \quad &: \text{iff} \quad i = j,\\
R^0 &:= \big\{ ((i_0,a_0),\dots,(i_r,a_r)) \bigm| (i_0,\dots,i_r) \in R^{\mathfrak{I}} \big\},\\
R^1 &:= \big\{ ((i,a_0),\dots,(i,a_r)) \bigm| (a_0,\dots,a_r) \in R^{\mathfrak{M}_i} \big\}.
\end{aligned}$$

Example. Let $\mathfrak{I} = (I, <)$ and $\mathfrak{M}_i = (M_i, <)$, $i \in I$, be linear orders. We can express the ordered sum $\mathfrak{N} := \sum_{i\in I} \mathfrak{M}_i$ by a generalised sum followed by an interpretation. Since $a < b$ holds in $\mathfrak{N}$ iff either

$$\begin{aligned} &a \in M_i \text{ and } b \in M_k \text{ for } i < k\,,\\ \text{or}\quad &a,\ b \in M_i \text{ and } a < b \text{ in } \mathfrak{M}_i\,, \end{aligned}$$

we obtain an interpretation $\mathcal{I} : \sum_{i\in I} \mathfrak{M}_i \leq_{\mathrm{MSO}} \bigcup_{i\in\mathfrak{I}} \mathfrak{M}_i$ by setting

$$\varphi_{\leq}(x, y) := x <^{\circ} y \vee (x \sim y \wedge x <^{1} y)\,.$$

Below we will state a theorem which shows that the monadic theory of a generalised sum can be computed from the theories of its components and the index structure. For the precise statement we need a refinement of the quantifier hierarchy.

Definition 1.3.17. (a) Let $\bar{k} \in \omega^{<\omega}$. We define the fragment $\mathrm{MSO}_{\bar{k}}$ of monadic second-order logic by induction on $|\bar{k}|$. $\mathrm{MSO}_{\bar{k}}$

$\mathrm{MSO}_{\varepsilon}$ consists of all formulae of the form $\exists y_0 \cdots \exists y_n \varphi(\bar{X}, \bar{y})$ where φ is quantifier free; and $\mathrm{MSO}_{\bar{k}m}$ is the set of all formulae of the form $\exists Y_0 \cdots Y_{m-1} \varphi(\bar{X}, \bar{Y})$ were φ is a boolean combination of $\mathrm{MSO}_{\bar{k}}$-formulae.

(b) Let $\mathfrak{M}$ be a structure, $\bar{P} \subseteq \wp(M)$, and $\bar{k} \in \omega^{<\omega}$. The *$\bar{k}$-type* of $\bar{P}$ in $\mathfrak{M}$ is the set *$\bar{k}$-type*

$$\mathrm{tp}_{\bar{k}}(\bar{P}/\mathfrak{M}) := \{\, \varphi(\bar{X}) \in \mathrm{MSO}_{\bar{k}} \mid \mathfrak{M} \vDash \varphi(\bar{P}) \,\}\,.$$

We are able to reduce the $\mathrm{MSO}_{\bar{k}}$-theory of $\bigcup_i \mathfrak{M}_i$ to the $\mathrm{MSO}_{\bar{m}}$-theory of a certain expansion of the index structure $\mathfrak{I}$.

Definition 1.3.18. Let $\mathfrak{I}$ be a σ-structure and $(\mathfrak{M}_i)_{i\in I}$ a sequence of τ-structures. For sets $P_0, \ldots, P_{n-1} \subseteq \bigcup_{i\in I} M_i$ and $\bar{k} \in \omega^{<\omega}$, we define the *$\bar{k}$-expansion* $\mathfrak{I}_{\bar{k}}(\bar{P}) := (\mathfrak{I}, \bar{Q})$ of $\bar{P}$ as the expansion of $\mathfrak{I}$ by unary predicates *$\bar{k}$-expansion* $\mathfrak{I}_{\bar{k}}(\bar{P})$

$$Q_t := \left\{\, i \in I \;\middle|\; \mathrm{tp}_{\bar{k}}(\bar{P}|_{M_i}/\mathfrak{M}_i) = t \,\right\} \qquad \text{for every } \bar{k}\text{-type } t\,.$$

Shelah [70] has proved the following theorem which shows that generalised sums can be regarded as a kind of MSO-functor.

Theorem 1.3.19. *Let σ and τ be finite signatures, $\mathfrak{I}$ a σ-structure, and $(\mathfrak{M}_i)_{i\in I}$ a sequence of τ-structures. Let $\mathfrak{N} := \bigcup_{i\in\mathfrak{I}} M_i$.*

For every $\bar{k} \in \omega^{<\omega}$ we can compute a sequence $\bar{m} \in \omega^{<\omega}$ such that, for every $\bar{P} \subseteq N$, the $\bar{k}$-type $\mathrm{tp}_{\bar{k}}(\bar{P}/\mathfrak{N})$ can be determined effectively from the $\mathrm{MSO}_{\bar{m}}$-theory of $\mathfrak{I}_{\bar{k}}(\bar{P})$.

1.4 Tree automata

The original proofs of the decidability of the monadic theories of (ω, suc) and $\mathfrak{T}_2 := (2^{<\omega}, \mathrm{suc}_0, \mathrm{suc}_1)$ by, respectively, Büchi and Rabin made use of finite automata (for an overview see [74, 76, 42]). Let us briefly recall their results. When dealing with binary trees we will use the following automaton model.

nondeterministic parity automaton

Definition 1.4.1. A *nondeterministic parity automaton* for binary trees is a tuple $\mathcal{A} = (Q, \Sigma, \Delta, q_0, \Omega)$ where Q is a finite set of states, Σ is a finite input alphabet, $\Delta \subseteq Q \times \Sigma \times Q \times Q$ is the transition relation, q_0 is the initial state, and $\Omega : Q \to [n]$ assigns to every state q a priority $\Omega(q)$.

run

A *run* of $\mathcal{A}$ on a tree $T : 2^{<\omega} \to \Sigma$ is a tree $\rho : 2^{<\omega} \to Q$ such that $\rho(\varepsilon) = q_0$ and

$$\big(\rho(w), T(w), \rho(w0), \rho(w1)\big) \in \Delta \quad \text{for all } w \in 2^{<\omega}.$$

accepting run

ρ is *accepting* if, for every maximal chain $C \subseteq 2^{<\omega}$, the number

$$\min\{\, c < \omega \mid (\Omega \circ \rho)^{-1}(c) \cap C \text{ is infinite} \,\} \qquad \text{is even.}$$

accepted

Finally, a labelled tree $T : 2^{<\omega} \to \Sigma$ is *accepted* by $\mathcal{A}$ if there exists an accepting run of $\mathcal{A}$ on T.

In order to decide $\mathrm{Th}_{\mathrm{MSO}}(\mathfrak{T}_2, \bar{P})$, for unary predicates $\bar{P}$, we have to encode the P_i by a labelling of $2^{<\omega}$.

Definition 1.4.2. For sets $P_0, \dots, P_{n-1} \subseteq 2^{<\omega}$, let $T_{\bar{P}}$ be the $\wp[n]$-labelled binary tree with

$$T(w) := \{\, i < n \mid w \in P_i \,\} \qquad \text{for } w \in 2^{<\omega}.$$

If $P_i = \{a_i\}$ is a singleton we also write $T_{\bar{a}}$.

The theorem of Rabin can now be stated in the following way:

Theorem 1.4.3 (Rabin)**.** *For each* MSO*-formula $\varphi(\bar{X}, \bar{x})$, we can effectively construct a nondeterministic parity automaton $\mathcal{A}$ recognising the language $\{\, T_{\bar{P},\bar{a}} \mid \mathfrak{T}_2 \vDash \varphi(\bar{P}, \bar{a}) \,\}$.*

For the proof of Muchnik's theorem we need a more general automaton model that runs on iterations of structures. In particular, the inputs are arbitrarily branching trees and, because of the clone relation cl, we need transition functions that depend on the current position in the input tree. Walukiewicz [78] introduced a fairly powerful class of automata which satisfies our needs. Since these automata are actually too general we have to restrict them to a suitable subclass in the next section.

Let $\mathcal{B}^+(X)$ be the set of infinitary positive boolean formulae over X, i.e., all formulae constructed from X with disjunction and conjunction. An *interpretation* of a formula $\varphi \in \mathcal{B}^+(X)$ is a set $I \subseteq X$ of atoms we consider true.

Definition 1.4.4. A *tree automaton* is a tuple *tree automaton*

$$\mathcal{A} = (Q, \Sigma, M, \delta, q_0, W)$$

where the input is a tree $M^{<\omega} \to \Sigma$, Q is the set of states, q_0 is the initial state, $W \subseteq Q^\omega$ is the acceptance condition, and

$$\delta : Q \times \Sigma \to \mathcal{B}^+(Q \times M)^{M^{<\omega}}$$

is the transition function which assigns to each state q and every input symbol c a function $\delta(q, c) : M^{<\omega} \to \mathcal{B}^+(Q \times M)$. Frequently we will write $\delta(q, c, w)$ instead of $\delta(q, c)(w)$.

Note that the transition function and acceptance condition of these automata are not finite. To obtain finite automata we will represent the transition function by an MSO-formula and consider only parity acceptance conditions in the next section.

In order to define the language accepted by such an automaton we introduce games.

Definition 1.4.5. A *game* $\mathcal{G} = (V_0, V_1, E, W)$ is a graph whose universe $V := V_0 \cup V_1$ is partitioned into positions for, respectively, player 0 and player 1. $W \subseteq V^\omega$ is the winning condition. We assume that every position has an outgoing edge. *game*

The game $\mathcal{G}$ starts at a given position v_0. In each turn that player the current position v belongs to selects an outgoing edge $(v, u) \in E$ and the game continues in position u. The resulting sequence $\pi \in V^\omega$ is called a *play*. Player 0 wins a play π if $\pi \in W$. Otherwise, player 1 wins. *play*

A *strategy* for player i is a function σ that assigns to every prefix $v_0, \dots, v_n$ of a play with $v_n \in V_i$ a successor $v_{n+1} = \sigma(v_0, \dots, v_n)$ such that $(v_n, v_{n+1}) \in E$. σ is *positional* if $\sigma(wv) = \sigma(w'v)$ for all sequences $wv, w'v$ whose last position is the same. A *winning strategy* is a strategy σ such that, whenever player i plays according to σ, then the resulting play is winning for him, regardless of the moves of the opponent. *strategy* *positional strategy* *winning strategy*

Below the winning conditions will mostly have the following form:

Definition 1.4.6. A function $\Omega : \Sigma \to [n]$ induces the *parity condition* $W \subseteq \Sigma^\omega$ which consists of all sequences $(c_i)_{i<\omega} \in \Sigma^\omega$ such that the least number appearing infinitely often in the sequence $(\Omega(c_i))_{i<\omega}$ is even. *parity condition*

parity automaton

parity game

A *parity automaton* is a tree automaton $\mathcal{A} = (Q, \Sigma, M, \delta, q_0, W)$ where W is a parity condition. In this case we sometimes write $\mathcal{A} = (Q, \Sigma, M, \delta, q_0, \Omega)$. Similarly, a *parity game* $\mathcal{G} = (V_0, V_1, E, \Omega)$ is a game with a parity winning condition.

The importance of parity winning conditions stems from the fact that all games with a parity condition are determined and the corresponding winning strategies are positional [37, 55].

Theorem 1.4.7 (Determinacy of parity games)**.** *For every parity game $\mathcal{G} = (V_0, V_1, E, \Omega)$ there exists a partition $W_0 \uplus W_1$ of the universe such that player i has a positional winning strategy σ_i for all plays starting in a position $v \in W_i$.*

Furthermore, Walukiewicz [78] has shown that the winning region W_0 of a parity game (V_0, V_1, E, Ω) can be defined by a μ-calculus formula. In monadic fixed point logic it takes the form (assuming an even number of priorities)

$$\mathrm{LFP}_{Z_n,x} \cdots \mathrm{GFP}_{Z_1,x} \bigvee_{k \leq n} \eta_k(x, \bar{Z})$$

$$\text{with } \eta_k := \Omega_k x \wedge [V_0 x \to \exists y(Exy \wedge Z_k y)] \wedge [V_1 x \to \forall y(Exy \to Z_k y)]$$

where $\Omega_k = \Omega^{-1}(k)$ is the set of positions of priority k. For a detailed definition of fixed point logic see [34]. The formula $\mathrm{LFP}_{Z,x}\varphi(x, Z)$ denotes the least set P such that we have $a \in P$ iff the formula $\varphi(a, P)$ holds. Analogously, GFP defines the greatest such set. Both of these operators can obviously be expressed in monadic second-order logic.

detailed game $\mathcal{G}(\mathcal{A}, T)$

Definition 1.4.8. Let $\mathcal{A} = (Q, \Sigma, M, \delta, q_0, W)$ be an automaton and $T : M^{<\omega} \to \Sigma$ a tree. The *detailed game* $\mathcal{G}(\mathcal{A}, T)$ is defined as follows:

(a) The set of vertices is

$$\big(Q \cup \mathcal{B}^+(Q \times M)\big) \times M^{<\omega}.$$

V_0 consists of all pairs $(q, w) \in Q \times M^{<\omega}$ and all pairs of the form (φ, w) where φ is either atomic or a disjunction, and V_1 consists of all pairs where φ is a conjunction.

(b) The initial position is (q_0, ε).

(c) Each node (q, w) has the successor $\big(\delta(q, T(w), w), w\big)$. The successors of nodes $(\bigwedge \Phi, w)$ and $(\bigvee \Phi, w)$ consist of all positions (φ, w) with $\varphi \in \Phi$. Finally, the successor of nodes $\big((q, a), w\big)$ with atomic formulae is (q, wa).

(d) Let $(\xi_i, w_i)_{i<\omega}$ be a play. Consider the subsequence $(\xi_{i_k}, w_{i_k})_{k<\omega}$ of positions where $\xi_{i_k} = q_k$ is a state. The play is winning if the sequence $q_0 q_1 \ldots$ is in W.

The language $L(\mathcal{A})$ recognised by $\mathcal{A}$ is the set of all trees T such that player 0 has a winning strategy for the game $\mathcal{G}(\mathcal{A}, T)$.

Sometimes it is more convenient to use a simpler game where several moves of the same kind are replaced by a single one. Both versions of the game are obviously equivalent.

Definition 1.4.9. Let $\mathcal{A} = (Q, \Sigma, M, \delta, q_0, W)$ be an automaton and $T : M^{<\omega} \to \Sigma$ a tree. Assume that δ is in disjunctive normal form. The *abridged game* $\check{\mathcal{G}}(\mathcal{A}, T)$ is defined by replacing conditions (a) and (c) in the above definition by:

abridged game $\check{\mathcal{G}}(\mathcal{A}, T)$

(a′) The sets of vertices are $V_0 := Q \times M^{<\omega}$ and $V_1 := \wp(Q \times M) \times M^{<\omega}$.

(c′) Each node $(q, w) \in V_0$ with $\delta(q, T(w), w) = \bigvee_i \bigwedge \Phi_i$ has the successors (Φ_i, w) for each i. The successors of a node $(\Phi, w) \in V_1$ are the nodes (q, wa) for $(q, a) \in \Phi$.

In the remainder of this section we will present results of Walukiewicz [78] showing that automata, as defined above, are closed under union, complement, and projection. This property is needed in the next section in order to translate formulae into automata. We start with unions.

Definition 1.4.10. Let $\mathcal{A}_i = (Q_i, \Sigma, M, \delta_i, q_i^0, W_i)$, $i = 1, 2$, be tree automata. Their *sum* is the automaton

$$\mathcal{A}_0 + \mathcal{A}_1 := \big(Q_1 \cup Q_2 \cup \{q^0\}, \Sigma, M, \delta, q^0, W\big)$$

where

$$\begin{aligned} \delta(q, c, w) &:= \delta_i(q, c, w) && \text{for } q \in Q_i\,, \\ \delta(q^0, c, w) &:= \delta_0(q_0^0, c, w) \vee \delta_1(q_1^0, c, w)\,, \end{aligned}$$

and W consists of all sequences $q_0 q_1 q_2 \ldots$ such that $q_0 = q^0$ is the initial state and $q_i^0 q_1 q_2 \ldots \in W_i$ for some i.

Lemma 1.4.11. $L(\mathcal{A}_0 + \mathcal{A}_1) = L(\mathcal{A}_0) \cup L(\mathcal{A}_1)$.

Proof. Note that $\mathcal{G}(\mathcal{A}_0 + \mathcal{A}_1, T)$ consists of disjoint copies of $\mathcal{G}(\mathcal{A}_0, T)$ and $\mathcal{G}(\mathcal{A}_1, T)$, and a new initial position from which player 0 has to choose one of the two subgames. Obviously, each winning strategy for player 0 in $\mathcal{G}(\mathcal{A}_0, T)$ or $\mathcal{G}(\mathcal{A}_1, T)$ is also a winning strategy in $\mathcal{G}(\mathcal{A}_0 + \mathcal{A}_1, T)$. On the other hand, if σ is a winning strategy for player 0 in the compound game it is also winning in either $\mathcal{G}(\mathcal{A}_0, T)$ or $\mathcal{G}(\mathcal{A}_1, T)$ depending on which subgame player 0 chooses in his first move. □

Complementation is easy as well.

Definition 1.4.12. Let $\mathcal{A} = (Q, \Sigma, M, \delta, q_0, W)$ be an automaton. Its complement is the automaton $\bar{\mathcal{A}} := (Q, \Sigma, M, \bar{\delta}, q_0, \bar{W})$ with

$$\bar{\delta}(q, c, w) := \overline{\delta(q, c, w)} \qquad \text{and} \qquad \bar{W} := Q^\omega \setminus W\,.$$

Here $\overline{\varphi}$ denotes the dual of φ, i.e., the formula where each $\wedge$ is replaced by $\vee$ and vice versa.

Lemma 1.4.13. $T \in L(\bar{\mathcal{A}})$ iff $T \notin L(\mathcal{A})$.

Proof. Let $\mathcal{G}(\bar{\mathcal{A}}, T) = (\bar{V}_0, \bar{V}_1, \bar{E}, \bar{W})$. Note that in $\mathcal{G}(\bar{\mathcal{A}}, T)$ the roles of player 0 and 1 are exchanged. $\bar{V}_0$ consists of all former V_1-nodes, and $\bar{V}_1$ contains all V_0-nodes except for the atomic ones. Since the latter have exactly one successor it is irrelevant which player they are assigned to. Thus, each choice of player 0 in the old game is made by player 1 in the new one and vice versa. Hence, a winning strategy σ for player 0 in $\mathcal{G}(\mathcal{A}, T)$ is a strategy for player 1 in $\mathcal{G}(\bar{\mathcal{A}}, T)$ which ensures that the resulting play induces a sequence in $W = Q^\omega \setminus \bar{W}$. Thus, σ is winning for 1. The other direction follows by symmetry. □

The closure under projections is the hardest part to prove. The projection $\Pi(L)$ of a tree language L is the set of all trees $T : M^{<\omega} \to \Sigma$ such that there exists a tree $T' : M^{<\omega} \to \Sigma \times [2]$ in L with

$$T'(w) = (T(w), c_w) \quad \text{for some } c_w \in [2] \text{ and all } w \in M^{<\omega}.$$

The proof is divided into several steps. We prove closure under projection for nondeterministic automata, and show that each alternating automaton can be transformed into an equivalent nondeterministic one.

Definition 1.4.14. An automaton $\mathcal{A} := (Q, \Sigma, M, \delta, q_0, W)$ is *nondeterministic* if each formula $\delta(q, c, w)$ is in disjunctive normal-form $\bigvee_i \bigwedge_k (q_{ik}, a_{ik})$ where, for each fixed i, all the a_{ik} are different.

Definition 1.4.15. Let $\mathcal{A} = (Q, \Sigma \times [2], M, \delta, q_0, W)$ be a nondeterministic automaton. Define $\mathcal{A}_\Pi := (Q, \Sigma, M, \delta_\Pi, q_0, W)$ where

$$\delta_\Pi(q, c, w) := \delta(q, (c, 0), w) \vee \delta(q, (c, 1), w)\,.$$

Lemma 1.4.16. $L(\mathcal{A}_\Pi) = \Pi(L(\mathcal{A}))$

Proof. ($\supseteq$) Let σ be a winning strategy for player 0 in $\mathcal{G}(\mathcal{A}, T)$. $\mathcal{G}(\mathcal{A}_\Pi, \Pi(T))$ contains additional vertices of the form $(\varphi_0 \vee \varphi_1, w)$ where $\varphi_i = \delta(q, (c, i), w)$. By defining

$$\sigma(\varphi_0 \vee \varphi_1, w) := \varphi_i \quad \text{for the } i \text{ with } T(w) = (c, i)$$

we obtain a strategy for player 0 in the new game. This strategy is winning since, if one removes the additional vertices from a play according to the extended strategy, a play according to σ in the original game is obtained which is winning by assumption.

($\subseteq$) Let σ be a winning strategy for player 0 in $\mathcal{G}(\mathcal{A}_\Pi, T)$. We have to define a tree $T' \in L(\mathcal{A})$ with $T = \Pi(T')$. Since $\mathcal{A}_\Pi$ is nondeterministic the game has the following structure: At each position $((q, a), w)$ with

$$\delta(q, T(w), w) = \bigvee_i \bigwedge_k (q_{ik}, a_{ik})$$

player 0 chooses some conjunction $\bigwedge_k (q_{ik}, a_{ik})$ out of which player 1 picks a successor (q_{ik}, a_{ik}). Thus, for each word $w \in M^{<\omega}$ there is at most one state q such that a play according to σ reaches the position (q, w). Suppose that $\sigma(\varphi_0 \vee \varphi_1, w) = (\varphi_i, w)$ where $\varphi_0 \vee \varphi_1 = \delta(q, T(w), w)$. We define T' by $T'(w) := (T(w), i)$. □

It remains to show how to translate alternating automata to nondeterministic ones. To do so we need to introduce some notation for operations on transition relations.

Definition 1.4.17. Let $\varphi \in \mathcal{B}^+(Q \times M)$.

(a) The *collection* of φ is defined as follows. Let $\bigvee_i \bigwedge_k (q_{ik}, a_{ik})$ be the disjunctive normal form of φ. *collection* $\operatorname{collect}(\varphi)$

$$\operatorname{collect}(\varphi) := \bigvee_i \bigwedge_{a \in M} \big(Q_i(a), a\big) \in \mathcal{B}^+(\wp(Q) \times M)$$

where $Q_i(a) := \{\, q_{ik} \mid a_{ik} = a \,\}$.

(b) Let $q' \in Q'$. The *shift* of φ by the state q' is the formula $\operatorname{sh}_{q'} \varphi \in \mathcal{B}^+(Q' \times Q \times M)$ obtained from φ by replacing all atoms (q, a) by (q', q, a). *shift* $\operatorname{sh}_q \varphi$

(c) For $S \subseteq Q \times Q$ let $(S)_2$

$$(S)_2 := \{\, q \mid (q', q) \in S \text{ for some } q' \,\}.$$

The translation is performed in two steps. First, the alternating automaton is transformed into a nondeterministic one with an obscure non-parity acceptance condition. Then, the result is turned into a normal nondeterministic parity automaton. The construction used for the first step is the usual one. For each node of the input tree the automaton stores the set of states of the original automaton from which the corresponding subtree must be accepted. That is, for universal choices of the alternating automaton, all successors are remembered, whereas for existential choices, only one successor is picked nondeterministically. What makes matters slightly more complicated is the fact that, in order to define the acceptance condition, the new automaton has to remember not only the set of current states but their predecessors as well, i.e., its states are of the form (q', q) where q is the current state of the original automaton and q' is the previous one.

Definition 1.4.18. Let $\mathcal{A} = (Q, \Sigma, M, \delta, q_0, W)$ be an alternating automaton.

$$\mathcal{A}_n := \big(\wp(Q \times Q), \Sigma, M, \delta_n, \{(q_0, q_0)\}, W_n\big)$$

is the automaton where

$$\delta_{\mathrm{n}}(S, c, w) := \operatorname{collect} \bigwedge_{q \in (S)_2} \operatorname{sh}_q \delta(q, c, w)\,.$$

trace A sequence $(q_i)_{i<\omega} \in Q^\omega$ is called a *trace* of $(S_i)_{i<\omega} \in \wp(Q \times Q)^\omega$ if $(q_i, q_{i+1}) \in S_i$ for all $i < \omega$. W_{n} consists of all sequences $(S_i)_{i<\omega} \in \wp(Q \times Q)^\omega$ such that every trace of $(S_i)_{i<\omega}$ is in W.

Lemma 1.4.19. *$\mathcal{A}_{\mathrm{n}}$ is a nondeterministic automaton with $L(\mathcal{A}_{\mathrm{n}}) = L(\mathcal{A})$.*

Proof. The definition of collect ensures that $\mathcal{A}_{\mathrm{n}}$ is nondeterministic.

($\supseteq$) Let $T \in L(\mathcal{A})$ and let σ be the corresponding winning strategy for player 0 in $\breve{\mathcal{G}}(\mathcal{A}, T)$. To define a strategy σ_{n} in $\breve{\mathcal{G}}(\mathcal{A}_{\mathrm{n}}, T)$ consider a position $(S, w) \in \breve{\mathcal{G}}(\mathcal{A}, T)$. Let $\sigma(q, w) = (\Phi_q, w)$ for $q \in (S)_2$. We define $\sigma_{\mathrm{n}}(S, w) := (\operatorname{collect} \bigwedge \Phi, w)$ where

$$\Phi = \bigcup_{q \in (S)_2} \operatorname{sh}_q \Phi_q\,.$$

This is valid since $(\operatorname{collect} \bigwedge \Phi, w)$ is a successor of (q, w).

To show that σ_{n} is a winning strategy consider the result $(S_i)_{i<\omega}$ of a play according to σ_{n}. If $(\Phi, w) \in \sigma_{\mathrm{n}}(S_i, w)$ and $(S_{i+1}, a) \in \Phi$, then for each $(q, q') \in S_{i+1}$ it is the case that $(q', a) \in \Phi_q$. Thus, all traces of $(S_i)_{i<\omega}$ are plays according to σ and therefore winning.

($\subseteq$) Let σ_{n} be a – not necessarily memoryless – winning strategy for player 0 in $\breve{\mathcal{G}}(\mathcal{A}_{\mathrm{n}}, T)$. We construct a winning strategy for player 0 in $\breve{\mathcal{G}}(\mathcal{A}, T)$ as follows. Let π_{n} be the prefix of a play according to σ_{n} in $\breve{\mathcal{G}}(\mathcal{A}_{\mathrm{n}}, T)$ with last position (S, w), and let π be the play according to σ. By induction we ensure that the last position in π is of the form (q, w) for some $q \in (S)_2$. Let $(\Phi_{\mathrm{n}}, w) = \sigma_{\mathrm{n}}(\pi_{\mathrm{n}})$ and define

$$\Phi := \{\, (q', a) \mid (S', a) \in \Phi_{\mathrm{n}} \text{ and } ((q, q'), a) \in S' \text{ for some } S' \,\}\,.$$

Then $\bigwedge \Phi$ is a conjunction in $\delta(q, T(w), w)$, by definition of δ_{n}, and we can set $\sigma(\pi) := (\Phi, w)$. The answer of player 0 to this move consists of some position (q', wa) for $(q', a) \in \Phi$. Suppose that in $\breve{\mathcal{G}}(\mathcal{A}_{\mathrm{n}}, T)$ player 1 chooses the position (S_a, wa) where S_a is the unique state such that $(S_a, a) \in \Phi_{\mathrm{n}}$. Since $(q, q') \in S_a$ the induction hypothesis is satisfied for the extended plays $\pi(\Phi, w)(q', wa)$ and $\pi_{\mathrm{n}}(\Phi_{\mathrm{n}}, w)(S_a, a)$.

It follows that each play π according to σ in $\breve{\mathcal{G}}(\mathcal{A}, T)$ is a trace of some play π_{n} according to σ_{n} and therefore winning by construction of $\mathcal{A}_{\mathrm{n}}$. □

The automaton $\mathcal{A}_{\mathrm{n}}$ constructed above does not have a parity acceptance condition. Since we intend to consider only parity automata in the next section, we have to construct a nondeterministic automaton

with such an acceptance condition. It is easy to see that, provided that the original automaton does have a parity acceptance condition, there is some parity automaton on infinite words $\mathcal{B} = (P, \wp(Q\times Q), \delta, p^\circ, \Omega)$ which recognises $W_\mathrm{n} \subseteq \wp(Q\times Q)^\omega$. Let $\mathcal{A}_\mathrm{p}$ be the product automaton of $\mathcal{A}_\mathrm{n}$ and $\mathcal{B}$, that is,

$$\mathcal{A}_\mathrm{p} = \big(P \times \wp(Q \times Q),\ \Sigma,\ M,\ \delta_\mathrm{p},\ (p^\circ, q^\circ_\mathrm{n}),\ \Sigma_\mathrm{p}\big)$$

where

$$\delta_\mathrm{p}((p, S), c, w) = \mathrm{sh}_{p'}\, \delta_\mathrm{n}(S, c, w) \qquad \text{for } p' := \delta(p, S)$$

and $\Omega_\mathrm{p}(p, S) = \Omega(p)$.

Lemma 1.4.20. *$\mathcal{A}_\mathrm{p}$ is a nondeterministic parity automaton recognising the language $L(\mathcal{A}_\mathrm{p}) = L(\mathcal{A}_\mathrm{n})$.*

Proof. Let σ be a winning strategy for player o in $\check{\mathcal{G}}(\mathcal{A}_\mathrm{n}, T)$. We define a corresponding strategy σ' in $\check{\mathcal{G}}(\mathcal{A}_\mathrm{p}, T)$ by

$$\sigma'((p, S), w) := \big(\mathrm{sh}_{p'}\, \Phi, w\big)$$

where $(\Phi, w) = \sigma(S, w)$ and $p' = \delta(p, S)$. That way every play

$$\big((p_\circ, S_\circ), w_\circ\big)\big(\Phi'_\circ, w_\circ\big)\big((p_1, S_1), w_1\big)\big(\Phi'_1, w_1\big)\dots$$

in $\check{\mathcal{G}}(\mathcal{A}_\mathrm{p}, T)$ according to σ' is induced by a play

$$(S_\circ, w_\circ)(\Phi_\circ, w_\circ)(S_1, w_1)(\Phi_1, w_1)\dots$$

in $\check{\mathcal{G}}(\mathcal{A}_\mathrm{n}, T)$ according to σ. Further, $(p_i)_{i<\omega}$ is the run of $\mathcal{B}$ on $(S_i)_{i<\omega}$. Since the second play is winning, the first one is so as well, by definition of $\mathcal{B}$. Hence, σ' is a winning condition. The other direction is proved analogously. □

In the next section we will define a restricted class of automata where we only allow transition functions that are MSO-definable. In order to transfer the results of this section we need to extract the required closure properties of the set of allowed transition functions from the above proofs.

Theorem 1.4.21. *Let $\mathcal{T}$ be a class of functions $f : M^{<\omega} \to \mathcal{B}^+(Q \times M)$ where M and Q may be different for each $f \in \mathcal{T}$. If $\mathcal{T}$ is closed under disjunction, conjunction, dual, shift, and collection then the class of automata with transition functions $\delta : Q \times \Sigma \to \mathcal{T}$ is closed under union, complement, and projection, and every such automaton can be transformed into a nondeterministic one.*

1.5 Translating formulae into automata

The type of automata defined in the previous section is much too powerful. For the proof of Muchnik's theorem we have to find a subclass whose expressive power corresponds exactly to the logic in question. Actually, we will prove an extension of Muchnik's theorem for stronger logics. Since, in general, a version of this theorem for one logic does not imply the corresponding version for another logic, even if the latter is strictly weaker, we have to state the theorem for each logic separately. To avoid duplicating the proofs we introduce the following notions.

$\mathfrak{L}$ *extends* MSO

Definition 1.5.1. A logic $\mathfrak{L}$ *extends* MSO if it contains MSO and it is closed under boolean operations and set quantification.

If $\mathfrak{L}$ is a logic extending MSO then we denote by $\mathfrak{L}$ + GSO the extension of $\mathfrak{L}$ by guarded second-order quantification.

Definition 1.5.2. The following class of logics is considered below.

$$\mathcal{L} := \{\text{MSO},\ \text{GSO},\ \text{MSO}(\exists^\omega),\ \text{GSO}(\exists^\omega),\ \text{MSO + C},\ \text{GSO + C}\}\,.$$

For guarded relations on the iteration of a structure we define the following notations.

Definition 1.5.3. Let $\mathfrak{M} = (M, \bar{R})$, $S \subseteq (M^{<\omega})^s$, and $w \in M^{<\omega}$. Define

$$S|_w := \{\, a \in M \mid wa \in S \,\}\,.$$

local relation

S is called *local* if $S = \bigcup \{\, wS_w \mid w \in M^{<\omega} \,\}$, i.e., if every tuple $\bar{c} \in S$ is of the form $(wa_0, \ldots, wa_{n-1})$ for some $w \in M^{<\omega}$, and $a_0, \ldots, a_{n-1} \in M$.

Remark. Let $\mathfrak{M}$ be a structure with iteration $\mathfrak{M}^*$. If every tuple of $S \subseteq M^{<\omega}$ is guarded by a relation R^* then S is local. In particular, every guarded relation $S \subseteq M^{<\omega}$ can be written as union $S = S_0 \uplus S_1$ where S_1 is local and every tuple of S_0 is guarded by suc.

Let $\mathfrak{L}$ be a logic extending MSO. In order to evaluate $\mathfrak{L}$-formulae over the iteration of a structure we translate them into automata where the transition function is defined by $\mathfrak{L}$-formulae. This can be done in such a way that the resulting class of automata is expressively equivalent to $\mathfrak{L}$.

$\langle\!\langle \varphi; \bar{S} \rangle\!\rangle_{\mathfrak{M}}$

Definition 1.5.4. Let $\mathfrak{L}$ be an extension of MSO, $\mathfrak{M}$ a structure, $\bar{S}$ relations over $M^{<\omega}$, $\varphi(X, \bar{Y}; \bar{Z})$ an $\mathfrak{L}$-formula, and $n < \omega$. The function

$$\langle\!\langle \varphi; \bar{S} \rangle\!\rangle_{\mathfrak{M}} : M^{<\omega} \to \mathcal{B}^+([n] \times M)$$

is defined by

$$\langle\!\langle \varphi; \bar{S} \rangle\!\rangle_{\mathfrak{M}}(\varepsilon) := \bigvee \Big\{ \bigwedge \{ (q, b) \mid b \in Q_q \} \;\Big|\; Q_0, \dots, Q_{n-1} \subseteq M \text{ such that } \mathfrak{M} \models \varphi(\varnothing, \bar{Q}; \bar{S}|_\varepsilon) \Big\},$$

$$\langle\!\langle \varphi; \bar{S} \rangle\!\rangle_{\mathfrak{M}}(wa) := \bigvee \Big\{ \bigwedge \{ (q, b) \mid b \in Q_q \} \;\Big|\; Q_0, \dots, Q_{n-1} \subseteq M \text{ such that } \mathfrak{M} \models \varphi(\{a\}, \bar{Q}; \bar{S}|_{wa}) \Big\}.$$

Let $\mathcal{T}^n_{\mathfrak{M}}$ be the set of all functions of the form $\langle\!\langle \varphi; \bar{S} \rangle\!\rangle_{\mathfrak{M}}$.

We consider automata where the transition functions are of the form $\langle\!\langle \varphi; \bar{S} \rangle\!\rangle_{\mathfrak{M}}$.

Definition 1.5.5. Let $\mathfrak{L}$ be an extension of MSO.

(a) An $\mathfrak{L}$-*automaton* is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, \Omega)$ where $Q = [n]$ for some $n < \omega$ and $\delta : Q \times \Sigma \to \mathfrak{L}$. *$\mathfrak{L}$-automaton*

(b) A Σ-labelled structure $(\mathfrak{M}^*, \bar{S})$ with local relations $\bar{S}$ is *accepted* by $\mathcal{A}$ if the automaton $\mathcal{A}_{\mathfrak{M}} := (Q, \Sigma, M, \delta_{\mathfrak{M}}, q_0, \Omega)$ accepts $\mathfrak{M}^*$, where $\delta : Q \times \Sigma \to \mathcal{T}^n_{\mathfrak{M}}$ is defined by $\delta_{\mathfrak{M}}(q, c) := \langle\!\langle \delta(q, c); \bar{S} \rangle\!\rangle_{\mathfrak{M}}$.

In order to translate formulae into automata, the latter must be closed under all operations available in the respective logic.

Proposition 1.5.6. *Let $\mathfrak{L}$ be an extension of* MSO. *$\mathfrak{L}$-automata are closed under boolean operations and projection.*

Proof. By Theorem 1.4.21, it is sufficient to show closure under disjunction, conjunction, dual, shift, and collection. To do so we will frequently need to convert between interpretations $I \subseteq Q \times M$ of boolean formulae $\langle\!\langle \varphi; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w) \in \mathcal{B}^+(Q \times M)$ and sets $\bar{Q}$ such that $\mathfrak{M} \models \varphi(C, \bar{Q})$. Given $I \subseteq Q \times M$ define

$$Q_i(I) := \{ a \in M \mid (q_i, a) \in I \}$$

for $i < n$, and given $Q_0, \dots, Q_{n-1} \subseteq M$ let

$$I(\bar{Q}) := \{ (q_i, a) \mid a \in Q_i,\ i < n \}.$$

Note that $I(\bar{Q}(I)) = I$ and $Q_i(I(\bar{Q})) = Q_i$. Then

$$I \models \langle\!\langle \varphi; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w) \quad \text{iff} \quad \mathfrak{M} \models \varphi(C, \bar{Q}(I); \bar{R}|_w)$$

and vice versa. (Here and below C denotes the set consisting of the last element of w.)

(disjunction) For the disjunction of two $\mathfrak{L}$-definable functions we can simply take the disjunction of their definitions since

$$I \models \langle\!\langle \varphi_0; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w) \vee \langle\!\langle \varphi_1; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w)$$

iff $I \vDash \langle\!\langle \varphi_i; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w)$ for some i

iff $\mathfrak{M} \vDash \varphi_i(C, \bar{Q}(I); \bar{R})$ for some i

iff $\mathfrak{M} \vDash \varphi_0(C, \bar{Q}(I); \bar{R}) \vee \varphi_1(C, \bar{Q}(I); \bar{R})$

iff $I \vDash \langle\!\langle \varphi_0 \vee \varphi_1; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w)$.

(dual) The definition of the dual operation is slightly more involved.

$I \vDash \overline{\langle\!\langle \varphi; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w)}$

iff $Q \times M \setminus I \nvDash \langle\!\langle \varphi; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w)$

iff $J \vDash \langle\!\langle \varphi; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w)$ implies $J \cap I \neq \varnothing$

iff $\mathfrak{M} \vDash \varphi(C, \bar{P}; \bar{R})$ implies $P_i \cap Q_i(I) \neq \varnothing$ for some i

iff $\mathfrak{M} \vDash \forall \bar{P}\big(\varphi(C, \bar{P}; \bar{R}) \to \bigvee_{i<n} P_i \cap Q_i \neq \varnothing\big)$.

(conjunction) follows from (disjunction) and (dual).

(shift) For a shift we simply need to renumber the states. If the pair (q_i, q_k) is encoded as number $ni + k$ we obtain

$$\varphi(C, Q_{ni+0}, \dots, Q_{ni+n-1}; \bar{R}).$$

(collection) The collection of a formula can be defined the following way:

$I \vDash \operatorname{collect} \langle\!\langle \varphi; \bar{R} \rangle\!\rangle_{\mathfrak{M}}(w)$

iff there are $Q'_S \subseteq Q_S(I)$ such that $\bar{Q}'$ partitions M and

$\mathfrak{M} \vDash \varphi(C, \bar{P}; \bar{R})$ where

$a \in P_i$: iff $i \in S$ for the unique $S \subseteq [n]$ with $a \in Q'_S$

iff there are $\bar{Q}'$ partitioning M such that $\mathfrak{M} \vDash \varphi(C, \bar{P}; \bar{R})$ where

$P_i := \bigcup \{ Q'_S \mid i \in S \}$

iff $\mathfrak{M} \vDash \varphi(C, \bar{P}; \bar{R})$ for some $P_i \subseteq \bigcup \{ Q_S \mid i \in S \}$ with

$P_i \cap Q_S = \varnothing$ for all S with $i \notin S$

iff $\mathfrak{M} \vDash \exists \bar{P}\big(\varphi(C, \bar{P}; \bar{R}) \wedge \bigwedge_{i<n} P_i \subseteq \bigcup \{ Q_S \mid i \in S \}$

$\wedge \bigwedge_{S \subseteq [n]} \bigwedge_{i \notin S} P_i \cap Q_S = \varnothing\big)$. □

For proper extensions $\mathfrak{L}$ of MSO, we further have to prove that $\mathfrak{L}$-automata are closed under the addional operations available in $\mathfrak{L}$.

Proposition 1.5.7. *Let $\mathfrak{L}$ be an extension of* MSO. $\mathfrak{L}$ + GSO*-automata are closed under guarded quantification.*

Proof. We have seen that every guarded relations can be written as union $T = T_0 \cup T_1$ where T_1 is local and every tuple in T_0 is guarded by suc. Consequently, we can replace every second-order

quantifier $\exists T$ by two quantifiers $\exists^{\mathrm{suc}} T_0 \exists^{\mathrm{loc}} T_1$ where the former ranges over relations guarded by suc and the latter over local relations.

First, suppose that we existentially quantify a relation T where every k-tuple $\bar{a} \in T$ is contained in an edge $(w_0, w_1) \in \mathrm{suc}$. We can encode $\bar{a}$ by the element w_1 and a function $h : [k] \to [2]$ such that $a_i = w_{h(i)}$. Consequently, we can replace the quantifier by a sequence of 2^k monadic quantifiers $\exists X_h$ where the index h ranges over $[2]^{[k]}$.

It remains to consider formulae $\exists^{\mathrm{loc}} T\, \psi(\bar{X}; \bar{S}, T)$ where we quantify over a nonmonadic local relation T. Let $\mathcal{A} = (Q, \Sigma, \delta_{\mathcal{A}}, q_0, \Omega)$ be a nondeterministic automaton equivalent to ψ. Since T ranges over local relations we have $\mathfrak{M}^* \models \exists^{\mathrm{loc}} T\, \psi(\bar{P}; \bar{S}, T)$ if and only if there are sets $T_w \subseteq M$ such that $\mathfrak{M}^* \models \psi(\bar{P}; \bar{S}, T)$ where $T := \bigcup_w wT_w$. By induction hypothesis, this is equivalent to $\mathcal{A}$ accepting the structure $(\mathfrak{M}^*, \bar{P}, \bar{S}, T)$.

We claim that this is the case if and only if $(\mathfrak{M}^*, \bar{P}, \bar{S})$ is accepted by the automaton $\mathcal{B} = (Q, \Sigma, \delta_{\mathcal{B}}, q_0, \Omega)$ where

$$\delta_{\mathcal{B}}(q, c) := \exists T \delta_{\mathcal{A}}(q, c)\,.$$

Before we prove that $\mathcal{B}$ is the desired automaton, we first show that it is also nondeterministic.

Suppose otherwise. There exists a model I of $\langle\!\langle \exists T \delta(q, c); \bar{S} \rangle\!\rangle_{\mathfrak{M}}(w)$ which is minimal and contains pairs (q_0, a), $(q_1, a) \in I$ for some $q_0 \neq q_1$. Since

$$\mathfrak{M} \models \exists T \delta(q, c)(C, \bar{Q}(I); \bar{S}|_w, T)$$

we find some $T' \subseteq M$ such that

$$\mathfrak{M} \models \delta(q, c)(C, \bar{Q}(I); \bar{S}|_w, T')\,.$$

Setting $T := wT'$ it follows that

$$I \models \langle\!\langle \delta(q, c); \bar{S}, T \rangle\!\rangle_{\mathfrak{M}}(w)\,.$$

As $\mathcal{A}$ is nondeterministic there exists a model $I_0 \subset I$ such that $Q_i(I_0) \cap Q_k(I_0) = \varnothing$ for $i \neq k$. But

$$I_0 \models \langle\!\langle \delta(q, c); \bar{S}, T \rangle\!\rangle_{\mathfrak{M}}(w)\,.$$

implies that

$$I_0 \models \langle\!\langle \exists T \delta(q, c); \bar{S} \rangle\!\rangle_{\mathfrak{M}}(w)$$

in contradiction to the minimality of I.

It remains to prove the above claim.

($\Rightarrow$) Let $\rho : M^{<\omega} \to Q$ be the run of $\mathcal{A}$ on $(\mathfrak{M}^*, \bar{P}, \bar{S}, T)$. Let $w \in M^{<\omega}$ and define $I_w := \{ (\rho(wa), a) \mid a \in M \}$. For all $w \in M^{<\omega}$ we have

$$\begin{aligned} & I_w \vDash \langle\!\langle \delta(q, c); \bar{S}, T \rangle\!\rangle_{\mathfrak{M}}(w) \\ \Rightarrow\ & \mathfrak{M} \vDash \delta(q, c)(C, \bar{Q}(I_w); \bar{S}|_w, T_w) \\ \Rightarrow\ & \mathfrak{M} \vDash \exists T \delta(q, c)(C, \bar{Q}(I_w); \bar{S}|_w, T) \\ \Rightarrow\ & I_w \vDash \langle\!\langle \exists T \delta(q, c); \bar{S} \rangle\!\rangle_{\mathfrak{M}}(w) . \end{aligned}$$

Consequently, ρ is also a run of $\mathcal{B}$ on $(\mathfrak{M}, \bar{P}, \bar{S})$.

($\Leftarrow$) Let $\rho : M^{<\omega} \to Q$ be the run of $\mathcal{B}$ on $(\mathfrak{M}^*, \bar{P}, \bar{S})$. For $w \in M^{<\omega}$ define $I_w := \{ (\rho(wa), a) \mid a \in M \}$ and fix some $T_w \subseteq M^k$ such that

$$\mathfrak{M} \vDash \delta(q, c)(C, \bar{Q}(I_w); \bar{S}|_w, T_w) .$$

Define $T := \bigcup_w wT_w$. Then $I_w \vDash \langle\!\langle \delta(q, c); \bar{S}, T \rangle\!\rangle_{\mathfrak{M}}(w)$. Hence, ρ is a run of $\mathcal{A}$ on $(\mathfrak{M}, \bar{P}, \bar{S}, T)$. □

Lemma 1.5.8. *Let $\mathfrak{L}$ be an extension of* MSO. *There exists an $\mathfrak{L}(\exists^\omega)$-automaton recognising the predicate $|X| \geq \aleph_0$.*

Proof. By König's lemma, there are two possible scenarios for infinite sets X_i. The prefix closure $\downarrow X_i$ may contain an infinite path, or there is some $w \in \downarrow X_i$ such that $wa \in \downarrow X_i$ for infinitely many elements $a \in M$. The automaton for the predicate $|X_i| \geq \aleph_0$ has states $Q := \{q_0, q_1\}$ and priority function $\Omega(q_0) := 0$, $\Omega(q_1) := 1$. In state q_0 it looks for infinitely many elements $x \in X_i$, whereas in state q_1 it looks for at least one such element. We define the transition function δ such that

$$\delta_{\mathfrak{M}}(q_0, c, w) = \bigvee_{a \in M} \big((q_0, a) \wedge (q_1, a) \big) \vee \bigvee_{\substack{M_0 \subseteq M \\ |M_0| \geq \aleph_0}} \bigwedge_{a \in M_0} (q_1, a) ,$$

$$\delta_{\mathfrak{M}}(q_1, c, w) = \begin{cases} \text{true} & \text{if } i \in c , \\ \bigvee_{a \in M} (q_1, a) & \text{otherwise} , \end{cases}$$

by setting

$$\begin{aligned} \delta(q_0, c) &= \exists x (Q_0 x \wedge Q_1 x) \vee |Q_1| \geq \aleph_0 , \\ \delta(q_1, c) &= \begin{cases} \text{true} & \text{if } i \in c , \\ \exists x Q_1 x & \text{otherwise} , \end{cases} \end{aligned}$$

□

Lemma 1.5.9. *Let $\mathfrak{L}$ be an extension of* MSO. *There exists an $\big(\mathfrak{L} + \mathrm{C}\big)$-automaton recognising the predicate $|X| \equiv k \pmod m$.*

Proof. Since there is an $\mathfrak{L}(\exists^\omega)$-automaton for $|X_i| \geq \aleph_0$ we may assume that X_i is finite when constructing an automaton for the predicate $|X_i| \equiv k \pmod m$.

Let $Q := \{ q_k \mid k < m \}$, $\Omega(q_0) := 0$, and $\Omega(q_k) := 1$ for $k \neq 0$. We label an element w by q_k if

$$|X \cap wM^{<\omega}| \equiv k \pmod{m}.$$

If n_k is the number of successors wa such that $|X \cap waM^{<\omega}| \equiv k \pmod{m}$ then we have

$$|X \cap wM^{<\omega}| \equiv \sum_{k<m} kn_k + |X \cap \{w\}| \pmod{m}.$$

Obviously, we only need to know n_k modulo m. Consequently, we define

$$\delta(q_k, c) = \begin{cases} \bigvee_{\bar{n} \in N_{k-1}} \bigwedge_{l<m} |Q_l| \equiv n_l \pmod{m} & \text{if } i \in c, \\ \bigvee_{\bar{n} \in N_k} \bigwedge_{l<m} |Q_l| \equiv n_l \pmod{m} & \text{otherwise}, \end{cases}$$

where

$$N_k := \left\{ \bar{n} \in [m]^k \;\middle|\; \sum_{l<m} ln_l \equiv k \pmod{m} \right\}.$$

□

With these preparations we can state the equivalence result. We say that an automaton $\mathcal{A}$ is *equivalent* to an $\mathfrak{L}$-formula $\varphi(X_0, \dots, X_{m-1})$ where all free variables are monadic if

$$L(\mathcal{A}) = \{ \mathfrak{M}^*_{\bar{P}} \mid \mathfrak{M}^* \vDash \varphi(\bar{P}) \}$$

where $\mathfrak{M}^*_{\bar{P}}$ is the structure where every element $w \in M^{<\omega}$ is labelled by the set $\{ i < m \mid w \in P_i \}$.

Theorem 1.5.10. *Let $\mathfrak{L} \in \mathcal{L}$. For every formula $\varphi \in \mathfrak{L}$ there is an equivalent $\mathfrak{L}$-automaton and vice versa.*

Proof. ($\Rightarrow$) By induction on $\varphi(\bar{X})$ we construct an equivalent $\mathfrak{L}$-automaton $\mathcal{A} := (Q, \wp[m], \delta, q_0, \Omega)$. We have already seen that $\mathfrak{L}$-automata are closed under all operations of $\mathfrak{L}$. Hence, it only remains to construct automata for atomic formulae.

($X_i \subseteq X_j$) We have to check for every element w of the input tree T that $i \notin T(w)$ or $j \in T(w)$. Thus, we set $Q := \{q_0\}$ with $\Omega(q_0) := 0$ and define the transition function such that

$$\delta_{\mathfrak{M}}(q_0, c, w) = \begin{cases} \bigwedge_{a \in M} (q_0, a) & \text{if } i \notin c \text{ or } j \in c, \\ \text{false} & \text{otherwise}. \end{cases}$$

for each input structure $\mathfrak{M}^*$. This can be done by setting

$$\delta(q_0, c) := \begin{cases} \forall x Q_0 x & \text{if } i \notin c \text{ or } j \in c, \\ \text{false} & \text{otherwise}. \end{cases}$$

($R^* X_{i_1} \dots X_{i_k}$) Set $Q := \{q_0, \dots, q_k\}$ and $\Omega(q_i) := 1$. The automaton guesses a node in the input tree while in state q_0 and checks whether its children are in the relation R. That is,

$$\begin{aligned}\delta_{\mathfrak{M}}(q_0, c, w) &= \bigvee_{a \in M} (q_0, a) \\ &\quad \vee \bigvee \{ (q_1, a_1) \wedge \dots \wedge (q_k, a_k) \mid \bar{a} \in R^{\mathfrak{M}} \}, \\ \delta_{\mathfrak{M}}(q_j, c, w) &= \begin{cases} \text{true} & \text{if } i_j \in c, \\ \text{false} & \text{otherwise}, \end{cases} \qquad \text{for } 1 \leq j \leq k.\end{aligned}$$

The corresponding $\mathfrak{L}$-definition is

$$\begin{aligned}\delta(q_0, c) &:= \exists x Q_0 x \vee \exists \bar{x}(R\bar{x} \wedge Q_1 x_1 \wedge \dots \wedge Q_k x_k), \\ \delta(q_j, c) &= \begin{cases} \text{true} & \text{if } i_j \in c, \\ \text{false} & \text{otherwise}, \end{cases} \qquad \text{for } 1 \leq j \leq k.\end{aligned}$$

($SX_{i_1} \dots X_{i_k}$ for a relation variable S) The automaton is identical to the one for $R^* \bar{X}$. We set $Q := \{q_0, \dots, q_k\}$, $\Omega(q_i) := 1$, and define

$$\begin{aligned}\delta(q_0, c) &:= \exists x Q_0 x \vee \exists \bar{x}(S\bar{x} \wedge Q_1 x_1 \wedge \dots \wedge Q_k x_k), \\ \delta(q_j, c) &= \begin{cases} \text{true} & \text{if } i_j \in c, \\ \text{false} & \text{otherwise}, \end{cases} \qquad \text{for } 1 \leq j \leq k.\end{aligned}$$

($\mathrm{suc}(X_i, X_j)$) Let $Q := \{q_0, q_1\}$ and $\Omega(q_i) := 1$. We guess some element $w \in X_i$ having a successor in X_j.

$$\begin{aligned}\delta_{\mathfrak{M}}(q_0, c, w) &= \begin{cases} \bigvee_{a \in M} (q_0, a) & \text{if } i \notin c, \\ \bigvee_{a \in M} \big((q_0, a) \vee (q_1, a) \big) & \text{otherwise}, \end{cases} \\ \delta_{\mathfrak{M}}(q_1, c, w) &= \begin{cases} \text{true} & \text{if } j \in c, \\ \text{false} & \text{otherwise}. \end{cases}\end{aligned}$$

The corresponding $\mathfrak{L}$-definition is

$$\begin{aligned}\delta(q_0, c) &:= \begin{cases} \exists x Q_0 x & \text{if } i \notin c, \\ \exists x (Q_0 x \vee Q_1 x) & \text{otherwise}, \end{cases} \\ \delta(q_1, c) &:= \begin{cases} \text{true} & \text{if } j \in c, \\ \text{false} & \text{otherwise}. \end{cases}\end{aligned}$$

($\mathrm{cl}(X_i)$) Let $Q := \{q_0, q_1\}$ and $\Omega(q_i) := 1$. We guess some element wa such that its successor waa is in X_i.

$$\begin{aligned}\delta_{\mathfrak{M}}(q_0, c, w) &= \begin{cases} \bigvee_{a \in M} (q_0, a) & \text{if } w = \varepsilon, \\ \bigvee_{a \in M} (q_0, a) \vee (q_1, b) & \text{if } w = w'b, \end{cases} \\ \delta_{\mathfrak{M}}(q_1, c, w) &= \begin{cases} \text{true} & \text{if } i \in c, \\ \text{false} & \text{otherwise}. \end{cases}\end{aligned}$$

The corresponding $\mathfrak{L}$-definition is

$$\delta(q_0, c) := \exists x Q_0 x \vee \exists x(Cx \wedge Q_1 x)\,,$$
$$\delta(q_1, c) := \begin{cases} \text{true} & \text{if } i \in c\,, \\ \text{false} & \text{otherwise}\,. \end{cases}$$

Note that this is the only place where the transition function actually depends on the current vertex.

($\Leftarrow$) Let $\mathcal{A} = (Q, \Sigma, \delta, o, \Omega)$ be an $\mathfrak{L}$-automaton and fix an input structure $\mathfrak{M}^*$. W.l.o.g. assume that $\mathcal{A}$ is nondeterministic. $\mathfrak{M}^*$ is accepted by $\mathcal{A}$ if there is an accepting run $\rho : M^{<\omega} \to Q$ of $\mathcal{A}$ on $\mathfrak{M}^*$. This can be expressed by an $\mathfrak{L}$-formula $\varphi(\bar{X})$ in the following way: we quantify existentially over tuples $\bar{Q}$ encoding ρ (i.e., $Q_i = \rho^{-1}(i)$), and then check that at each position $w \in M^{<\omega}$ a valid transition is used and that each path in ρ is accepting. □

Before proceeding to the proof of Muchnik's theorem let us note an immediate corollary to the equivalence result.

Theorem 1.5.11. *If $\mathfrak{L}_0$, $\mathfrak{L}_1 \in \mathcal{L}$ then $\mathfrak{L}_0 \leq \mathfrak{L}_1$ on $\mathfrak{M}$ implies $\mathfrak{L}_0 \leq \mathfrak{L}_1$ on $\mathfrak{M}^*$.*

Proof. Let $\varphi_0 \in \mathfrak{L}_0$ and $\mathcal{A}_0$ be the corresponding $\mathfrak{L}_0$-automaton. For every formula $\delta_0(q, c) \in \mathfrak{L}_0$ there is an equivalent $\mathfrak{L}_1$-formula. Hence, we can translate $\mathcal{A}_0$ into an $\mathfrak{L}_1$-automaton $\mathcal{A}_1$. The $\mathfrak{L}_1$-formula φ_1 equivalent to $\mathcal{A}_1$ is the desired translation of φ_0 into $\mathfrak{L}_1$. □

1.6 Muchnik's theorem

With these preparations we are finally able to prove the theorem of Muchnik. In fact, we will prove it not only for MSO but for every logic in the class $\mathcal{L}$ defined above. Again we follow the proof of Walukiewicz [78].

Theorem 1.6.1. *Let $\mathfrak{L} \in \mathcal{L}$. For every sentence $\varphi \in \mathfrak{L}$ one can effectively construct a sentence $\varphi^* \in \mathfrak{L}$ such that*

$$\mathfrak{M} \models \varphi^* \quad \text{iff} \quad \mathfrak{M}^* \models \varphi \qquad \textit{for all structures } \mathfrak{M}\,.$$

Corollary 1.6.2. *Let $\mathfrak{M}$ be a structure. The $\mathfrak{L}$-theory of $\mathfrak{M}^*$ is decidable if and only if we can decide* $\mathrm{Th}_{\mathfrak{L}}(\mathfrak{M})$.

The proof of Muchnik's theorem is split into several steps. Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \Omega)$ be the $\mathfrak{L}$-automaton equivalent to φ. W.l.o.g. assume that $Q = [n]$ for an even number n and that $\Omega(k) = k$ for

all $k \in Q$. Note that the input alphabet $\Sigma = \{\varnothing\}$ of $\mathcal{A}$ is unary since φ is a sentence. We construct a formula φ^* stating that player 0 has a winning strategy in the game $\check{\mathcal{G}}(\mathcal{A}, \mathfrak{M}^*)$. It follows that

$$\mathfrak{M} \models \varphi^* \quad \text{iff} \quad \mathfrak{M}^* \in L(\mathcal{A}) \quad \text{iff} \quad \mathfrak{M}^* \models \varphi .$$

1.6.1 The game structure.

We construct φ^* by modifying the formula of Walukiewicz so that it can be evaluated in the structure $\mathfrak{M}$. To do so we embed the game $\check{\mathcal{G}}(\mathcal{A}, \mathfrak{M}^*)$ in $\mathfrak{M}$. First, we reduce the second component of a position (X, w) from a sequence $w \in M^{<\omega}$ to a single symbol $a \in M$. Let $\mathcal{G}'(\mathcal{A}, \mathfrak{M})$ be the game obtained from $\check{\mathcal{G}}(\mathcal{A}, \mathfrak{M}^*)$ by identifying all nodes of the form (q, wa) and $(q, w'a)$, i.e.:

(a) The positions of player 0 are $V_0 \cup \{(q_0, \varepsilon)\}$ where $V_0 := Q \times M$, those of player 1 are $V_1 := \wp(Q \times M)$.

(b) The initial position is (q_0, ε).

(c) Let $\langle\!\langle \delta(q, \varnothing) \rangle\!\rangle_{\mathfrak{M}}(a) = \bigvee_i \bigwedge \Phi_i$ for $a \in M \cup \{\varepsilon\}$. The node $(q, a) \in V_0$ has the successors Φ_i for all i. Nodes $\Phi \in V_1$ have their elements $(q, a) \in \Phi$ as successors.

(d) A play $(q_0, a_0), \Phi_0, (q_1, a_1), \Phi_1, \ldots$ is winning if the sequence $(q_i)_{i<\omega}$ satisfies the parity condition Ω.

Lemma 1.6.3. *Player* 0 *has a winning strategy from the vertex* (q, wa) *in the game* $\check{\mathcal{G}}(\mathcal{A}, \mathfrak{M}^*)$ *if and only if he has one from the vertex* (q, a) *in the game* $\mathcal{G}'(\mathcal{A}, \mathfrak{M})$.

Proof. The unravellings of $\check{\mathcal{G}}(\mathcal{A}, \mathfrak{M}^*)$ and $\mathcal{G}'(\mathcal{A}, \mathfrak{M})$ from the respective vertices are isomorphic. □

In the second step we encode the game $\mathcal{G}'(\mathcal{A}, \mathfrak{M})$ as the structure

$$\mathfrak{G}(\mathcal{A}, \mathfrak{M}) := \big(V_0 \cup V_1,\ E,\ \mathrm{eq}_2,\ V_0,\ V_1,\ (S_q)_{q \in Q}, R_0, \ldots\big)$$

where (V_0, V_1, E) is the game graph and

$$\begin{aligned} &\mathrm{eq}_2(q, a)(q', a') && : \text{iff } a = a' , \\ &S_q(q', a) && : \text{iff } q' = q , \\ &R_i(q_0, a_0) \ldots (q_r, a_r) && : \text{iff } (a_0, \ldots, a_r) \in R_i^{\mathfrak{M}} . \end{aligned}$$

Note that these relations only contain elements of V_0.

Let $\mathfrak{G}(\mathcal{A}, \mathfrak{M})|_{V_0}$ denote the restriction of $\mathfrak{G}(\mathcal{A}, \mathfrak{M})$ to V_0. We can embed $\mathfrak{G}(\mathcal{A}, \mathfrak{M})|_{V_0}$ in $\mathfrak{M}$ via an interpretation. Let MSO_0^+ be the set of quantifier-free, positive MSO-formulae.

Lemma 1.6.4. *There exists an injective* $\mathrm{MSO}^+_{\mathrm{o}}$*-interpretation* $\mathcal{I}$ *such that*

$$\mathcal{I} : \mathfrak{G}(\mathcal{A}, \mathfrak{M})|_{V_{\mathrm{o}}} \leq_{\mathrm{MSO}^+_{\mathrm{o}}} n \times \mathfrak{M} \qquad \text{for every structure } \mathfrak{M}.$$

Proof. The universe and the relation V_{o} can be defined by true, and the relations E and V_1 by false. For the remaining relations we have

$$\varphi_{\mathrm{eq}_2}(x, y) := x =_1 y,$$
$$\varphi_{S_q}(x) := x =_{\mathrm{o}} q,$$
and $$\varphi_{R_i}(\bar{x}) := R^1_i \bar{x}. \qquad \square$$

In order to use this interpretation to calculate the winning positions we have to show that we can translate $\mathfrak{L}$-formulae from $\mathfrak{G}(\mathcal{A}, \mathfrak{M})|_{V_{\mathrm{o}}}$ to $\mathfrak{M}$.

Lemma 1.6.5. *Products by finite structures and injective* $\mathrm{MSO}^+_{\mathrm{o}}$*-interpretations are* $\mathfrak{L}$*-functors for every logic* $\mathfrak{L} \in \mathcal{L}$.

It remains to devise a way to speak about the whole structure $\mathfrak{G}(\mathcal{A}, \mathfrak{M})$ in its restriction to V_{o}. This can be done by encoding elements $\Phi \in V_1 = \wp(V_{\mathrm{o}})$ as sets $\Phi \subseteq V_{\mathrm{o}}$. All we have to do is to define the edge relation. We split E into three parts

$$E_{\mathrm{o}} \subseteq V_{\mathrm{o}} \times V_1, \quad E_1 \subseteq V_1 \times V_{\mathrm{o}}, \quad \text{and} \quad E_2 \subseteq \{(q_{\mathrm{o}}, \varepsilon)\} \times V_1$$

which we define separately by formulae $\varepsilon_{\mathrm{o}}(x, Y)$, $\varepsilon_1(X, y)$, and $\varepsilon_2(Y)$.

Lemma 1.6.6. *There are* $\mathfrak{L}$*-formulae* $\varepsilon_{\mathrm{o}}(x, Y)$, $\varepsilon_1(X, y)$, *and* $\varepsilon_2(Y)$ *defining the edge relations* E_{o}, E_1, *and* E_2, *respectively.*

Proof. Since $\big(\Phi, (q, a)\big) \in E_1$ iff $(q, a) \in \Phi$ we set

$$\varepsilon_1(Y, x) := Yx.$$

The definition of ε_{o} is more involved. Let $\delta_q(C, \bar{Q}) := \langle\!\langle \delta(q, \varnothing) \rangle\!\rangle_{\mathfrak{M}}$. We have

$$((q, a), \Phi) \in E_{\mathrm{o}} \quad \text{iff} \quad \mathfrak{M} \vDash \delta_q(\{a\}, \bar{Q})$$

where $Q_i := \{\, b \mid (i, b) \in \Phi \,\}$. In order to evaluate δ_q we need to define $\mathfrak{M}$ inside $\mathfrak{G}(\mathcal{A}, \mathfrak{M})$. Since the latter consists of $|Q|$ copies of $\mathfrak{M}$ with universes $(S_q)_{q \in Q}$, we pick one such copy and relativise δ_q to it. For simplicity we choose S_q corresponding to the first component of (q, a).

$$((q, a), \Phi) \in E_{\mathrm{o}} \quad \text{iff} \quad \mathfrak{G}(\mathcal{A}, \mathfrak{M})|_{V_{\mathrm{o}}} \vDash \delta_q^{S_q}\big(\{(q, a)\}, \bar{Q}'\big)$$

where $Q'_i := \{ (q, b) \mid (i, b) \in \Phi \}$. This condition can be written as

$$\mathfrak{G}(\mathcal{A}, \mathfrak{M})|_{V_0} \models \exists C \exists \bar{Q} \Big(\delta_q^{S_q}(C, \bar{Q}) \wedge C = \{(q, a)\} \\ \wedge \bigwedge_{i \in Q} Q_i = \{ (q, b) \mid (i, b) \in \Phi \} \Big).$$

Thus, we define

$$\varepsilon_0(x, Y) := \bigvee_{q \in Q} \big(S_q x \wedge \varepsilon_0^q(x, Y) \big)$$

where

$$\varepsilon_0^q(x, Y) := \exists C \exists \bar{Q} \Big(\delta_q^{S_q}(C, \bar{Q}) \wedge C = \{x\} \\ \wedge \bigwedge_{i \in Q} Q_i = \{ (q, b) \mid (i, b) \in Y \} \Big).$$

The condition $Q_i = \{ (q, b) \mid (i, b) \in Y \}$ can be expressed by the formula

$$\forall z \big(Q_i z \leftrightarrow \exists z' (S_q z \wedge S_i z' \wedge z =_1 z') \big).$$

In the same way we define

$$\varepsilon_2(Y) := \exists \bar{Q} \Big(\delta_{q_0}^{S_{q_0}}(\varnothing, \bar{Q}) \wedge \bigwedge_{i \in Q} Q_i = \{ (q_0, b) \mid (i, b) \in Y \} \Big). \qquad \square$$

1.6.2 The winning set.

It remains to evaluate the formula

$$\mathrm{LFP}_{Z_n, x} \cdots \mathrm{GFP}_{Z_1, x} \bigvee_{k \leq n} \eta_k(x, \bar{Z})$$

with $\eta_k := \Omega_k x \wedge [V_0 x \to \exists y (Exy \wedge Z_k y)] \wedge [V_1 x \to \forall y (Exy \to Z_k y)]$

which defines the winning set in the original game graph $\mathcal{G}'(\mathcal{A}, \mathfrak{M})$. Since in the given game the nodes of V_0 and V_1 are strictly alternating, we remain in V_0 if we take two steps each time. Hence, we can replace η_i by

$$\eta'_i := \Omega_i x \wedge V_0 x \wedge \exists y \big(V_1 y \wedge Exy \wedge \forall z (Eyz \to Z_i z) \big).$$

Lemma 1.6.7. *The formulae*

$$\mathrm{GFP}_{Z_1, x} \bigvee_{i \leq n} \eta_i \quad \textit{and} \quad \mathrm{GFP}_{Z_1, x} \bigvee_{i \leq n} \eta'_i$$

define the same subset of V_0 in $\mathfrak{G}(\mathcal{A}, \mathfrak{M})$ for each assignment of the free variables.

Finally, interpreting elements of V_1 by subsets of V_0, as explained above, we obtain

$$\eta_i'' := \Omega_i x \wedge V_0 x \wedge \exists Y\big[Y \subseteq V_0 \wedge \varepsilon_0(x, Y) \\ \wedge \forall z(\varepsilon_1(Y, z) \to Z_i z)\big]$$

Thus, we can express that player 0 has a winning strategy in $\mathcal{G}'(\mathcal{A}, \mathfrak{M})$ from position (q_0, ε) by the formula

$$\varphi^* := \exists Y\big[\varepsilon_2(Y) \wedge \forall x\big(\varepsilon_0(Y, x) \to \mathrm{LFP}_{Z_1,x} \cdots \mathrm{GFP}_{Z_n,x} \bigvee_{i\leq n} \eta_i''\big)\big].$$

This concludes the proof of Theorem 1.3.8.

2 Clique width

Initially, research on simple monadic theories was focused on linear orders and trees culminating in the development of the composition method by Shelah and Gurevich [70, 43, 45, 44]. Later on, these results were used in the study of graph grammars (see e.g. [21, 38, 30, 29]) where graphs can be associated with their derivation trees. Since it is only a small step from graphs to arbitrary relational structures we will try to develop a theory of structures with simple monadic theory by generalising these results. In the present chapter we start by giving a survey of known results about clique width and related notions.

2.1 Clique width and NLC-width

The notion of clique width arose in the study of graph grammars. In the following we will present three different kinds: *hyperedge replacement grammars* and *vertex replacement grammars*, HR- and VR-grammars for short, as considered by Courcelle [24], and NLC-grammars (*node-label-controlled grammars*) studied by Wanke [79].

Let C be a set of colours. Consider the following operations on undirected graphs whose vertices are labelled with colours from C:

- a denotes the trivial graph whose single vertex is coloured a;
- $a - b$ is the graph consisting of a single edge between vertices of colour a and b, respectively;
- $\mathfrak{G}_0 + \mathfrak{G}_1$ is the disjoint union of $\mathfrak{G}_0$ and $\mathfrak{G}_1$;
- The recolouring $\rho_\beta(\mathfrak{G})$ with $\beta : C \to C$ changes each colour a to $\beta(a)$; ρ_β
- $\alpha_{a,b}(\mathfrak{G})$ adds edges from all a-coloured vertices to every vertex of colour b; $\alpha_{a,b}$
- $\vartheta_a(\mathfrak{G})$ is the graph obtained from $\mathfrak{G}$ by identifying all vertices that have the colour a; ϑ_a
- $\mathfrak{G}_0 \oplus_S \mathfrak{G}_1$ with $S \subseteq C \times C$ denotes the disjoint union of $\mathfrak{G}_0$ and $\mathfrak{G}_1$ where a-coloured vertices of $\mathfrak{G}_0$ are connected by an edge to b-coloured vertices of $\mathfrak{G}_1$ iff $(a, b) \in S$. $\oplus_S$

HR-term
VR-term
NLC-term

A term $T \subseteq 2^{<\omega}$ is called an *HR-term*, if it is build up from the operations $a - b$, ρ_β, ϑ_a, and $\alpha_{a,b}$. T is a *VR-term*, if it consists of the operations a, $+$, ρ_β, and $\alpha_{a,b}$. If T is constructed from a, ρ_β, and $\oplus_S$, then it is called an *NLC-term*.

The above operations can easily be generalised to arbitrary transition systems by adding to $\alpha_{a,b}$ and $\oplus_S$ information about the direction and label of the new edges. For simplicity, we have refrained from doing so, but we will, nevertheless, state some of the results below for this more general case. In Chapter 3 we will present a generalisation to arbitrary relational structures.

Example. Cliques K_n have clique width 2. We can define VR-terms t_n with value $\mathrm{val}(t_n) = K_n$ using the colours a and b by setting

$$t_1 := a\,, \quad \text{and} \quad t_{n+1} := \rho_{b\to a}\alpha_{b,a}(b + t_n)\,.$$

Their NLC-width is 1. The corresponding terms are

$$t_1 := a\,, \quad \text{and} \quad t_{n+1} := a \oplus_{\{(a,a)\}} t_n\,.$$

The operators for HR-terms were chosen in such a way that they correspond to a well-known complexity measure. Courcelle [23] has shown that the number of colours one needs to construct a HR-term for a given graph roughly corresponds to its tree width.

Theorem 2.1.1. *A countable graph has finite tree width if and only if it is the value of an HR-term.*

In a similar way the other two kinds of terms give rise to complexity measures of graphs which were first defined, respectively, by Courcelle, Engelfriet, and Rozenberg [30], and by Wanke [79].

clique width
$\mathrm{cwd}\,\mathfrak{G}$
NLC-width

Definition 2.1.2. The *clique width* of a graph $\mathfrak{G}$ is the minimal size of a set C of colours such that there is a VR-term denoting $\mathfrak{G}$ which uses only colours from C. The *NLC-width* is defined analogously using NLC-terms.

The following observation by Johansson [48] shows that these two measures are nearly the same.

Lemma 2.1.3. *Let k be the clique width of a graph $\mathfrak{G}$ and m its NLC-width. Then $m \leq k \leq 2m$.*

The characterisation we aim to generalise is the following result of Courcelle [24] relating clique width with interpretations in the binary tree.

Theorem 2.1.4. *A countable graph $\mathfrak{G} = (V, E)$ has finite clique width if and only if $\mathfrak{G} \leq_{\mathrm{MSO}} (2^{<\omega}, \preceq, P)$ for some unary predicate $P \subseteq 2^{<\omega}$.*

As a corollary we will obtain in Section 2.4 a similar characterisation for finite tree width.

2.2 Prefix-recognisable graphs

When studying algorithmic properties of graphs we have the requirement that every graph has to be encoded by a finite object. We can use the fact that graphs of finite clique width are denoted by VR-terms in order to find such finite representations. Obvious candidates for terms with a finite encoding are the so called *regular terms* which can be obtained as unravellings of finite graphs. Equivalently, we could define them as those terms that are solutions of finite systems of equations of the form

regular term

$$x_0 = t_0(\bar{x}), \quad \ldots, \quad x_{n-1} = t_{n-1}(\bar{x})$$

where the t_i are VR-terms with free variables $\bar{x}$ and we require that none of them is of the form $t_i = x_k$ for some variable x_k.

Definition 2.2.1. A graph is *VR-equational* if it is denoted by a regular VR-term. Similarly, *HR-equational* graphs are the value of regular HR-terms.

VR-equational
HR-equational

Example. If we colour the first element by a and the other ones by b we can define $(\omega, \mathrm{suc}, <)$ by

$$x_0 = \alpha^{<}_{a,b}(x_1), \qquad x_1 = \rho_{c\to b}\alpha^{\mathrm{suc}}_{a,c}(a + x_2), \qquad x_2 = \rho_{a\to c}(x_0)\,.$$

We conclude this section with a presentation of several equivalent definitions of the class of VR-equational graphs.

Definition 2.2.2. A directed countable graph is *prefix-recognisable* if it is isomorphic to a graph $(S, (E_\lambda)_\lambda)$ where $S \subseteq \Sigma^{<\omega}$ is a regular language over a finite alphabet Σ and each edge relation E_λ is a finite union of relations of the form

prefix-recognisable

$$W(U \times V) := \{\, (wu, wv) \mid u \in U,\ v \in V,\ w \in W \,\}$$

for regular languages U, V, and W.

Actually in the common definition the reverse order $(U \times V)W$ is used. The above definition was chosen as it fits better to the usual conventions regarding trees.

Example. The structure $(\omega, \mathrm{suc}, \leq)$ is prefix-recognisable. If we represent the universe by $a^{<\omega}$ the relations take the form

$$\mathrm{suc} = a^{<\omega}(\varepsilon \times a) \qquad \text{and} \qquad \leq\ = a^{<\omega}(\varepsilon \times a^{<\omega})\,.$$

Proposition 2.2.3 (Barthelmann [3])**.** *Let $\mathfrak{G}$ be a graph. The following statements are equivalent:*

(1) $\mathfrak{G}$ *is prefix-recognisable.*

(2) $\mathfrak{G}$ *is VR-equational.*

(3) $\mathfrak{G} \leq_{\mathrm{MSO}} \mathfrak{T}_2$.

Originally, Caucal defined prefix-recognisable graphs in a different way. In order to obtain a class of graphs with decidable MSO-theory he introduced two operations on graphs that preserve MSO-decidability and applied them to the binary tree $\mathfrak{T}_2$.

Definition 2.2.4. Let $\mathfrak{G} = (V, (E_\lambda)_{\lambda \in \Lambda})$ be a graph with universe $V \subseteq 2^{<\omega}$.

rational restriction

(1) A *rational restriction* of $\mathfrak{G}$ is a structure of the form

$$\mathfrak{G}|_C := (V \cap C, (E_\lambda \cap C \times C)_\lambda)$$

for some regular language $C \subseteq 2^{<\omega}$.

inverse rational substitution

(2) Let $\bar{\Lambda}$ be a disjoint copy of Λ and expand $\mathfrak{G}$ by the relations $E_{\bar{\lambda}} := (E_\lambda)^{-1}$ for $\bar{\lambda} \in \bar{\Lambda}$. Given a set of labels Ξ and a mapping h associating to every $\xi \in \Xi$ a regular language $h(\xi) \subseteq (\Lambda \cup \bar{\Lambda})^{<\omega}$, the *inverse rational substitution* $h^{-1}(\mathfrak{G})$ is the graph $(V, (E'_\xi)_{\xi \in \Xi})$ where E'_ξ consists of those pairs (u, v) such that in the expansion of $\mathfrak{G}$ there is a path from u to v labelled by some word in $h(\xi)$.

Example. The structure $(\omega, \mathrm{suc}, \leq)$ can be written as $h^{-1}(\mathfrak{T}_2)|_C$ with $C := 1^{<\omega}$ and $h(\mathrm{suc}) := 1$, $h(\leq) := 1^{<\omega}$.

Proposition 2.2.5 (Caucal [15])**.** *A graph $\mathfrak{G}$ is prefix-recognisable if and only if it is isomorphic to $h^{-1}(\mathfrak{T}_2)|_C$ for some regular language C and a mapping h such that $h(\xi) \subseteq \{0, 1, \bar{0}, \bar{1}\}^{<\omega}$ is regular for all ξ.*

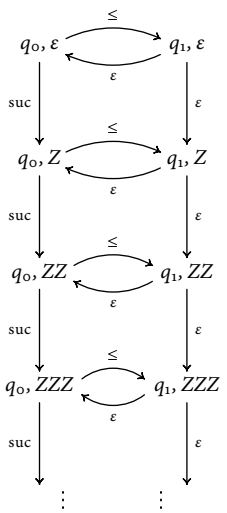
In a similar way to the characterisation of context-free graphs as configuration graphs of pushdown automata one can describe the class of prefix-recognisable graphs via a suitable model of automaton. To do so one considers pushdown automata with ε-transitions where each configuration has either no outgoing ε-transitions or no outgoing non-ε-transitions. Then the ε-transitions are "factored out" in the following way: one takes only those vertices without outgoing ε-transitions and adds an a-transition between two vertices iff in $\mathfrak{G}$ there is a path between them consisting of one a-transition followed by arbitrarily many ε-transitions.

ε-pushdown graph

Definition 2.2.6. An *ε-pushdown graph* is a graph $\mathfrak{G}$ obtained from the configuration graph of a pushdown automaton with ε-transitions by

- factoring out the ε-transitions in the way described above and
- restricting the resulting graph to a regular subset of the vertices.

Example. A pushdown automaton for $(\omega, \mathrm{suc}, \leq)$ has the configuration on the right.



Proposition 2.2.7 (Stirling [73])**.** *A graph $\mathfrak{G}$ is prefix-recognisable if and only if it is isomorphic to an ε-pushdown graph.*

The various equivalent definitions of the class of prefix-recognisable graphs are summarised in the next theorem.

Theorem 2.2.8. *Let $\mathfrak{G}$ be a countable graph. The following statements are equivalent:*

(1) *$\mathfrak{G}$ is prefix-recognisable.*

(2) *$\mathfrak{G} \leq_{\mathrm{MSO}} \mathfrak{T}_2$.*

(3) *$\mathfrak{G}$ is VR-equational.*

(4) *$\mathfrak{G} \cong h^{-1}(\mathfrak{T}_2)|_C$ for some rational substitution h and a regular language C.*

(5) *$\mathfrak{G}$ is isomorphic to an ε-pushdown graph.*

2.3 Finite subgraphs of prefix-recognisable graphs

Instead of a single countable graph we can also use VR-terms to define a class of finite graphs. The graph operations mentioned above can be applied to sets of graphs in a canonical way. If we add the union of classes as new operation, we can define classes of finite graphs by systems of equations

$$\begin{aligned} x_0 &= t_0^0(\bar{x}) \cup \cdots \cup t_{n_0}^0(\bar{x}), \\ &\vdots \\ x_k &= t_0^k(\bar{x}) \cup \cdots \cup t_{n_k}^k(\bar{x}). \end{aligned}$$

One can show [19, 3, 20] that each such system has a least solution, called the *canonical solution*, provided there is no equation of the form $x_i = x_k$. One way to obtain this solution consists in taking the value of the infinite term which is the least solution of the system in the corresponding term algebra.

canonical solution

Definition 2.3.1. A class of finite graphs is called *VR-equational* if it is the canonical solution of a system of VR-equations.

VR-equational

Analogously to the corresponding theorem above one can characterise VR-equational classes by interpretations.

Theorem 2.3.2 (Engelfriet [38]). *A class $\mathcal{K}$ of finite graphs is VR-equational if and only if there exists a regular class of finite binary trees $\mathcal{T}$ such that $\mathcal{K} \leq_{\mathrm{MSO}} \mathcal{T}$.*

A countable graph is of finite clique width iff the clique width of its finite induced subgraphs is bounded. Below we will show that there is no such characterisation for prefix-recognisable graphs. First we show that one direction still holds. The class of finite induced subgraphs of a prefix-recognisable graph can be interpreted in the class of all finite binary trees. For the proof, we need some technical lemmas.

Lemma 2.3.3. *For all* MSO*-formulae $\varphi(\bar{x})$ there exists some formula $\hat{\varphi}(\bar{x}) \in \mathrm{MSO}$ which is equivalent modulo $\mathrm{Th}_{\mathrm{MSO}}(\mathfrak{T}_2)$ to φ such that for all subtrees $\mathfrak{S} \subseteq \mathfrak{T}_2$ and all elements $\bar{a} \subseteq S$ it holds that*

$$\mathfrak{S} \models \hat{\varphi}(\bar{a}) \quad \text{iff} \quad \mathfrak{T}_2 \models \hat{\varphi}(\bar{a}) .$$

Proof. Given φ, consider the corresponding tree automaton $\mathcal{A}$. We construct an automaton $\mathcal{A}'$ which takes labelled subtrees of $\mathfrak{T}_2$ as input and simulates the work of $\mathcal{A}$ on those. Whenever a node with some missing successors is encountered $\mathcal{A}'$ makes sure that from the state which would be assigned to these missing vertices the tree $T_\varnothing$ is accepted. Finally, let $\hat{\varphi}(\bar{x})$ be the formula associated with $\mathcal{A}'$. It follows that

$$\mathfrak{S} \models \hat{\varphi}(\bar{a}) \;\text{ iff }\; T_{\bar{a}}|_S \in L(\mathcal{A}') \;\text{ iff }\; T_{\bar{a}} \in L(\mathcal{A}) \;\text{ iff }\; \mathfrak{T}_2 \models \varphi(\bar{a}) .$$

□

Lemma 2.3.4. *Let $\mathfrak{G} \leq_{\mathrm{MSO}} \mathfrak{T}_2$ be a prefix-recognisable graph. The class of finite induced subgraphs of $\mathfrak{G}$ can be obtained from the class of all finite labelled binary trees via an* MSO*-interpretation.*

Proof. Let $\mathcal{I} = \langle \delta(x),\ \varepsilon(x,y),\ (\varphi_R(\bar{x}))_R \rangle$ be the interpretation of $\mathfrak{G}$ in $\mathfrak{T}_2$. By the preceding lemma, we can assume that

$$\psi^{\mathfrak{S}} = \psi^{\mathfrak{T}_2}|_S$$

for all finite subtrees $\mathfrak{S} \subseteq \mathfrak{T}_2$ where ψ is one of δ, ε, or φ_R. Let $\mathfrak{H} \subseteq \mathfrak{G}$ be a finite induced subgraph of $\mathfrak{G}$. Define the set $P := \mathcal{I}(H) \subseteq 2^{<\omega}$ and let $\mathfrak{S} \subseteq \mathfrak{T}_2$ be a subtree of $\mathfrak{T}_2$ whose universe contains P. Then

$$\mathcal{I}' = \langle \delta^P(x),\ \varepsilon^P(x,y),\ (\varphi_R^P(\bar{x}))_R \rangle$$

is an interpretation of $\mathfrak{H}$ in $(\mathfrak{S}, P)$. Conversely, each subtree of the form $(\mathfrak{S}, P)$ interprets a substructure of $\mathfrak{G}$. □

Lemma 2.3.5. *The class of finite labelled binary trees can be obtained from some regular class of finite unlabelled binary trees via an* MSO*-interpretation.*

Proof. Let $h : 2^{<\omega} \to 2^{<\omega}$ be the homomorphism defined by $h(0) = 00$ and $h(1) = 11$. Let $(\mathfrak{S}, P)$ be a labelled tree. We encode each node x of $\mathfrak{S}$ by $h(x)$. The unary predicate P is encoded by appending 01 to those vertices that are in P. Thus, the universe of the corresponding unlabelled tree is the prefix closure of $h(S)$ together with $h(P)01$. Clearly, the class of all such trees is regular, and the desired interpretation is given by

$$\begin{aligned}
\delta(x) &:= \text{“}x \in (00 + 11)^{<\omega}\text{”}, \\
\varepsilon(x, y) &:= x = y, \\
\varphi_{\mathrm{suc}_c}(x, y) &:= y = xcc, \qquad \text{for } c \in [2], \\
\varphi_P(x) &:= \exists y(y = x01). \qquad \square
\end{aligned}$$

Combining the preceding lemmas we have obtained the following characterisation of the class of subgraphs of a prefix-recognisable graph.

Proposition 2.3.6. *For each prefix-recognisable graph, the class of its finite induced subgraphs is generated by a* VR*-grammar.*

Proof. By the two preceding lemmas, the class can be obtained from a regular class of finite trees by an MSO-interpretation. According to Theorem 2.3.2, this is equivalent to being generated by a VR-grammar. $\square$

The converse fails. There are classes generated by VR-grammars that cannot be obtained as the class of subgraphs of a prefix-recognisable graph.

Lemma 2.3.7. *There exists a class $\mathcal{K}$ that is generated by a* VR*-grammar and that is the class of finite induced subgraphs of some infinite graph but which cannot be obtained from any prefix-recognisable graph in this way.*

Proof. Let $\mathcal{K}$ be the set of all finite forests $(T, \leq, P)$ where $P \subseteq T$ is an unary predicate that contains only leaves. $\mathcal{K}$ is defined by the grammar

$$x = 0 \cup 1 \cup x + x \cup r(x)$$

where

- 0 denotes a leaf which is not in P,
- 1 denotes a leaf which is in P,

- $x + y$ denotes the disjoint union of x and y, and
- $r(x)$ denotes the tree obtained from the forest x by adding a new least element.

All of these operations can be expressed by VR-terms.

To show that $\mathcal{K}$ is the class of finite induced subgraphs of some graph we have to verify the hereditary property and the joint embedding property (see e.g. [47]). $\mathcal{K}$ is closed under substructures and, if $s, t \in \mathcal{K}$, then s and t can be embedded in $r(s + t) \in \mathcal{K}$.

On the other hand, there is no prefix-recognisable graph such that $\mathcal{K}$ is the class of its finite induced subgraphs since every prefix-recognisable forest has, up to isomorphism, only finitely many different connected components each of which is a regular tree. □

2.4 Clique width and tree width

We conclude this chapter with a survey of the relation between tree width and clique width. The following results extend a theorem of Courcelle [25] which lists conditions implying that a graph of bounded clique width also has a bounded tree width. Most of them have independently been obtained by Gurski and Wanke [46] using nearly the same proofs.

For the proofs, we need a characterisation of clique width by decompositions of a structure in a way similar to tree decompositions for tree width. If we ignore the colours for a moment, a VR-term consist purely of disjoint unions. That is, when traversing a term T from the root to its leaves we observe a progression of decompositions of the structure denoted by T. The hierarchical decomposition thus obtained can then be augmented by information about the colouring.

Definition 2.4.1. Let $\mathfrak{M} = (M, (E_\lambda)_\lambda, \bar{P})$ be a countable transition system.

clique refinement (a) A *clique refinement* of $\mathfrak{M}$ of width n is a family $(\mathcal{R}^v)_{v \in T}$ indexed by a binary tree $T \subseteq 2^{<\omega}$ where each component $\mathcal{R}^v = (U^v, \bar{C}^v)$ consists of a nonempty set $U^v \subseteq M$ and a partition $C^v_0 \dot\cup \cdots \dot\cup C^v_{n-1} = U^v$ such that the following conditions are satisfied:

(1) $U^\varepsilon = M$ and $|U^v| = 1$ for leaves $v \in T$.

(2) $U^v = U^{v0} \dot\cup U^{v1}$ if $v \in T$ is not a leaf.

(3) For each C^v_a and every $u \preceq v$, there is some b such that $C^v_a \subseteq C^u_b$.

(4) Let $x \in C^v_a$ and $y \in C^v_b$ with $x \in U^{vc}$ and $y \in U^{v(1-c)}$ for some $c < 2$. If $(x, y) \in E_\lambda$ then $C^v_a \times C^v_b \subseteq E_\lambda$.

regular (b) We call $\mathcal{R} = (\mathcal{R}^v)_{v \in T}$ *regular* if, up to isomorphism, there are

only finitely many different $\mathcal{R}^v$. That is, there exists a congruence $\approx$ on T of finite index such that $u \approx v$ implies

- $ux \in T$ iff $vx \in T$ for all words $x \in 2^{<\omega}$,
- $C^u_a \neq \varnothing$ iff $C^v_a \neq \varnothing$ for all colours a,
- $C^u_a \supseteq C^{ux}_b$ iff $C^v_a \supseteq C^{vx}_b$ for all a, b, and x,
- $C^u_a \times C^u_b \subseteq E_\lambda$ iff $C^v_a \times C^v_b \subseteq E_\lambda$ for all a, b, and λ.

Proposition 2.4.2. *Let $\mathfrak{M} = (M, (E_\lambda)_\lambda, \bar{P})$ be a countable transition system.*

(a) *$\mathfrak{M}$ has a clique width of at most n if and only if there is a clique refinement of $\mathfrak{M}$ of width n.*

(b) *$\mathfrak{M}$ is prefix-recognisable if and only if there exists a regular clique refinement of $\mathfrak{M}$ of finite width.*

Proof. ($\Leftarrow$) Let $(U^v, \bar{C}^v)_{v \in T}$ be a clique refinement of $\mathfrak{M}$. We construct VR-terms $t_v(x_0, x_1)$ with n colours such that, if we define infinite terms T_w, $w \in 2^{<\omega}$, by $T_w = t_w(T_{w0}, T_{w1})$, then the term T_ε denotes $\mathfrak{M}$.

In order to define t_v, fix a mapping $\beta_c : [n] \to [n]$, for $c < 2$, such that $C^{vc}_i \subseteq C^v_{\beta_c(i)}$. Let

$$t_v(x_0, x_1) := \mathrm{add}\big(\rho_{\beta_0}(x_0) + \rho_{\beta_1}(x_1)\big)$$

where add is the composition of all $\alpha^\lambda_{i,k}$ such that $C^v_i \times C^v_k \subseteq E_\lambda$.

Finally, note that, if the refinement is regular, then the resulting term is also regular which implies that $\mathfrak{M} \leq_{\mathrm{MSO}} \mathfrak{T}_2$.

($\Rightarrow$) Let T be a (possibly infinite) term denoting $\mathfrak{M}$. We decompose T into finite terms t_w, $w \in 2^{<\omega}$, of the form

$$t_w(x_0, x_1) = \tau_0 \cdots \tau_r(x_0 + x_1) \qquad \text{or} \qquad t_w = \tau_0 \cdots \tau_r a$$

where either $\tau_i = \rho_\beta$ or $\tau_i = \alpha^\lambda_{a,b}$ for some a, b, β, and λ. As above, define T_w by $T_w = t_w(T_{w0}, T_{w1})$. We construct a clique refinement $(U^v, \bar{C}^v)_v$ as follows. Let U^w be the set of vertices of the graph denoted by T_w. We set

$$C^v_a := \{\, x \in U^v \mid x \text{ is coloured } a \,\}.$$

Again, if T is a regular term, then the clique refinement we have obtained is regular. □

Courcelle and Olariu [32] have shown that every graph of bounded tree width also has bounded clique width. The slightly better bounds below were obtained by Corneil and Rotics [18]. They also gave the example with the exponential lower bound.

Theorem 2.4.3. *Let $\mathfrak{G} = (V, E)$ be a finite undirected graph.*

$$\operatorname{cwd}(\mathfrak{G}) \leq 3 \cdot 2^{\operatorname{twd}(\mathfrak{G})-1}.$$

Theorem 2.4.4. *For every $k < \omega$ there exists a finite graph $\mathfrak{G}$ with*

$$\operatorname{twd}(\mathfrak{G}) = k \quad \textit{and} \quad \operatorname{cwd}(\mathfrak{G}) \geq 2^{\lfloor k/2 \rfloor - 1}.$$

Of course, graphs of bounded clique width do not need to have bounded tree width. For instance, the complete graph K_κ has clique width $\operatorname{cwd}(K_\kappa) = 2$ while its tree width is $\operatorname{twd}(K_\kappa) + 1 = \kappa$, for every cardinal κ. Below we will try to bound the tree width in terms of the clique width and one additional parameter, say, the maximal degree. In the following we denote by $K_{\kappa,\lambda}$ the *directed* graph $(A \mathbin{\dot\cup} B,\ A \times B)$ where A and B are sets of cardinality, respectively, κ and λ.

$K_{\kappa,\lambda}$

Definition 2.4.5. Let $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \bar{P})$ be a transition system. The *maximal degree* $\Delta(\mathfrak{M})$ of $\mathfrak{M}$ is the supremum of the cardinals κ such that some edge relation E_λ contains a subgraph of the form $K_{1,\kappa}$ or $K_{\kappa,1}$

maximal degree $\Delta(\mathfrak{M})$

Theorem 2.4.6. *Let $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \bar{P})$ be a countable transition system.*

$$\operatorname{twd}(\mathfrak{M}) + 1 \leq \Delta(\mathfrak{M}) \operatorname{cwd}(\mathfrak{M}).$$

Proof. Let $(U^\nu, \bar{C}^\nu)_{\nu\in T}$ be a clique refinement of $\mathfrak{M}$ of minimal width. $C^\nu_a \times C^\nu_b \subseteq E_\lambda$ implies $|C^\nu_a|, |C^\nu_b| \leq \Delta(\mathfrak{M})$. Let $x \in C^\nu_a$ be adjacent to some $y \notin U^\nu$. By definition, there is some prefix $\nu_\circ \preceq \nu$ such that $C^\nu_a \subseteq C^{\nu_\circ}_b$, $y \in C^{\nu_\circ}_c$, and $C^{\nu_\circ}_b \times C^{\nu_\circ}_c \subseteq E_\lambda$ or $C^{\nu_\circ}_c \times C^{\nu_\circ}_b \subseteq E_\lambda$ for some colours b, c. Hence, $|C^\nu_a| \leq |C^{\nu_\circ}_b| \leq \Delta(\mathfrak{M})$. Thus, the number of elements $x \in U^\nu$ which are adjacent to some vertex outside of U^ν is bounded by $\Delta(\mathfrak{M}) \operatorname{cwd}(\mathfrak{M})$.

We obtain a tree decomposition $(F_\nu)_{\nu\in T}$ of width less than

$$\Delta(\mathfrak{M}) \operatorname{cwd}(\mathfrak{M})$$

by setting

$$F_\nu := \bigcup \big\{\, C^\nu_a \;\big|\; |C^\nu_a| \leq \Delta(\mathfrak{M}) \,\big\}.$$

$(F_\nu)_{\nu\in T}$ is a tree decomposition since, for any edge $(x, y) \in E_\lambda$, there are some $\nu \in T$ and colours a, b such that $x \in C^\nu_a$, $y \in C^\nu_b$, and $C^\nu_a \times C^\nu_b \subseteq E_\lambda$. By assumption, $|C^\nu_a|, |C^\nu_b| \leq \Delta(\mathfrak{M})$ which implies that $C^\nu_a, C^\nu_b \subseteq F_\nu$. □

Lemma 2.4.7. *For each $\kappa \leq \aleph_0$ there is a graph $\mathfrak{G}_\kappa$ with*

$$\operatorname{twd}(\mathfrak{G}_\kappa) = \kappa, \quad \Delta(\mathfrak{G}_\kappa) = 1, \quad \textit{and} \quad \operatorname{cwd}(\mathfrak{G}_\kappa) = \kappa + 1.$$

Proof. Let $\mathfrak{G}_\kappa := (V, E_\mathrm{h}, E_\mathrm{v})$ be the grid of size $\kappa \times \kappa$ where

$$\begin{aligned} V &:= \kappa \times \kappa\,, \\ E_\mathrm{h} &:= \{\, (\langle i,k\rangle, \langle i+1,k\rangle) \mid i+1, k < \kappa \,\}\,, \\ E_\mathrm{v} &:= \{\, (\langle i,k\rangle, \langle i,k+1\rangle) \mid i, k+1 < \kappa \,\}\,. \end{aligned}$$

Its tree width is κ and Golumbic and Rotics [40] proved that $\operatorname{cwd} \mathfrak{G}_\kappa = \kappa + 1$. □

Remark. If there are only n vertices of degree more than δ then one can construct a tree decomposition of width

$$\operatorname{twd}(\mathfrak{G}) + 1 \leq \delta \operatorname{cwd}(\mathfrak{G}) + n$$

in the same way as above by adding those n vertices to every component of the decomposition.

The second parameter we consider is the maximal cardinal κ such that $\mathfrak{M}$ contains $K_{\kappa,\kappa}$.

Definition 2.4.8. Let $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \bar{P})$ be a transition system. $\beta(\mathfrak{M})$

$$\beta(\mathfrak{M}) := \sup\{\, \kappa \mid K_{\kappa,\kappa} \subseteq E_\lambda \text{ for some } \lambda \,\}\,.$$

Theorem 2.4.9. *Let $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \bar{P})$ be a countable transition system.*

(a) $\operatorname{twd}(\mathfrak{M}) + 1 \leq 2\beta(\mathfrak{M}) \operatorname{cwd}(\mathfrak{M})$.

(b) *If $\mathfrak{M}$ is prefix-recognisable and $\beta(\mathfrak{M}) < \aleph_0$ then $\mathfrak{M}$ is HR-equational.*

Proof. (a) To simplify notation we assume that, for each $\lambda \in \Lambda$, there is some $\lambda^- \in \Lambda$ with $E_{\lambda^-} = E_\lambda^{-1}$. Fix a clique refinement $(U^\nu, \bar{C}^\nu)_{\nu\in T}$ of $\mathfrak{M}$ of width $k := \operatorname{cwd}(\mathfrak{M})$ and let $n := \beta(\mathfrak{M})$. By assumption, if $C_a^\nu \times C_b^\nu \subseteq E_\lambda$ then $|C_a^\nu| \leq n$ or $|C_b^\nu| \leq n$. For $\nu \in T$ define

$$\begin{aligned} I^\nu &:= \bigcup \Big\{\, C_a^\nu \;\Big|\; |C_a^\nu| \leq n \text{ and } C_a^\nu \times \{x\} \subseteq E_\lambda \text{ for some } x \in M \,\Big\}\,, \\ O_a^\nu &:= \Big\{\, x \in M \setminus U^\nu \;\Big|\; \{x\} \times C_b^{\nu_\mathrm{o}} \subseteq E_\lambda \text{ for some } C_b^{\nu_\mathrm{o}} \supseteq C_a^\nu \text{ with } |C_b^{\nu_\mathrm{o}}| > n \,\Big\}\,, \\ O^\nu &:= \bigcup_{a<k} O_a^\nu\,. \end{aligned}$$

We have $|I^\nu| \leq nk$. Since $O_a^\nu \times C_b^{\nu_\mathrm{o}} \subseteq E_\lambda$ where $\nu_\mathrm{o} \preceq \nu$ is the greatest prefix of ν such that $|C_b^{\nu_\mathrm{o}}| > n$ for $C_b^{\nu_\mathrm{o}} \supseteq C_a^\nu$, it follows that $|O_a^\nu| \leq n$ and, hence, also $|O^\nu| \leq nk$. We claim that the family

$$F_\nu := I^\nu \cup O^\nu\,,\ \nu \in T\,,$$

is a tree decomposition of $\mathfrak{M}$. For any edge $(x, y) \in E_\lambda$, there are some colours a, b and a node $v \in T$ such that

$$(x, y) \in C_a^v \times C_b^v \subseteq E\,.$$

If $|C_a^v|, |C_b^v| \leq n$ then $x, y \in I^v \subseteq F^v$. Otherwise, w.l.o.g. assume that $|C_a^v| \leq n$ and $|C_b^v| > n$. There is some $w > v$ with $U^w = \{y\}$. By construction, $x \in O^w \subseteq F_w$ and $y \in I^w \subseteq F_w$.

It remains to prove that all components F_v which contain some given vertex x are connected. We consider the following cases.

If $x \in I^{v_0}$ and $x \in I^{v_1}$ then $v_0 \preceq v_1$ and $x \in I^w$ for all $v_0 \preceq w \preceq v_1$, or vice versa.

If $x \in O^v$ there exists some $v_0 \preceq v$ with $x \in C_a^{v_0}$ and $C_a^{v_0} \times C_b^{v_0} \subseteq E_\lambda$ for some colour b with $|C_b^{v_0}| > n$ and $C_b^{v_0} \cap U^v \neq \varnothing$. Let $w \in T$ be the node with $U^w = \{x\}$. It follows that $x \in O^u$ for all $w \sqcap v \prec u \preceq v$ and $x \in I^u$ for all $w \preceq u \preceq v_0$.

Suppose that $x \in O^{v'}$ for another node $v' \in T$. By the same argument as above we obtain a vertex v'_0 such that $x \in O^u$ for all $w \sqcap v' \prec u \preceq v'$ and $x \in I^u$ for all $w \preceq u \preceq v'_0$. Since $v_0 \preceq v'_0$ or $v'_0 \preceq v_0$ the claim follows.

(b) Let $(U^v, C^v)_v$ be a regular clique refinement of $\mathfrak{M}$. In the same way as in (a) we obtain a tree decomposition $(F_v)_v$ that, furthermore, is also regular. Hence, $\mathfrak{M}$ is HR-equational. □

Corollary 2.4.10. *Let $\mathfrak{G}$ be a countable planar undirected graph.*

$$\operatorname{twd}(\mathfrak{G}) + 1 \leq 4 \operatorname{cwd}(\mathfrak{G})\,.$$

Proof. Since $\mathfrak{G}$ is planar it does not contain $K_{3,3}$. □

Since the tree width of the $n \times n$ grid is n it follows that its clique width is greater than $n/4$. In fact, Golumbic and Rotics [40] have shown that the precise value is $n + 1$.

The next example shows that the above bounds are tight up to a factor of 2.

Example. For each $n \in \mathbb{N}$, let $\mathfrak{G}_n = (V_n, E_n)$ be the graph with

$$V_n := [3] \times [n]\,,$$
$$\text{and} \quad E_n := \big\{\, (\langle i, k\rangle, \langle i', k'\rangle) \;\big|\; i \neq i' \,\big\}\,.$$
$$\text{Then} \quad \operatorname{twd}(\mathfrak{G}_n) = 2n\,, \qquad \Delta(\mathfrak{G}_n) = 2n\,,$$
$$\text{and} \quad \operatorname{cwd}(\mathfrak{G}_n) = 2\,, \qquad \beta(\mathfrak{G}_n) = n\,.$$

Another consequence of Theorem 2.4.9 is a charaterisation of tree width via interpretations.

Theorem 2.4.11. *A countable graph $\mathfrak{G} = (V, E)$ has finite tree width if and only if $\mathfrak{G}^{\mathcal{I}} \leq_{\mathrm{MSO}} (2^{<\omega}, \preceq, P)$ for some unary predicate $P \subseteq 2^{<\omega}$.*

Proof. In one direction, we can construct an MSO-interpretation of the incidence structure $\mathfrak{G}^{\mathcal{I}}$ in the HR-term denoting $\mathfrak{G}$. For the other one, note that $\mathfrak{G}^{\mathcal{I}}$ does not contain the subgraph $K_{3,3}$. It follows that

$$\operatorname{twd}(\mathfrak{G}) \leq \operatorname{twd}(\mathfrak{G}^{\mathcal{I}}) \leq 4 \operatorname{cwd}(\mathfrak{G}^{\mathcal{I}}) < \aleph_0 . \qquad \square$$

Finally, we consider uniformly sparse graphs (see Definition 1.2.10). The following result of Courcelle [28] characterises sparse graphs.

Lemma 2.4.12. *A transition system $\mathfrak{M} = (M, (E_\lambda)_\lambda, \bar{P})$ is uniformly k-sparse if and only if there exist functions $f_\lambda : E_\lambda \to M$, $\lambda \in \Lambda$, such that $f_\lambda(x, y) \in \{x, y\}$ and $\left|f_\lambda^{-1}(x)\right| \leq k$ for all $x, y \in M$.*

Theorem 2.4.13. *Every countable transition system $\mathfrak{M}$ is k-sparse for $k := \big(\beta(\mathfrak{M}) \operatorname{cwd}(\mathfrak{M})\big)$.*

Proof. We have to construct functions $f_\lambda : E_\lambda \to M$ with $f_\lambda(x, y) \in \{x, y\}$ such that $|f_\lambda^{-1}(x)| \leq \beta(\mathfrak{M}) \operatorname{cwd}(\mathfrak{M})$ for all $x \in M$.

Let $(U^\nu, \bar{C}^\nu)_{\nu \in T}$ be a clique refinement of $\mathfrak{M}$ of width $k := \operatorname{cwd}(\mathfrak{M})$ and set $n := \beta(\mathfrak{M})$. By assumption, if $C_a^\nu \times C_b^\nu \subseteq E_\lambda$ then $|C_a^\nu| \leq n$ or $|C_b^\nu| \leq n$.

If $e := (x, y) \in C_a^\nu \times C_b^\nu \subseteq E_\lambda$ with $|C_a^\nu| > n$ then we set $f_\lambda(e) := x$. For each vertex x there can be at most n vertices y such that the edge (x, y) satisfies this condition, since otherwise $C_a^\nu \times Y \subseteq E$ where Y is the set of all such y and $\nu \in T$ is the longest word such that $|C_a^w| > n$ for all $w \preceq \nu$.

Similarly, if $|C_b^\nu| > n$ we set $f_\lambda(e) := y$. For all other edges we have $e := (x, y) \in C_a^\nu \times C_b^\nu \subseteq E_\lambda$ with $|C_a^\nu|, |C_b^\nu| \leq n$. Let $\nu_0 \preceq \nu$ be the maximal prefix of ν such that $C_a^\nu \subseteq C_c^{\nu_0}$ or $C_b^\nu \subseteq C_c^{\nu_0}$ for some c with $|C_c^{\nu_0}| > n$. In the first case we set $f_\lambda(e) := x$, in the other one, $f_\lambda(e) := y$. Let ν_1 be the successor of ν_0 with $\nu_1 \preceq \nu$. For all such edges e with $f_\lambda(e) = x$ it follows that $y \in C_d^{\nu_1}$ for some $d \neq c$ with $|C_d^{\nu_1}| \leq n$. Consequently, there are at most $(k-1)n$ such edges.

It follows that $\left|f_\lambda^{-1}(x)\right| \leq n + (k-1)n = kn$ as desired. $\square$

3 Partition Width

In this chapter and the next one we develop a model theory of structures having a simple monadic second-order theory by extending the concept of NLC-width from graphs to relational structures of arbitrary cardinality. The resulting complexity measure will be called the *partition width* of a structure. Mirroring the development of the notions of tree width, NLC-width, and clique width we will proceed in three steps: (1) we define *terms* denoting structures; (2) we introduce a notion of *decomposition* of a structure; and (3) we give a characterisation via *interpretations*.

3.1 Infinite terms

We start by generalising NLC-terms to infinite terms describing relational structures of arbitrary cardinality. Moving from binary relations to relations of higher arity requires that we colour not only single elements but all tuples up to this arity.

graded set of colours

Definition 3.1.1. A *graded set of colours* is a set C that is partitioned into finite nonempty sets C_n, $n < \omega$. Colours $c \in C_n$ are said to be of *arity* n.

arity

C-colouring

A *C-colouring* of a structure $\mathfrak{M}$ is a function χ mapping every n-tuple $\bar{a} \in M^n$ to some colour $\chi(\bar{a}) \in C_n$. The empty tuple is also coloured. We call the pair $(\mathfrak{M}, \chi)$ a *C-coloured structure*.

C-coloured structure

Analogously to the NLC-composition $\oplus_S$ we define two operators $\sum^{\Theta}$ and $\bigcup^{\Theta}$ to compose a family of C-coloured structures $(\mathfrak{M}_i, \chi_i)$, $i < \kappa$, one for ordered families and one for unordered ones. In both cases the resulting structure will consist of the union of the $\mathfrak{M}_i$. Additionally, we will update the colouring and add new tuples to the relations of $\mathfrak{M}$. If $\bar{a}$ is a tuple of $\mathfrak{M}$ then the colours of its parts $\bar{a} \cap M_i$, for $i < \kappa$, will determine both, its new colour and whether we add $\bar{a}$ to a relation R. We record this information in an *update instruction* $(n, \kappa, \bar{I}, \bar{c}, d, S)$ where $I_i := \{ k \mid a_k \in M_i \}$ is the partition of $\bar{a}$ induced by the union, $c_i := \chi_i(\bar{a}|_{I_i})$ is the colour of the tuple $\bar{a} \cap M_i$, d is the new colour of $\bar{a}$, and S contains all relation symbols to which $\bar{a}$ is added.

Definition 3.1.2. Let τ be a signature and C a graded set of colours.

update instruction

(a) An *update instruction* is a tuple $(n, \alpha, \bar{I}, \bar{c}, d, S)$ where

- $n < \omega$ is a natural number and α is an arbitrary ordinal;
- $\bar{I}$ is a partition $\biguplus_{i<\alpha} I_i = [n]$ of $[n]$ into α classes (of which all but finitely many are empty);
- $\bar{c} \in C^\alpha$ is a sequence of α colours such that the arity of c_i is $|I_i|$ (which implies that the sum of their arities is n);
- $d \in C_n$ is a colour of arity n; and
- $S \subseteq \tau$ is a set of n-ary relation symbols.

arity

The number n is called the *arity* of the instruction.

ordered κ-update

(b) An *ordered κ-update* is a set Θ of update instructions that contains exactly one instruction $(n, \kappa, \bar{I}, \bar{c}, d, S)$, for all n, $\bar{I}$, and $\bar{c}$. Each such set Θ induces a family of functions

$$\Theta_n(\bar{I}; \bar{c}) = (d, S) \quad : \text{iff} \quad (n, \kappa, \bar{I}, \bar{c}, d, S) \in \Theta\,.$$

symmetric update

(c) A *symmetric update* is a set Θ of update instructions with the following properties:

- Θ contains exactly one instruction $(n, s, \bar{I}, \bar{c}, d, S)$ for all $n < \omega$, every $s \leq n$, all partitions $\bar{I} = I_0 \uplus \cdots \uplus I_{s-1}$ where each of the I_i is nonempty, and all appropriate $\bar{c} \in C^s$.
- For all permutations $\sigma \in S_s$ we have

$$\begin{aligned} &\big(n,\ s, \langle I_{\sigma 0}, \ldots, I_{\sigma(s-1)}\rangle, \langle c_{\sigma 0}, \ldots, c_{\sigma(s-1)}\rangle,\ d,\ S\big) \in \Theta \\ \text{iff}\quad &\big(n,\ s, \langle I_0, \ldots, I_{s-1}\rangle, \quad \langle c_0, \ldots, c_{s-1}\rangle, \quad d,\ S\big) \in \Theta\,. \end{aligned}$$

The family of functions induced by Θ is

$$\Theta^s_n(\bar{I}; \bar{c}) = (d, S) \quad : \text{iff} \quad (n, s, \bar{I}, \bar{c}, d, S) \in \Theta\,.$$

We use ordered updates to define a sum operation $\sum^\Theta$ where the ordering of the structures matters, whereas symmetric updates are used to define an operation $\bigcup^\Theta$ that is invariant under permutations of its arguments. For every symmetric sum there exists an equivalent ordered one, while the converse only holds if we are allowed to use more colours. (Basically, we need to colour each structure with a different copy of the colours.) Below we will use ordered sums only for finitely many arguments.

Definition 3.1.3. Let $(\mathfrak{M}_i, \chi_i)$, $i < \kappa$, be a sequence of C-coloured structures.

ordered sum $\sum^\Theta$

(a) Let Θ be an ordered κ-update. The *ordered sum*

$$\sum_{i<m}^{\Theta} (\mathfrak{M}_i, \chi_i)$$

of $(\mathfrak{M}_i, \chi_i)$, $i < \kappa$, with respect to Θ is the structure $(\mathfrak{N}, \eta)$ obtained from the disjoint union of the $\mathfrak{M}_i$ by the following operation:

For every n-tuple $\bar{a} \in N^n$, $n < \omega$, if

$$\Theta_n(\bar{I}; \bar{c}) = (d, S)$$

where

$$I_i := \{\, k < n \mid a_k \in M_i \,\} \quad \text{and} \quad c_i := \chi_i(\bar{a}|_{I_i}) \quad \text{for } i < \kappa,$$

then we add $\bar{a}$ to all relations $R \in S$ and set the new colour to $\eta(\bar{a}) := d$.

(b) Let Θ be a symmetric update. The *symmetric sum*

symmetric sum $\bigcup^{\Theta}$

$$\bigcup_{i<\kappa}^{\Theta} (\mathfrak{M}_i, \chi_i)$$

of $(\mathfrak{M}_i, \chi_i)$, $i < \kappa$, with respect to Θ is the structure $(\mathfrak{N}, \eta)$ obtained from the disjoint union of the $\mathfrak{M}_i$ by the following operation:

For every n-tuple $\bar{a} \in N^n$, $n < \omega$, containing elements from $M_{j_0}, \dots, M_{j_{s-1}}$, if

$$\Theta^s_n(\bar{I}; \bar{c}) = (d, S)$$

where

$$I_i := \{\, k < n \mid a_k \in M_{j_i} \,\} \quad \text{and} \quad c_i := \chi_i(\bar{a}|_{I_i}) \quad \text{for } i < s,$$

then we add $\bar{a}$ to all relations $R \in S$ and set the new colour to $\eta(\bar{a}) := d$.

Note that this definition does not depend on the ordering of $j_0, \dots, j_{s-1}$ since Θ is invariant under permutations.

(c) For every sequence of colours $c_n \in C_n$, $n < \omega$, let $\bar{c}$ denote the C-coloured structure $(\mathfrak{D}, \zeta)$ with universe $D := [1]$ and empty relations $R := \varnothing$ where the only n-tuple is coloured with c_n.

Example. Consider three structures with universes $\{x, x'\}$, $\{y, y'\}$, and $\{z, z'\}$, and colouring

$$\begin{aligned}
\chi(x) &= a, & \chi(y) &= b, & \chi(z) &= c,\\
\chi(x') &= b, & \chi(y') &= c, & \chi(z') &= a,\\
\chi(x, x') &= e, & \chi(y, y') &= e, & \chi(z, z') &= f,\\
\chi(x', x) &= e, & \chi(y', y) &= f, & \chi(z', z) &= f.
\end{aligned}$$

(a) Let Θ be a symmetric update. The following examples show how the new colour and relations of a tuple are determined.

$$\begin{aligned}
(x, y): &\quad \Theta^2_2\big(\{0\}, \{1\};\, a, b\big)\\
(y, x): &\quad \Theta^2_2\big(\{0\}, \{1\};\, b, a\big) = \Theta^2_2\big(\{1\}, \{0\};\, a, b\big)\\
(y', y): &\quad \Theta^1_2\big(\{0, 1\};\, f\big)\\
(y, x, y'): &\quad \Theta^2_3\big(\{1\}, \{0, 2\};\, a, e\big)\\
(y, z, x): &\quad \Theta^3_3\big(\{1\}, \{2\}, \{0\};\, c, a, b\big)
\end{aligned}$$

(b) For an ordered 3-update Θ we have:

$$\begin{aligned}(x,y):&\quad \Theta_2\big(\{0\},\{1\},\varnothing;\ a,b,1\big)\\ (y,x):&\quad \Theta_2\big(\{1\},\{0\},\varnothing;\ a,b,1\big)\\ (y',y):&\quad \Theta_2\big(\varnothing,\{0,1\},\varnothing;\ 1,f,1\big)\\ (y,x,y'):&\quad \Theta_3\big(\{1\},\{0,2\},\varnothing;\ a,e,1\big)\\ (y,z,x):&\quad \Theta_3\big(\{2\},\{0\},\{1\};\ a,b,c\big)\end{aligned}$$

where 1 denotes the colour of the empty tuple.

Having decided on the operations we can start building terms. Since we want to support uncountable structures we consider terms as infinitely branching trees of ordinal height.

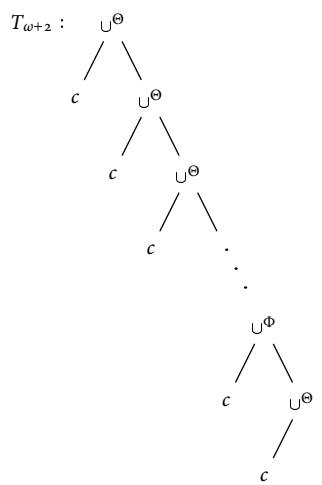
$\Upsilon^{<}_{C,\tau}$ $\Upsilon_{C,\tau}$

Definition 3.1.4. For a graded set of colours C and a signature τ, let $\Upsilon^{<}_{C,\tau}$ be the signature consisting of all operations of the form $\tilde{c}$ and $\sum^{\Theta}$ with colours from C and relation symbols from τ. Similarly, $\Upsilon_{C,\tau}$ consists of $\tilde{c}$ and $\bigcup^{\Theta}$.

Example. Let $C_1 = \{a,b,c\}$, $C_n = \{1\}$, for $n \neq 1$, and

$$\begin{aligned}\Theta := \big\{&(1,\ 1,\ \langle\{0\}\rangle,\ \langle a\rangle,\ b,\ \varnothing),\\ &(1,\ 1,\ \langle\{0\}\rangle,\ \langle b\rangle,\ b,\ \varnothing),\\ &(1,\ 1,\ \langle\{0\}\rangle,\ \langle c\rangle,\ a,\ \varnothing),\\ &(2,\ 2,\ \langle\{0\},\{1\}\rangle,\ \langle c,a\rangle,\ 1,\ \{\mathrm{suc},\leq\}),\\ &(2,\ 2,\ \langle\{1\},\{0\}\rangle,\ \langle a,c\rangle,\ 1,\ \{\mathrm{suc},\leq\}),\\ &(2,\ 2,\ \langle\{0\},\{1\}\rangle,\ \langle c,b\rangle,\ 1,\ \{\leq\}),\\ &(2,\ 2,\ \langle\{1\},\{0\}\rangle,\ \langle b,c\rangle,\ 1,\ \{\leq\}),\\ &\dots\big\}\end{aligned}$$

(where we left out the irrelevant entries). Let Φ be the update obtained from Θ by replacing the instruction $(1,\ 1,\ \langle\{0\}\rangle,\ \langle c\rangle,\ a,\ \{\leq\})$ by $(1,\ 1,\ \langle\{0\}\rangle,\ \langle c\rangle,\ b,\ \{\leq\})$.



For each ordinal α, we can define a term T_α denoting the structure $(\alpha, \mathrm{suc}, \leq)$ where the colour of the first element is a and the other elements are coloured by b. (A formal definition of the value of a term can be found below.) For $\beta < \alpha$, we set

$$T_\alpha(\mathrm{o}^\beta) := \begin{cases}\cup^{\Theta} & \text{if } \beta \text{ is a successor},\\ \cup^{\Phi} & \text{if } \beta \text{ is a limit},\end{cases} \qquad \text{and} \qquad T_\alpha(\mathrm{o}^\beta 1) := c.$$

For instance,

$$T_4 = c \cup^{\Theta} (c \cup^{\Theta} (c \cup^{\Theta} \bigcup\nolimits^{\Theta}\{c\})).$$

When trying to evaluate an infinite term $T \subseteq \kappa^{<\alpha}$ for $\alpha > \omega$ in a bottom-up fashion, we face the difficulty that, after having obtained the value of a subterm whose root is at a limit depth, we have to propagate this value to its predecessors. To do so, we start at the predecessor in question and trace the value back until we reach the already evaluated subterm.

Definition 3.1.5. Fix a relation $\leq$ well-ordering each colour set C_n such that colours of different arities are incomparable.

(1) For sequences of colours $(c_i)_{i<\alpha}$, $(d_i)_{i<\alpha}$ we define the ordering componentwise.

$$(c_i)_i \leq (d_i)_i \quad :\text{iff} \quad c_i \leq d_i \text{ for all } i < \alpha\,,$$
$$\text{and} \quad (c_i)_i < (d_i)_i \quad :\text{iff} \quad (c_i)_i \leq (d_i)_i \text{ and } (d_i)_i \not\leq (c_i)_i\,.$$

(2) Let T be a term, $v \in T$ a node, and $\alpha := |v|$. A *colour trace to* v is a sequence $(c_i)_{i<\alpha+1}$ of colours of the same arity which satisfies the following conditions: *colour trace*

(a) If $\alpha = \beta + 1$ is a successor then $(c_i)_{i<\beta+1}$ is a colour trace to the predecessor u of v and the operation at u changes the colour of tuples from $c_{\beta+1}$ to c_β.

(b) If α is a limit then each subsequence $(c_i)_{i<\beta+1}$ for $\beta < \alpha$ is a colour trace to the corresponding prefix of v, and c_α is the minimal colour c such that the set $\{\, \beta < \alpha \mid c_\beta = c \,\}$ is unbounded below α.

Example. For the terms T_α in the previous example, the colour traces are of the form $bb\ldots bbac$, $bb\ldots bba$, or $bb\ldots bb$.

With these notions we can define a subclass of terms to which we can assign a value. Basically, we call a term T *well-formed* if its value $\mathrm{val}(T)$ (which we introduce below) is well-defined.

Definition 3.1.6. A term T is *well-formed* if the following conditions are satisfied: *well-formed term*

(1) For each $v \in T$, the set of colour traces to v is linearly ordered by $\leq$.

(2) For every leaf v labelled $\tilde{c}$ and all arities n there exists a colour trace $(d_i)_{i<\alpha+1}$ to v with $d_\alpha = c_n$.

(3) For all finite sequences of vertices v^k, $k < m$, and all colour traces $(c^k_i)_i$ to v^k, there exists a colour trace $(d_i)_{i<\alpha+1}$ to $u := v^0 \sqcap \cdots \sqcap v^{m-1}$ such that d_α is the result of the operation at u applied to the colours $c^k_{\alpha+1}$.

Lemma 3.1.7. *Let T be a well-formed term. For every $v \in T$ and all colours $c \in C$ there is at most one colour trace $(c_\beta)_{\beta<\alpha+1}$ to v with $c_\alpha = c$.*

Proof. Let $(c_\beta)_{\beta<\alpha+1}$ and $(d_\beta)_{\beta<\alpha+1}$ be colour traces to ν with $c_\alpha = d_\alpha$. We prove by induction on α that $(c_\beta)_\beta = (d_\beta)_\beta$. The case $\alpha = 0$ is trivial.

If $\alpha = \beta + 1$ is a successor ordinal then the operation at ν maps $c_\alpha = d_\alpha$ to $c_\beta = d_\beta$ and the claim follows by induction hypothesis.

Suppose that α is a limit and that $(c_\beta)_\beta \neq (d_\beta)_\beta$. By symmetry, we may assume that $(c_\beta)_\beta < (d_\beta)_\beta$. By definition, the set

$$S := \{\, \beta < \alpha \mid d_\beta = d_\alpha \,\}$$

is unbounded below α. Let e be the minimal colour such that the subset $S' := \{\, \beta \in S \mid c_\beta = e \,\}$ is also unbounded. Such a colour exists since there are only finitely many colours of the given arity.

By definition of a colour trace we have $e \geq c_\alpha$. Since $c_\beta \leq d_\beta$ for all $\beta < \alpha$ it follows that $e = c_\beta \leq d_\beta = d_\alpha = c_\alpha$ for $\beta \in S'$. Consequently, $c_\beta = d_\beta$ for all $\beta \in S'$. Since S' is unbounded the induction hypothesis implies that $c_\beta = d_\beta$ for all $\beta < \alpha$. Contradiction. □

Definition 3.1.8. Let $T \subseteq \kappa^{<\alpha}$ be a well-formed term and $L \subseteq T$ the set of its leaves.

(a) To every tuple $\bar{a} \in L^n$ we associate a colour trace $\chi(\bar{a})$ by induction on $|\bar{a}|$. If $a_0 = \cdots = a_{n-1}$ and the node a_0 is labelled by d then $\chi(\bar{a}) = (c_\beta)_{\beta<\alpha+1}$ is the (unique) colour trace to a_0 that ends in $c_\alpha = d_n$.

Otherwise, let $\nu := \bigsqcap \bar{a}$. There is a partition $I_0 \cup \cdots \cup I_{s-1} = [n]$ of the indices such that

- $\nu \prec a_i \sqcap a_k$ if i and k belong to the same class I_l, and
- $\nu = a_i \sqcap a_k$ for i and k belonging to different classes.

The node ν is labelled by either $\sum^\Theta$ or $\bigcup^\Theta$ for some update Θ. Let $\chi(\bar{a}|_{I_i}) = (c^i_\beta)_\beta$, for $i < s$, and let $\alpha := |\nu|$. We either have

$$(d, S) = \Theta_n\big(\bar{I}'; (\hat{c}^i_\alpha)_{i<\kappa}\big)$$

or $$(d, S) = \Theta^s_n\big(\bar{I}; c^0_\alpha, \ldots, c^{s-1}_\alpha\big),$$

where $(\hat{c}^i_\alpha)_{i<\kappa}$ is the sequence of length κ obtained from $c^0_\alpha, \ldots, c^{s-1}_\alpha$ by inserting the colour of the empty tuple at the appropriate places. We let $\chi(\bar{a}) = (c_\beta)_{\beta<\alpha+1}$ be the (unique) colour trace to ν with $c_\alpha = d$.

value val(T)

(b) The *value* val(T) of T is the structure whose universe $M := L$ consists of all leaves of T. A tuple $\bar{a} \in M^n$ with associated colour trace $\chi(\bar{a}) = (c_\beta)_{\beta<\alpha+1}$ belongs to a relation R iff there is some node $\nu \preceq \bigsqcap \bar{a}$ labelled by an operation $\sum^\Theta$ or $\bigcup^\Theta$ that adds tuples coloured $c_{|\nu|}$ to R.

In the following we will tacitly assume that all terms are well-formed.

What structures can be the value of a $\Upsilon_{C,\tau}$-term? If $\mathfrak{M}$ is a finite structure with $|M^n| \leq |C_n|$, for all $n < \omega$, then, by assigning different colours to each tuple $\bar{a} \subseteq M$, we can easily construct a term denoting $\mathfrak{M}$.

But, if M is infinite, this does not need to be the case. In the next lemma we prove that every structure denoted by an $\Upsilon_{C,\tau}$-term T can be interpreted in some tree, namely, the term T itself. The converse is shown in Section 3.4.

One remaining technicality we have to deal with is to fix an encoding of terms as structures. In order to allow infinite signatures we encode a Υ-term $T \subseteq \kappa^{<\alpha}$ as a structure $(T, \preceq, \bar{P})$ with universe T, prefix ordering $\preceq$, and unary predicates $\bar{P}$ coding the functions in Υ. Each operator is encoded by several predicates:

$$P_d := \{\, v \in T \mid v \text{ is labelled by some } \bar{c} \text{ with } d \in \bar{c} \,\},$$
$$P_{(n,\alpha,\bar{I},\bar{c},d,R)} := \{\, v \in T \mid v \text{ is labelled by } \textstyle\sum^{\Theta} \text{ or } \bigcup^{\Theta} \text{ for some } \Theta \text{ containing } (n, \alpha, \bar{I}, \bar{c}, d, S) \text{ with } R \in S \,\}.$$

Proposition 3.1.9. *For all signatures τ and every set C of colours there are* MSO*-interpretations $\mathcal{V}$ and $\mathcal{V}^<_k$, $k < \aleph_0$, such that*

$$\mathcal{V}^<_k : \mathrm{val}(T) \leq_{\mathrm{MSO}} (T, \preceq, \bar{P}, (\mathrm{suc}_i)_{i<k}) \quad \textit{for all } \Upsilon^<_{C,\tau}\textit{-terms } T \subseteq k^{<\alpha},$$
$$\textit{and} \quad \mathcal{V} : \mathrm{val}(T) \leq_{\mathrm{MSO}} (T, \preceq, \bar{P}) \quad \textit{for all } \Upsilon_{C,\tau}\textit{-terms } T \subseteq \kappa^{<\alpha}.$$

If the arity of τ is bounded then there even exist MSO_m*-interpretations for some m.*

Proof. The universe of $\mathrm{val}(T)$ consists of the set of leaves of T, which is definable. The above definition of the relations of $\mathrm{val}(T)$ can be translated immediately into MSO once we have shown how to encode colour traces. If colour traces $(c_i)_{i<\alpha} \in C_n^\alpha$ to some node $v \in T$ are represented by sets $(X_d)_{d \in C_n}$ such that $u \preceq v$ belongs to X_d iff $c_{|u|} = d$, then there is an MSO-formula which expresses that the sequence of colours encoded in some tuple $\bar{X}$ is indeed a colour trace.

The quantifier rank of these formulae depends only on $|C_n|$ and the arity of the relations involved. □

3.2 Partition refinements

Given some structure $\mathfrak{M}$, how can we find out how large the set C of colours needs to be in order that there is some $\Upsilon_{C,\tau}$-term denoting $\mathfrak{M}$? We will derive a structural criterion answering this question by using a suitable variant of clique refinements and showing that

every structure denoted by an $\Upsilon_{C,\tau}$-term admits such a decomposition, called a *partition refinement*, and that, vice versa, every partition refinement yields a term.

We have chosen the operations of our terms in such a way that we do not need to explicitly store information about the colours and the refinements become particularly simple.

partial partition refinement

Definition 3.2.1. (a) A *partial $\kappa^{<\alpha}$-partition refinement* of a structure $\mathfrak{M}$ is a family $(U_\nu)_{\nu\in T}$ of nonempty subsets $U_\nu \subseteq M$ indexed by a tree $T \subseteq \kappa^{<\alpha}$ such that the following conditions are satisfied:

(1) $U_\varepsilon = M$ and for every $a \in M$ there is some leaf $\nu \in T$ with $a \in U_\nu$.

(2) Each U_ν is the disjoint union of its successors $U_{\nu\beta}$, $\nu\beta \in T$, $\beta < \kappa$.

(3) If $|\nu|$ is a limit ordinal then $U_\nu = \bigcap_{u<\nu} U_u$.

granularity

The *granularity* of a partial partition refinement $(U_\nu)_\nu$ is the supremum of the cardinalities $|U_\nu|$ of its leaves ν.

partition refinement

(b) A *$\kappa^{<\alpha}$-partition refinement* is a partial $\kappa^{<\alpha}$-partition refinement of granularity 1.

We can retrieve the colouring from a given partition refinement since the colour of a tuple corresponds to its type as explained below. As the colours are only needed to connect tuples $\bar{a} \subseteq U_\nu$ in some component U_ν with tuples $\bar{b} \subseteq \overline{U_\nu}$ in the complement we define a notion of type consisting only of formulae containing both, a free variable and some parameter.

Definition 3.2.2. Let $\mathfrak{M}$ be a structure, $\bar{a} \subseteq M$, and $U \subseteq M$. Let

Δ-type

$\Delta \subseteq \mathrm{FO}$. The *$\Delta$-type* of $\bar{a}$ over U is the set

$$\mathrm{tp}_\Delta(\bar{a}/U) := \{\, \varphi(\bar{x};\bar{c}) \mid \mathfrak{M} \models \varphi(\bar{a};\bar{c}),\ \varphi \in \Delta,\ \bar{c} \subseteq U \,\},$$

external Δ-type

and the *external Δ-type* of $\bar{a}$ over U is defined by

$$\mathrm{etp}_\Delta(\bar{a}/U) := \{\, \varphi(\bar{x};\bar{c}) \in \mathrm{tp}_\Delta(\bar{a}/U) \mid \text{every atom of } \varphi \text{ contains a variable and some parameter } c \in U \,\}.$$

We denote the set of all Δ-types over U with n free variables by $S^n_\Delta(U)$ and its subset of external types by $ES^n_\Delta(U)$. In case $\Delta = \mathrm{FO}_k$ we simply write $\mathrm{tp}_k(\bar{a}/U)$ and $S^n_k(U)$.

For sets $\bar{A} \subseteq \wp(M)$ and monadic formulae $\Delta \subseteq \mathrm{MSO}$ we also define

monadic Δ-type

the *monadic Δ-type* of $\bar{A}$ over U and its external variant by

$$\mathrm{mtp}_\Delta(\bar{A}/U) := \{\, \varphi(\bar{X};\bar{C}) \mid \mathfrak{M} \models \varphi(\bar{A};\bar{C}),\ \varphi \in \Delta,\ \bar{C} \subseteq \wp(U) \,\},$$

$$\mathrm{emtp}_\Delta(\bar{A}/U) := \{\, \varphi(\bar{X};\bar{C}) \in \mathrm{mtp}_\Delta(\bar{A}/U) \mid \text{every atom of } \varphi \text{ contains a variable and some parameter } C \subseteq U \,\}.$$

The set of all monadic Δ-types over U with n free variables is denoted by $MS^n_\Delta(U)$.

Definition 3.2.3. Let $\mathfrak{M}$ be a structure and $U \subseteq M$. For tuples $\bar a, \bar b \subseteq M$ we define

$\bar a \approx^\Delta_U \bar b$, $\bar a \simeq^\Delta_U \bar b$
$\bar A \approx^\Delta_U \bar B$, $\bar A \simeq^\Delta_U \bar B$

$$\begin{aligned} \bar a \approx^\Delta_U \bar b \quad &:\text{iff} \quad \operatorname{tp}_\Delta(\bar a/U) = \operatorname{tp}_\Delta(\bar b/U)\,, \\ \bar a \simeq^\Delta_U \bar b \quad &:\text{iff} \quad \operatorname{etp}_\Delta(\bar a/U) = \operatorname{etp}_\Delta(\bar b/U)\,. \end{aligned}$$

For sets $\bar A, \bar B \subseteq \wp(M)$ we reuse the these symbols and write

$$\begin{aligned} \bar A \approx^\Delta_U \bar B \quad &:\text{iff} \quad \operatorname{mtp}_\Delta(\bar A/U) = \operatorname{mtp}_\Delta(\bar B/U)\,, \\ \bar A \simeq^\Delta_U \bar B \quad &:\text{iff} \quad \operatorname{emtp}_\Delta(\bar A/U) = \operatorname{emtp}_\Delta(\bar B/U)\,. \end{aligned}$$

The [*external*] [*monadic*] Δ-*type index* of a set X over U is

type index
$\operatorname{ti}^n_\Delta(X/U)$, $\operatorname{mti}^n_\Delta(X/U)$
$\operatorname{eti}^n_\Delta(X/U)$, $\operatorname{emti}^n_\Delta(X/U)$

$$\begin{aligned} \operatorname{ti}^n_\Delta(X/U) &:= |X^n/{\approx^\Delta_U}|\,, & \operatorname{mti}^n_\Delta(X/U) &:= |\wp(X)^n/{\approx^\Delta_U}|\,, \\ \operatorname{eti}^n_\Delta(X/U) &:= |X^n/{\simeq^\Delta_U}|\,, & \operatorname{emti}^n_\Delta(X/U) &:= |\wp(X)^n/{\simeq^\Delta_U}|\,. \end{aligned}$$

Again, in case $\Delta = \mathrm{FO}_k$ we simply write $\approx^k_U$, $\operatorname{ti}^n_k(X/U)$, and so on.

Remark. Note that, for undirected graphs, the relations $\simeq^{\mathrm{o}}_U$ coincides with the relation $\sim_{\overline{U}}$ defined by Courcelle in [27].

For the most part we will concentrate on atomic external types $\operatorname{etp}_{\mathrm{o}}(\bar a/U)$ and the corresponding index $\operatorname{eti}^n_{\mathrm{o}}(X/U)$.

Example. Consider the binary tree $\mathfrak{T} = (2^{\leq\omega}, \preceq)$ and fix a vertex $w \in 2^{\leq\omega}$. If $v \in {\uparrow}w := \{\, v \in 2^{\leq\omega} \mid w \preceq v \,\}$ then

$$\begin{aligned} &u \preceq v && \text{for all } u \in {\downarrow}w := \{\, v \in 2^{\leq\omega} \mid v \prec w \,\}\,, \\ \text{and}\quad &u \not\preceq v && \text{for all } u \in 2^{\leq\omega} \setminus ({\uparrow}w \cup {\downarrow}w)\,. \end{aligned}$$

Hence $\operatorname{eti}^1_{\mathrm{o}}({\uparrow}w/\overline{{\uparrow}w}) = 1$ since only one external atomic type over $2^{\leq\omega} \setminus {\uparrow}w$ is realised in ${\uparrow}w$. On the other hand, $\operatorname{eti}^1_{\mathrm{o}}(\overline{{\uparrow}w}/{\uparrow}w) = 2$ because there are two external atomic types over ${\uparrow}w$ realised in $2^{\leq\omega} \setminus {\uparrow}w$.

Below it will be shown that, when colouring a component U_v of a partition refinement, we can take as colours the classes of the relation $\simeq^{\mathrm{o}}_{\overline{U_v}}$, i.e., the atomic external types over the complement of U_v. Therefore, the number of n-ary colours we need equals $\operatorname{eti}^n_{\mathrm{o}}(U_v/\overline{U_v})$.

Definition 3.2.4. (1) The *n-ary partition width* of a partition refinement $(U_v)_{v\in T}$ is the number

partition width

$$\operatorname{pwd}_n(U_v)_v := \sup\{\, \operatorname{eti}^n_{\mathrm{o}}(U_v/\overline{U_v}) \mid v \in T \,\}\,,$$

and the *n-ary symmetric partition width* is

symmetric partition width

$$\operatorname{spwd}_n(U_v)_v := \sup\left\{ \operatorname{eti}^n_{\mathrm{o}}\big(\textstyle\bigcup_{i\in I} U_{vi} \,/\, \overline{\bigcup_{i\in I} U_{vi}}\big) \;\middle|\; v \in T, I \subseteq \kappa \right\}.$$

(2) The *n-ary partition width* $\operatorname{pwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ of a structure $\mathfrak{M}$ is defined inductively as follows: $\operatorname{pwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ is the minimal cardinal λ such that there exists a $\kappa^{<\alpha}$-partition refinement $(U_v)_v$ with

$$\operatorname{pwd}_n(U_v)_v = \lambda \quad \text{and} \quad \operatorname{pwd}_i(U_v)_v = \operatorname{pwd}_i(\mathfrak{M}, \kappa^{<\alpha}) \text{ for } i < n.$$

finite partition width

If $\kappa^{<\alpha} = 2^{<|M|^+}$ we omit the second parameter and simply write $\operatorname{pwd}_n \mathfrak{M}$. $\mathfrak{M}$ is said to be of *finite partition with* if $\operatorname{pwd}_n \mathfrak{M}$ is finite for all $n < \omega$.

The *n-ary symmetric partition width* $\operatorname{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ of $\mathfrak{M}$ is defined analogously. We set $\operatorname{spwd}_n \mathfrak{M} := \operatorname{spwd}_n\big(\mathfrak{M}, |M|^{<|M|^+}\big)$.

monadic partition width

(3) The *monadic* [*symmetric*] *partition widths* mpwd_n and smpwd_n of a partition refinement or a structure are defined similarly by replacing $\operatorname{eti}^n_{\mathrm{o}}$ by $\operatorname{emti}^n_{\mathrm{o}}$.

Remark. (1) Obviously, we have $\operatorname{pwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \operatorname{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$.

(2) In each partition refinement $(U_v)_{v\in T}$ we can remove all nodes $v \in T$ with exactly one successor. In that way we can transform any $\kappa^{<\alpha}$-partition refinement of a structure of cardinality λ into a $\kappa^{<\lambda^+}$-partition refinement.

(3) It is not clear whether there always exists a partition refinement $(U_v)_v$ such that $\operatorname{pwd}_n \mathfrak{M} = \operatorname{pwd}_n(U_v)_v$ for all n.

Lemma 3.2.5. *Every linear order $\mathfrak{M} = (M, \leq)$ has a $2^{<|M|^+}$-partition refinement $(U_v)_{v\in T}$ of monadic partition width $\operatorname{mpwd}_n(U_v)_v = 1$ where every U_v forms an interval of $\mathfrak{M}$.*

Proof. We define U_v by induction on $|v|$. Let $U_\varepsilon := M$. Given an interval U_v containing at least two different elements, we pick some $a \in U_v$ that is not the least element of U_v and set

$$U_{v0} := \{ b \in U_v \mid b < a \} \quad \text{and} \quad U_{v1} := \{ b \in U_v \mid b \geq a \}.$$

Finally, if $|v|$ is a limit ordinal, we set $U_v := \bigcap_{u\leq v} U_u$. □

Lemma 3.2.6. *For the tree $\mathfrak{T} := (\beta^{<\alpha}, \preceq)$ we have*

$$\operatorname{smpwd}_n(\mathfrak{T}, \beta^{<2\alpha}) = 1 \quad \textit{and} \quad \operatorname{smpwd}_n(\mathfrak{T}, 2^{<(\beta+2)\alpha}) = 1.$$

Proof. We define a $\beta^{<2\alpha}$-partition refinement $(U_v)_v$ by induction on v. Set $U_\varepsilon := \beta^{<\alpha}$. Suppose that U_v is already defined and of the form $\uparrow w := \{ x \in \beta^{<\alpha} \mid w \preceq x \}$ for some w. We define

$$U_{v0} := \{w\}, \quad U_{v1} = U_v \setminus \{w\}, \quad U_{v1i} := \uparrow wi \quad \text{for } i < \beta.$$

Then we have $\operatorname{emti}^n_{\mathrm{o}}(U_v/\overline{U_v}) = 1$ for all v, as desired.

The second claim is proved analogously. If $U_v = \uparrow w$ is already defined, we set

$$U_{v0} := \{w\}\,, \quad U_{v11^\gamma} := \bigcup_{i\geq\gamma} \uparrow wi\,, \quad U_{v11^\gamma 0} := \uparrow w\gamma \quad \text{for } \gamma < \beta\,. \qquad \square$$

We promised above that we will show how one can use types to define a canonical colouring. For the symmetric case we first need a technical lemma which relates infinite symmetric sums and symmetric partition width.

We say that a union $\bigcup_i X_i$ *induces* the equivalence relation *induced partition*

$$a \sim b \quad :\text{iff} \quad a \in X_i \Leftrightarrow b \in X_i \text{ for all } i\,.$$

When considering an n-tuple $\bar{a}$, this relation induces a partition $I_0 \cup \dots \cup I_s = [n]$ of the indices such that $a_i \sim a_k$ iff $i, k \in I_l$ for some l.

We call a tuple $\bar{a} \subseteq \bigcup_i X_i \cup U$ *fragmented* if the induced partition consists of at least two classes. Further, we say that a colouring χ of a set X is *compatible* with the equivalence relation $\simeq^0_U$ if *fragmented* *compatible*

$$\chi(\bar{a}) = \chi(\bar{b}) \quad \text{iff} \quad \bar{a} \simeq^0_U \bar{b} \qquad \text{for all } \bar{a}, \bar{b} \subseteq X\,.$$

Proposition 3.2.7. *Let $\mathfrak{M}$ be a structure of arity $r < \omega$, $Y := \bigcup_{i<\kappa} X_i \subseteq M$ a disjoint union, and $U \subseteq M$ disjoint from Y. For $I \subseteq \kappa$, define $X_I := \bigcup_{i\in I} X_i$ and $U_I := U \cup (Y \setminus X_I)$. Let $\sim$ be the equivalence relation induced by the union $\bigcup_i X_i$. Consider the following statements:*

(1) *There is a bound $\bar{w} \in \omega^\omega$ with $w_n \leq w_{n+1}$ such that*

$$\mathrm{eti}^n_0(X_I/U_I) \leq w_n \qquad \textit{for all } n < \omega \textit{ and } I \subseteq \kappa\,.$$

(2) *There exists a set of colours C and C-colourings η of Y and χ_i of X_i compatible with, respectively, $\simeq^0_U$ and $\simeq^0_{U_{\{i\}}}$ such that*

$$(\mathfrak{M}|_Y, \eta) = \bigcup_{i<\kappa}^{\Theta} (\mathfrak{M}|_{X_i}, \chi_i) \qquad \textit{for suitable } \Theta\,.$$

The following implications hold:

(2) $\Rightarrow$ (1) *with $w_n \leq n^{n+1}(c_n)^n$ where $c_n := \max_{i\leq n} |C_i|$.*

(1) $\Rightarrow$ (2) *with $|C_n| < (w_n(r-n)+1)R(K_n)^2_{r^3r+1}$ where*

$$K_n := w_n(rw_r)^r + R(w_n + 2(r-n) + 2)^3_8\,.$$

Proof. (2) $\Rightarrow$ (1) Define $\chi(\bar{a}) := \chi_i(\bar{a})$ for $\bar{a} \subseteq X_i$, $i < \kappa$. Let $I \subseteq \kappa$ and $\bar{a}, \bar{a}' \in (X_I)^n$. We claim that, if $\sim$ induces the same partition $J_0 \cup \dots \cup J_s = [n]$ of the indices of $\bar{a}$ and $\bar{a}'$ and if $\chi(\bar{a}|_{J_i}) = \chi(\bar{a}'|_{J_i})$ for all $i \leq s$, then $\bar{a} \simeq^0_{U_I} \bar{a}'$.

First suppose that $\mathfrak{M} \models \varphi(\bar{a}; \bar{b})$ for some atomic formula φ and parameters $\bar{b} \subseteq Y \setminus X_I$. Then $\bigcup^{\Theta}$ adds all tuples of colour $\eta(\bar{a}\bar{b}) = \eta(\bar{a}'\bar{b})$ to the corresponding relation. Hence, $\mathfrak{M} \models \varphi(\bar{a}'; \bar{b})$.

It remains to consider the case $\mathfrak{M} \models \varphi(\bar{a}; \bar{b}, \bar{c})$ where $\bar{b} \subseteq Y \setminus X_I$ and $\bar{c} \subseteq U$. $\eta(\bar{a}\bar{b}) = \eta(\bar{a}'\bar{b})$ implies $\bar{a}\bar{b} \simeq^{\circ}_{U} \bar{a}'\bar{b}$. Thus, $\mathfrak{M} \models \varphi(\bar{a}'; \bar{b}, \bar{c})$.

Setting $c_n := \max_{i \leq n} |C_i|$ it follows that

$$\begin{aligned} w_n &\leq \sum \big\{ \, |C_{|J_0|}| \cdot \dots \cdot |C_{|J_{s-1}|}| \;\big|\; J_0 \cup \dots \cup J_{s-1} = [n],\ s \leq n \big\} \\ &\leq \sum_{s \leq n} s^n (c_n)^s \leq n^{n+1} (c_n)^n \,. \end{aligned}$$

(1) ⇒ (2) We call a sequence $(f_n)_{n \leq r}$ of functions

$$f_n : \bigcup_{\alpha < \kappa} X_\alpha^n \to C_n$$

a *valid colouring* iff

$$(\mathfrak{M}|_Y, \eta) = \bigcup_{\alpha < \kappa}^{\Theta} (\mathfrak{M}|_{X_\alpha}, \chi_\alpha)$$

for some Θ where χ_α is the colouring of X_α induced by $(f_n)_n$. This condition is equivalent to the following one: $(f_n)_n$ is valid if and only if, for all tuples $\bar{a}$, $\bar{b} \in Y^n$ such that $\sim$ induces the same partition $J_0 \cup \dots \cup J_s$ of their indices, $f_{|J_i|}(\bar{a}|_{J_i}) = f_{|J_i|}(\bar{b}|_{J_i})$, $i \leq s$, and for every atomic formula $\varphi(\bar{x}; \bar{d})$ with parameters $\bar{d} \subseteq U$ such that $\bar{a}\bar{d}$ and $\bar{b}\bar{d}$ are fragmented, we have

$$\mathfrak{M} \models \varphi(\bar{a}; \bar{d}) \leftrightarrow \neg\varphi(\bar{b}; \bar{d}) \,.$$

For $\bar{a}_0 \in X_\alpha^n$ and $\bar{b}_0 \in X_\beta^n$, we write $\bar{a}_0 \nsim \bar{b}_0$ if there are tuples $\bar{a}_1 \subseteq Y \setminus X_\alpha$ and $\bar{b}_1 \subseteq Y \setminus X_\beta$ such that

- $\sim$ induces the same partition $J_0 \cup \dots \cup J_s$ of their indices,
- $f_{|J_i|}(\bar{a}_1|_{J_i}) = f_{|J_i|}(\bar{b}_1|_{J_i})$, for $i \leq s$, and
- for some atomic formula $\varphi(\bar{x}, \bar{y}; \bar{d})$ with parameters $\bar{d} \subseteq U$ such that $\bar{a}_0\bar{a}_1\bar{d}$ and $\bar{b}_0\bar{b}_1\bar{d}$ are fragmented, we have

$$\mathfrak{M} \models \varphi(\bar{a}_0, \bar{a}_1; \bar{d}) \leftrightarrow \neg\varphi(\bar{b}_0, \bar{b}_1; \bar{d}) \,.$$

We will call such tuples $\bar{a}_1$ and $\bar{b}_1$ *witnesses* of the fact that $\bar{a}_0 \nsim \bar{b}_0$.

By the above remark, it follows that $(f_n)_n$ is a valid colouring if and only if $\bar{a} \nsim \bar{b}$ implies $f_n(\bar{a}) \neq f_n(\bar{b})$ for all $\bar{a}$ and $\bar{b}$.

Let $(f_n)_n$ be a valid colouring such that $C_n := \operatorname{rng} f_n$ is of minimal size. Suppose that

$$m := |C_n| \geq (w_n(r-n)+1) R(K_n)^2_{r^3 r+1} \,.$$

We fix an arbitrary ordering of each C_n and we order colourings pointwise:

$$(f_n)_n \leq (g_n)_n \quad :\text{iff} \quad f_n(\bar{a}) \leq g_n(\bar{a}) \quad \text{for all } n \leq r,\ \bar{a} \in \bigcup_\alpha X_\alpha^n .$$

W.l.o.g. we may assume that $(f_n)_n$ is minimal w.r.t. this ordering. It follows that, for all $\bar{a} \in \bigcup_\alpha X_\alpha^n$ and every colour $c \in C_n$ with $c < f_n(\bar{a})$, there exists some tuple $\bar{b} \in f_n^{-1}(c)$ with $\bar{a} \nsim \bar{b}$ since, otherwise, the sequence $(g_n)_n$ defined by

$$g_n(\bar{x}) := \begin{cases} c & \text{if } \bar{x} = \bar{a}\,, \\ f_n(\bar{x}) & \text{otherwise}\,, \end{cases}$$

and $g_i := f_i$ for $i \neq n$, would be a strictly smaller valid colouring.

Further, it follows that $|\operatorname{rng} f_n|_{X_\alpha^n}| \leq w_n$ for all $\alpha < \kappa$ since, if $\bar{a} \simeq^{\mathrm{o}}_{U_{\{\alpha\}}} \bar{b}$ and $f_n(\bar{a}) < f_n(\bar{b})$, then we could change the colour of $\bar{b}$ to $f_n(\bar{a})$ and the colouring would still be valid.

(A) Fix a decreasing enumeration $c_0 > \cdots > c_{m-1}$ of C_n. We construct a sequence $(\bar{a}^i)_i$ such that $\bar{a}^i \nsim \bar{a}^k$ for $i \neq k$. By induction on i, we define

- an increasing sequence of indices $s_i \in [m]$;
- a decreasing sequence of sets $H_i \subseteq [m]$;
- sets $I_{it} \subseteq \kappa$, for $s_i < t < m$; and
- tuples $\bar{a}^i \in f_n^{-1}(c_{s_i}) \cap X^n_{I_{i-1,s_i}}$

such that

- $\bar{b} \nsim \bar{a}^i$ for all $\bar{b} \in f_n^{-1}(c_t) \cap X^n_{I_{it}}$, $s_i < t < m$, and
- $f_n^{-1}(c_t) \cap X^n_{I_{it}} \neq \varnothing$ for all $t \in H_i$.

Let $H_{-1} := [m]$ and $I_{-1,t} := \kappa$. For every i, we perform the following steps. If $H_{i-1} = \varnothing$ we stop. Otherwise, let $s_i := \max H_{i-1}$ and choose an arbitrary tuple $\bar{a}^i \in f_n^{-1}(c_{s_i}) \cap X^n_{I_{i-1,s_i}}$, say $\bar{a}^i \in X_\alpha^n$. Since $I_{i-1,s_i} \subseteq I_{ks_i}$ for $k < i$ and by induction hypothesis, we have $\bar{a}^i \nsim \bar{a}^k$, for every $k < i$, as desired.

To define I_{it}, $s_i < t < m$, fix some $\bar{b}_0 \in f_n^{-1}(c_t)$ such that $\bar{b}_0 \nsim \bar{a}^i$, say, $\bar{b}_0 \in X_\beta^n$. By definition, there exist an atomic formula $\varphi(\bar{x}, \bar{y}; \bar{d})$ with parameters $\bar{d} \subseteq U$ and tuples $\bar{a}_1$ and $\bar{b}_1$ such that $\bar{a}^i \bar{a}_1 \bar{d}$ and $\bar{b}_0 \bar{b}_1 \bar{d}$ are fragmented, $\sim$ induces the same partition $J_0 \cup \cdots \cup J_s$ of the indices of $\bar{a}_1$ and $\bar{b}_1$, $f_{|J_l|}(\bar{a}_1|_{J_l}) = f_{|J_l|}(\bar{b}_1|_{J_l})$, for $l \leq s$, and we have

$$\mathfrak{M} \vDash \varphi(\bar{a}^i, \bar{a}_1; \bar{d}) \leftrightarrow \neg\varphi(\bar{b}_0, \bar{b}_1; \bar{d})\,.$$

Let $J \subseteq \kappa$ be the minimal set such that $\bar{b}_1 \subseteq X_J$. If $\bar{b}' \in f_n^{-1}(c_t) \cap X^n_{\kappa \setminus J}$ then

$$\mathfrak{M} \vDash \varphi(\bar{b}', \bar{b}_1; \bar{d}) \leftrightarrow \varphi(\bar{b}_0, \bar{b}_1; \bar{d})$$

since $(f_n)_n$ is a valid colouring. This implies $\bar{b}' \nsim \bar{a}^i$. Therefore, we can set $I_{it} := I_{i-1,t} \setminus J$. We conclude the construction by setting

$$H_i := \{\, t \in H_{i-1} \setminus \{s_i\} \mid f_n^{-1}(c_t) \cap X^n_{I_{it}} \neq \varnothing \,\}\,.$$

The sequence $(\bar{a}^i)_{i<m_1}$ obtained this way satisfies $\bar{a}^i \nsim \bar{a}^k$ for $i \neq k$. It remains to determine its length m_1. We have

$$\begin{aligned} |H_i| &\geq |H_{i-1}| - w_n|J| - 1 \\ &\geq |H_{-1}| - (i+1)(w_n(r-n)+1) \\ &= m - (i+1)(w_n(r-n)+1)\,. \end{aligned}$$

We can define $\bar{a}^i$ provided $H_{i-1} \neq \varnothing$. This is the case if

$$i < \frac{m}{w_n(r-n)+1}\,.$$

Consequently,

$$m_1 \geq \frac{m}{w_n(r-n)+1} \geq R(K_n)^2_{r^{3r+1}}\,.$$

(B) Denote the index α such that $\bar{a}^i \in X^n_\alpha$ by α_i. For all $i < k$, we fix tuples $\bar{b}^{ik} \subseteq X_{\kappa\setminus\{\alpha_i\}}$ and $\bar{b}^{ki} \subseteq X_{\kappa\setminus\{\alpha_k\}}$ witnessing the fact that $\bar{a}^i \nsim \bar{a}^k$, that is,

$$\mathfrak{M} \vDash \varphi(\bar{a}^i, \bar{b}^{ik}; \bar{d}) \leftrightarrow \neg\varphi(\bar{a}^k, \bar{b}^{ki}; \bar{d})$$

for some atomic formula $\varphi(\bar{x}, \bar{y}; \bar{d})$. Let $J_0 \cup \cdots \cup J_s$ be the partition of the indices of $\bar{b}^{ik}$ (or of $\bar{b}^{ki}$) induced by $\sim$. Set

$$\bar{b}^{ik}_l := \bar{b}^{ik}|_{J_l}\,, \qquad \bar{b}^{ki}_l := \bar{b}^{ki}|_{J_l}\,,$$

and let $\beta^{ik}_l, \beta^{ki}_l < \kappa$ be the indices such that $\bar{b}^{ik}_l \subseteq X_{\beta^{ik}_l}$ and $\bar{b}^{ki}_l \subseteq X_{\beta^{ki}_l}$. Assume that we have chosen $\bar{b}^{ik}$ and $\bar{b}^{ki}$ such that the set

$$N := \{\, l \mid \bar{b}^{ik}_l = \bar{b}^{ki}_l \,\}$$

is maximal.

It follows that, for each $l \notin N$, we either have $\beta^{ik}_l = \alpha_k$ or there exists some index $\sigma(l) \neq l$ such that $\beta^{ik}_l = \beta^{ki}_{\sigma(l)}$. Otherwise, we could replace $\bar{b}^{ki}_l$ by $\bar{b}^{ik}_l$ and the resulting pair of tuples would still witness $\bar{a}^i \nsim \bar{a}^k$ in contradiction to the maximality of N.

Let $\sigma_{ik} : [s+1] \setminus N \to ([s+1] \setminus N) \cup \{*\}$ be the function such that

$$\beta^{ik}_l = \begin{cases} \alpha_k & \text{if } \sigma_{ik}(l) = *\,, \\ \beta^{ki}_{\sigma_{ik}(l)} & \text{otherwise}\,, \end{cases}$$

and define σ_{ki} analogously. The maximality of N further implies that there exists no sequence $l_0, \dots, l_t$ of indices such that $\sigma_{ik}(l_j) = l_{j+1}$, for $j < t$, and $\sigma_{ik}(l_t) = l_0$ since, otherwise, we could simultaneously replace each $\bar{b}^{ki}_{l_j}$ by $\bar{b}^{ik}_{l_j}$ and again obtain witnesses for $\bar{a}^i \nsim \bar{a}^k$ with strictly larger N.

It follows that $\beta^{ik}_l \in \{\alpha_k, \beta^{ki}_0, \dots, \beta^{ki}_s\}$, for every $l \notin N$, and there is some number j such that $\sigma^j_{ik}(l) = *$, i.e., $\beta^{ik}_{\sigma^{j-1}_{ik}(l)} = \alpha_k$.

For each pair $i < k$ of indices we record

- the partition $J_0 \uplus \cdots \uplus J_s$ of the indices of $\bar{b}^{ik}$ induced by $\sim$,
- the size $|N|$ of the set N defined above, and
- the functions σ_{ik} and σ_{ki}.

There exists a subset $I \subseteq \kappa$ of size

$$\begin{aligned} |I| &\geq m_1 := \max\{\, k \mid m \to (k)^2_{r^3r+1} \,\} \\ &\geq K_n = w_n(rw_r)^r + R(w_n + 2(r-n) + 2)^3_8 \end{aligned}$$

such that all pairs $i, k \in I$ with $i < k$ are coloured in the same way. W.l.o.g. we may assume that $I = [m_1]$.

(c) First, consider the case that $N = [s+1]$ for all $i, k \in I$. Let $B_{ik} \subseteq \kappa$ be the smallest set of indices such that $\bar{b}^{ik} = \bar{b}^{ki} \subseteq X_{B_{ik}}$. Clearly, $B_{ik} = B_{ki}$. Also note that, by definition of $\bar{b}^{ik}$ and $\bar{b}^{ki}$, we have $\alpha_i, \alpha_k \notin B_{ik}$. For each set $\{i, k, l\}$ of indices $i < k < l$, we record which of the following conditions hold:

$$\alpha_i \in B_{kl}\,, \quad \alpha_k \in B_{il}\,, \quad \alpha_l \in B_{ik}\,.$$

There exists a subset $I' \subseteq [m_1]$ of size

$$|I'| \geq m_2 := \max\{\, k \mid m_1 \to (k)^3_{2^3} \,\} \geq w_n + 2(r-n) + 2$$

such that all triples $i, k, l \in I'$ are coloured in the same way. W.l.o.g. we may assume that $I' = [m_2]$.

First we consider the case that $\alpha_l \in B_{ik}$ for all $i < k < l < m_2$. Then $\alpha_i \in B_{01}$, for $1 < i < m_2$. Furthermore, for $0 < i < k$, we have $\alpha_i \notin B_{0i}$ and $\alpha_k \in B_{0i} \setminus B_{0k}$ which implies that $\alpha_i \neq \alpha_k$. Hence,

$$m_2 \leq |B_{01}| + 2 \geq r - n + 2\,.$$

Contradiction. Analogously, if $\alpha_i \in B_{kl}$ or $\alpha_k \in B_{il}$, for $i < k < l$, then we obtain, respectively,

$$m_2 \leq |B_{m_2-2,m_2-1}| + 2 \quad \text{and} \quad m_2 \leq |B_{0,m_2-1}| + 2\,,$$

which lead to similar contradictions.

The only remaining case is that none of the above conditions holds, that is, we have $\alpha_i \notin B_{kl}$ for all pairwise distinct sets of indices i, k, l. Let $H := \{ \alpha_i \mid i < m_2 \}$. $\bar{b}^{ik} \subseteq U_H$ implies $\bar{a}^i \not\simeq^{\mathrm{o}}_{U_H} \bar{a}^k$, for all $i \neq k$. Consequently, we have

$$\mathrm{eti}^n_{\mathrm{o}}(X_H/U_H) \geq m_2 > w_n.$$

Contradiction.

(D) It remains to consider the case that $[s{+}1] \setminus N \neq \emptyset$. Let $l_{\mathrm{o}} \in \sigma_{1\mathrm{o}}^{-1}(*)$, i.e., $\beta^{ki}_{l_{\mathrm{o}}} = \alpha_i$, for all $i < k$, and define $l_{j+1} := \sigma_{\mathrm{o}1}(l_j)$. Let $l_{\mathrm{o}}, \dots, l_t$ be the sequence of indices obtained in this way where $l_t = *$. Note that, for $i < k$ and $j < t - 1$, we have $\beta^{ik}_{l_j} = \beta^{ki}_{\sigma_{\mathrm{o}1}(l_j)} = \beta^{ki}_{l_{j+1}}$. For notational convenience, we also set $\beta^{ki}_* := \beta^{ik}_{l_{t-1}} = \alpha_k$.

By induction on $j \leq t$, we construct a decreasing sequence of subsets $I_j \subseteq I$ of size

$$|I_j| \geq (|I| - 1)/(rw_r)^j$$

such that

$$\beta^{i\mathrm{o}}_{l_j} = \beta^{k\mathrm{o}}_{l_j} \quad \text{and} \quad f_{|J_{l_{j-1}}|}(\bar{b}^{\mathrm{o}i}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{\mathrm{o}k}_{l_{j-1}}) \quad \text{for all } i, k \in I_j\,.$$

For all indices $i, k \in I_t$ it follows that $\alpha_i = \beta^{i\mathrm{o}}_* = \beta^{k\mathrm{o}}_* = \alpha_k$. Since each tuple $\bar{a}^i$ has a different colour it further follows that $|I_t| \leq w_n$ which implies that

$$w_n \geq |I_t| \geq (|I| - 1)/(rw_r)^t > w_n\,.$$

Contradiction.

(E) We still have to construct the sets I_j. Let $I_{\mathrm{o}} := I \setminus \{\mathrm{o}\}$. Since $\beta^{i\mathrm{o}}_{l_{\mathrm{o}}} = \alpha_{\mathrm{o}} = \beta^{k\mathrm{o}}_{l_{\mathrm{o}}}$ our claim holds for $j = \mathrm{o}$. Suppose that $I_{\mathrm{o}}, \dots, I_{j-1}$ are already defined. Since $\beta^{i\mathrm{o}}_{l_{j-1}} = \beta^{k\mathrm{o}}_{l_{j-1}}$, for $i, k \in I_{j-1}$, there exists a subset $I'_j \subseteq I_{j-1}$ of size

$$|I'_j| \geq |I_{j-1}|/w_{|J_{l_{j-1}}|} \geq |I_{j-1}|/w_r$$

such that $f_{|J_{l_{j-1}}|}(\bar{b}^{i\mathrm{o}}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{k\mathrm{o}}_{l_{j-1}})$ for all $i, k \in I'_j$. It follows that

$$c := f_{|J_{l_{j-1}}|}(\bar{b}^{\mathrm{o}i}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{i\mathrm{o}}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{k\mathrm{o}}_{l_{j-1}}) = f_{|J_{l_{j-1}}|}(\bar{b}^{\mathrm{o}k}_{l_{j-1}})\,,$$

and, by the remarks in (B), we have $f^{-1}_{|J_{l_{j-1}}|}(c) \subseteq X_{\{\alpha_{\mathrm{o}}, \beta^{\mathrm{o}i}_{\mathrm{o}}, \dots, \beta^{\mathrm{o}i}_s\}}$. Therefore, there exists a subset $I_j \subseteq I'_j$ of size

$$|I_j| \geq |I'_j|/(s+2) \geq |I_{j-1}|/(rw_r) \geq (|I| - 1)/(rw_r)^j$$

such that $\beta^{oi}_{l_{j-1}} = \beta^{ok}_{l_{j-1}}$ for all $i, k \in I_j$. It follows that

$$\beta^{io}_{l_j} = \beta^{oi}_{l_{j-1}} = \beta^{ok}_{l_{j-1}} = \beta^{ko}_{l_j}$$

as desired. □

After these somewhat lengthy preparations we are finally able to prove that every structure denoted by a term has finite partition width and, conversely, every structure with finite partition width is denoted by a term.

Proposition 3.2.8. *Let C be a graded set of colours, τ a signature, and $n < \omega$.*

(1) $\mathrm{pwd}_n(\mathrm{val}(T), \kappa^{<\alpha}) < \aleph_0$ *for all $Y^<_{C,\tau}$-terms $T \subseteq \kappa^{<\alpha}$.*

(2) $\mathrm{spwd}_n(\mathrm{val}(T), \kappa^{<\alpha}) < \aleph_0$ *for every $Y_{C,\tau}$-term $T \subseteq \kappa^{<\alpha}$.*

Proof. (1) Consider the subterm T_v of T with root $v \in T$ and let U_v be the universe of $\mathrm{val}(T_v)$. We claim that $(U_v)_{v \in T}$ is the desired partition refinement.

Suppose that $\bar{a}, \bar{b} \in U_v^n$ are tuples such that, for all $I \subseteq [n]$, the subtuples $\bar{a}|_I$ and $\bar{b}|_I$ have the same colour at node v. Let $\varphi(\bar{x}, \bar{c})$ be an atomic formula with parameters $\bar{c} \subseteq \overline{U_v}$. If $\mathrm{val}(T) \vDash \varphi(\bar{a}, \bar{c})$ then there exists a node $u \prec v$ such that $\bar{a}, \bar{c} \subseteq U_u$ and the operation $\sum^\Theta$ at u adds all tuples with the colour of $(\bar{a}\bar{c})|_I$ to the relation in φ where I is the set of those indices that actually appear in φ. Since $(\bar{b}\bar{c})|_I$ has the same colour it follows that also $\mathrm{val}(T) \vDash \varphi(\bar{b}, \bar{c})$. Consequently, we have $\bar{a} \simeq^{\mathrm{o}}_{U_v} \bar{b}$.

(2) Define $(U_v)_v$ as above. By the preceding proposition, we have

$$\mathrm{eti}^n_{\mathrm{o}}\big(\textstyle\bigcup_{i \in I} U_{vi} \,/\, \overline{\bigcup_{i \in I} U_{vi}}\big) \leq n^{n+1}\big(\max_{i \leq n} |C_n|\big)^n.$$ □

Remark. Note that, for the case $n = 1$, the proof above implies that $\mathrm{pwd}_1(\mathrm{val}(T), \kappa^{<\alpha}) \leq |C_1|$.

Proposition 3.2.9. *Let $\mathfrak{M}$ be a τ-structure.*

(1) *Let $k < \aleph_0$. For every $k^{<\alpha}$-partition refinement $(U_v)_{v \in S}$ of $\mathfrak{M}$ of finite partition width, there exists a $Y^<_{C,\tau}$-term $T \subseteq k^{<\alpha}$ denoting $\mathfrak{M}$ where C is a set of colours with $|C_n| \leq \mathrm{pwd}_n(U_v)_v$ for $n < \omega$.*

(2) *If the arity of $\mathfrak{M}$ is finite and there exists a $\kappa^{<\alpha}$-partition refinement $(U_v)_{v \in S}$ of $\mathfrak{M}$ such that $\mathrm{spwd}_n(U_v)_v < \aleph_0$ for all n, then there is a $Y_{C,\tau}$-term $T \subseteq \kappa^{<\alpha}$ denoting $\mathfrak{M}$ for some set of colours C.*

Proof. (1) Let $w_n := \mathrm{pwd}_n(U_v)_v$. Let $T := S \cup \{\, w\mathrm{o} \mid w \text{ leaf of } S \,\}$ be the tree obtained from S by adding to every leaf of S a new vertex as successor. We construct a $Y^<_{C,\tau}$-term with domain T such that, for every $v \in S$, the subterm $T_v := \{\, w \in T \mid w \geq v \,\}$ will evaluate to the substructure $\mathfrak{M}|_{U_v}$ of $\mathfrak{M}$ induced by U_v.

In a first step, each such component U_ν will be coloured by a different set C^ν of colours with $|C^\nu_n| \leq w_n$. To obtain a single set of colours C we then define injective functions $\mu^\nu_n : C^\nu_n \to [w_n]$ and identify colours $c \in C^u_n$ and $d \in C^\nu_n$ iff $\mu^u_n(c) = \mu^\nu_n(d)$.

Colour each tuple $\bar{a} \subseteq U_\nu$ by its external type $\mathrm{etp}_\mathrm{o}(\bar{a}/\overline{U_\nu})$. If $\bar{a}_i \subseteq U_{\nu i}$, for $i < k$, then the type $\mathrm{etp}_\mathrm{o}(\bar{a}_\mathrm{o} \dots \bar{a}_{k-1}/\overline{U_\nu})$ is uniquely determined by $\mathrm{etp}_\mathrm{o}(\bar{a}_i/\overline{U_{\nu i}})$ for $i < k$. Hence, these colourings χ_ν enable us to express U_ν as ordered sum of the $U_{\nu i}$

$$(\mathfrak{M}|_{U_\nu}, \chi_\nu) = \sum_{i<k}^{\Theta_\nu} (\mathfrak{M}|_{U_{\nu i}}, \chi_{\nu i})$$

for a suitable set Θ_ν.

For non-leaves $\nu \in S$, we define the labelling of T by $T(\nu) := \sum^{\Theta_\nu}$. Then we have $T_\nu = \sum_{i<k}^{\Theta_\nu} T_{\nu i}$.

For leaves $\nu \in S$ with $U_\nu = \{a\}$ we set $T(\nu) := \sum^\Theta$ and $T(\nu\mathrm{o}) := \bar{c}$, i.e., $T_\nu = \sum^\Theta \bar{c}$, where $c_n := \mathrm{etp}_\mathrm{o}(a^n/M \setminus \{a\})$ and

$$\Theta := \{ (n, 1, [n], c_n, c_n, S_n) \mid n < \omega \}$$

with $S_n := \{ R \mid a^n \in R \}$.

It remains to define the functions $\mu^\nu_n : C^\nu_n \to [w_n]$ such that the resulting term $T := T_\varepsilon$ is well-formed. For $\nu \in T$, we denote by $\nu_\beta \leq \nu$ the prefix of ν of length $|\nu_\beta| = \beta$ and, for each type $p \in C^\nu_n$ over $\overline{U_\nu}$, we denote by p_β its restriction to $\overline{U_{\nu_\beta}}$.

For T to be well-formed it is sufficient to define μ^ν_n such that

- for each $p \in C^\nu_n$, the sequence $(\mu^{\nu_\beta}_n(p_\beta))_{\beta<|\nu|+1}$ forms a colour trace to ν;
- the colour traces to ν are linearly ordered.

We define μ^ν_n by induction on $|\nu|$. Let μ^ε_n be an arbitrary injective function $C^\varepsilon_n \to [w_n]$. (Note that $|C^\varepsilon_n| = 1$ since there is only one external type over the empty set.) Suppose that μ^u_n is already defined for all $|u| < \alpha$ and let $|\nu| = \alpha$.

First, consider the case that $\alpha = \beta + 1$ is a successor. Set $u := \nu_\beta$ and let $<$ be the ordering on C^u_n induced by the function μ^u_n. We order C^ν_n in the following way. If $p_\beta < p'_\beta$, for $p, p' \in C^\nu_n$, then we set $p < p'$ and, if $p_\beta = p'_\beta$, then we choose an arbitrary ordering between them. Finally, let μ^ν_n be some injective order preserving function $C^\nu_n \to [w_n]$.

It remains to consider limit ordinals α. Let $p \in C^\nu_n$ and let c be the minimal number such that the set $\{ \beta < \alpha \mid \mu^{\nu_\beta}_n(p_\beta) = c \}$ is unbounded. We set $\mu^\nu_n(p) := c$.

With these definitions, $(\mu^{\nu_\beta}_n p_\beta)_\beta$ satisfies both conditions on a colour trace, and we have ensured that all colour traces to some node ν are linearly ordered.

(2) In the symmetric case the proof is analogous except that, according to the above proposition, we have to use a suitable refinement of the colouring given by the external types. This poses no problem since the number of additional n-ary colours only depends on the arity of $\mathfrak{M}$ and $\mathrm{spwd}_i(U_\nu)_\nu$, for $i < \omega$, so the bound $\sup\{\, |C_i^\nu| \mid \nu \in S\}$ remains finite. □

We claimed above that partition width generalises the notion of clique-width or NLC-width. This is justified by the following lemma.

Lemma 3.2.10. *Let $\mathfrak{G} = (V, E)$ be a countable undirected graph of NLC-width k.*

$$\mathrm{pwd}_1(\mathfrak{G}, 2^{<\omega}) \leq k \leq \mathrm{cwd}\,\mathfrak{G} \leq 2 \cdot \mathrm{pwd}_1(\mathfrak{G}, 2^{<\omega})\,.$$

Proof. One direction follows since VR- and NLC-operations can be expressed by suitable $\Upsilon^{\leq}_{C,\tau}$-terms using the same set of colours. For the other one, fix a $\Upsilon^{\leq}_{C,\tau}$-term T denoting $\mathfrak{G}$ with $n := |C_1|$ colours of arity 1. We construct a VR-term using colours $[2n]$.

For $w \in 2^{<\omega}$, let T_w be the subterm of T with root w and let U_w be the universe of $\mathrm{val}(T_w)$. For every injective mapping φ of the atomic external 1-types over $\overline{U_w}$ realised in U_w into the set $[2n]$, we will construct a VR-term t_w^φ that denotes $\mathrm{val}(T_w)$ such that the colouring of elements $a \in U_w$ is the one induced by φ.

If w is a leaf with $U_w = \{a\}$ then we set

$$t_w^\varphi := \varphi\big(\mathrm{etp}_0(a/V \smallsetminus \{a\})\big)\,.$$

Otherwise, $T_w = T_{w0} +^\Theta T_{w1}$, and we set

$$t_w^\varphi := \rho_\beta \mathrm{add}(t_{w0}^{\psi_0} + t_{w1}^{\psi_1})$$

where ψ_0 and ψ_1 are mappings with disjoint ranges, β maps the colours induced by ψ_0 and ψ_1 to the ones required by φ, and add is a sequence of operations $\alpha_{a,b}$ adding all the necessary edges. □

3.3 The type equivalence

Before proceeding we need to collect some basic properties of type indices. In the following lemmas let $\mathfrak{M}$ be a fixed relational structure.

Recall that, when speaking of the quantifier rank of monadic second-order formulae, we consider the variant of MSO without first-order variables where the atomic formulae are of the form $X \subseteq Y$ and $R\bar{X}$, where the latter means that there exist some elements $a_i \in X_i$ such that $\bar{a} \in R$.

The first lemma summarises some immediate relations between the various kinds of type indices.

Lemma 3.3.1. *Let $X, U \subseteq M$ and $\bar{a}, \bar{b} \in M^n$.*

(1) *If $m \leq n$ and $\Gamma \subseteq \Delta$ then $\mathrm{ti}^m_\Gamma(X/U) \leq \mathrm{ti}^n_\Delta(X/U)$ and analogously for the external and monadic case.*

(2) $\mathrm{eti}^n_0(X/U) \leq \mathrm{ti}^n_0(X/U) \leq |S^n_0(\varnothing)| \cdot \mathrm{eti}^n_0(X/U)$,
$\mathrm{emti}^n_0(X/U) \leq \mathrm{mti}^n_0(X/U) \leq |MS^n_0(\varnothing)| \cdot \mathrm{emti}^n_0(X/U)$.

(3) $a_0 \dots a_{n-1} \approx^0_U b_0 \dots b_{n-1}$ iff $\{a_0\} \dots \{a_{n-1}\} \approx^0_U \{b_0\} \dots \{b_{n-1}\}$.

(4) *If the arity of $\mathfrak{M}$ is bounded by r then*

$$\mathrm{eti}^n_0(X/U) \leq \left(\mathrm{eti}^{r-1}_0(X/U)\right)^{2^n}.$$

Proof. (1) $\bar{a} \approx^\Delta_U \bar{b}$ implies $\bar{a}|_I \approx^\Gamma_U \bar{b}|_I$ for all sets of indices I.
(2) $\bar{a} \approx^0_U \bar{b}$ iff $\bar{a} \simeq^0_U \bar{b}$ and $\mathrm{tp}_0(\bar{a}) = \mathrm{tp}_0(\bar{b})$.
(3) For singletons $X_i = \{a_i\}$ we have $R\bar{X}$ iff $R\bar{a}$.
(4) Let $\bar{a}, \bar{b} \in X^n$ such that $\bar{a}|_I \simeq^0_U \bar{b}|_I$ for all $I \subseteq [n]$ of size $|I| < r$. If $\bar{a} \not\simeq^0_U \bar{b}$ then there is some atomic formula $\varphi(\bar{x}; \bar{c})$ with $\bar{c} \subseteq U$ such that

$$\mathfrak{M} \models \varphi(\bar{a}; \bar{c}) \leftrightarrow \neg\varphi(\bar{b}; \bar{c}).$$

Let $I \subseteq [n]$ be the set of those indices i such that the variable x_i appears in φ. Then $|I| < r$ and $\bar{a}|_I \not\simeq^0_U \bar{b}|_I$. Contradiction.

Since there are

$$\sum_{i=0}^{r-1} \binom{n}{i} \leq \sum_{i<n} \binom{n}{i} = 2^n$$

subsets of $[n]$ of size less than r the claim follows. □

Frequently, one would like to compute the type index of a boolean combination of sets from their respective type indices. For arbitrary structures this is only possible in special cases and even then quite complicated. In the case of transition systems, the situation is much simpler since, if all relations are at most binary, we have

$$\bar{a} \simeq^0_U \bar{b} \quad \text{iff} \quad a_i \simeq^0_c b_i \quad \text{for every } i \text{ and all } c \in U.$$

Lemma 3.3.2. *Let $\mathfrak{M}$ be a transition system with r binary relations. Let $X, Y, U \subseteq M$.*

(1) $\mathrm{eti}^1_0(X \cup Y/U) \leq \mathrm{eti}^1_0(X/U) + \mathrm{eti}^1_0(Y/U)$.

(2) $\mathrm{eti}^1_0(X \cap Y/\overline{X \cap Y}) \leq \mathrm{eti}^1_0(X/\overline{X}) \cdot \mathrm{eti}^1_0(Y/\overline{Y})$.

(3) $\mathrm{eti}^1_0(X \setminus Y/\overline{X \setminus Y}) \leq \mathrm{eti}^1_0(X/\overline{X}) 4^{r\,\mathrm{eti}^1_0(Y/\overline{Y})}$.

(4) $\mathrm{eti}^1_0(U/X) \leq 4^{r\,\mathrm{eti}^1_0(X/U)}$.

Proof. (1) Immediate.

(2) Let $a, b \in X \cap Y$. If $a \simeq^{\mathrm{o}}_{\overline{X}} b$ and $a \simeq^{\mathrm{o}}_{\overline{Y}} b$ then $a \simeq^{\mathrm{o}}_{\overline{X} \cup \overline{Y}} b$.

(3) By (2) we have $\mathrm{eti}^1_{\mathrm{o}}(X \cap \overline{Y}/\overline{X} \cup Y) \leq \mathrm{eti}^1_{\mathrm{o}}(X/\overline{X})\mathrm{eti}^1_{\mathrm{o}}(\overline{Y}/Y)$. The claim follows by (4).

(4) Let $m := \mathrm{eti}^1_{\mathrm{o}}(X/U)$ and fix representatives c_i, $i < m$, of the classes in $X/\simeq^{\mathrm{o}}_U$. By the above remark we have, for $a, b \in U$,

$$a \simeq^{\mathrm{o}}_X b \quad \text{iff} \quad a \simeq^{\mathrm{o}}_{c_i} b \quad \text{for all } i < m.$$

There are only $2r$ different atomic formulae $\varphi(x, y)$ containing both variables. Therefore, there are only 2^{2r} possible $\simeq^{\mathrm{o}}_{c_i}$-classes for each i and the claim follows. □

Remark. If $\mathfrak{M}$ is a transition system with r symmetric relations and s asymmetric ones, then (d) can be improved to

$$\mathrm{eti}^1_{\mathrm{o}}(U/X) \leq 2^{(r+2s)\,\mathrm{eti}^1_{\mathrm{o}}(X/U)}.$$

The general case is much more complicated. For instance, we can construct a structure $\mathfrak{M}$ such that $\mathrm{pwd}_n \mathfrak{M} \geq \aleph_0$, for all n, but there exists a single element $v \in M$ such that $\mathrm{pwd}_n \mathfrak{M}|_{M \setminus v} = 1$ for all $n < \omega$:

Let $(\mathbb{Z} \times \mathbb{Z}, E)$ be the infinite grid, and let v be a new vertex. We can set $\mathfrak{M} := (M, R)$ where

$$M := \mathbb{Z} \times \mathbb{Z} \cup \{v\}$$
$$\text{and} \quad R := \{\, (a, b, v) \mid (a, b) \in E \,\}.$$

Nevertheless, some results can be obtained.

Lemma 3.3.3. *Let $X, Y \subseteq M$ and $n < \omega$.*

(1) $$\mathrm{ti}^n_\Delta(X \cup Y/\overline{X \cup Y}) \leq \sum_{i \leq n} \binom{n}{i} \mathrm{ti}^i_\Delta(X/\overline{X})\, \mathrm{ti}^{n-i}_\Delta(Y \setminus X/\overline{Y \setminus X})$$
$$\leq 2^n\, \mathrm{ti}^n_\Delta(X/\overline{X})\, \mathrm{ti}^n_\Delta(Y \setminus X/\overline{Y \setminus X}).$$

The same holds for eti^n_Δ.

(2) $$\mathrm{eti}^n_{\mathrm{o}}(X \cup Y/\overline{X \cup Y})$$
$$\leq \sum_{n_{\mathrm{o}}+n_1+n_2=n} R\big(\mathrm{eti}^{n_{\mathrm{o}}+n_1}_{\mathrm{o}}(X/\overline{X}), \mathrm{eti}^{n_{\mathrm{o}}+n_2}_{\mathrm{o}}(Y/\overline{Y})\big).$$

where $R(m, n) := nR(3)^2_m - 1$.

Proof. (1) The second inequality holds by Lemma 3.3.1 (1). To prove the first one, let $i < n$, $\bar{a}, \bar{a}' \in X^i$, and $\bar{b}, \bar{b}' \in (Y \setminus X)^{n-i}$. Set $U := \overline{X \cup Y}$. We claim that

$$\bar{a} \approx^{\Delta}_{U \cup \bar{b}'} \bar{a}' \text{ and } \bar{b} \approx^{\Delta}_{U \cup \bar{a}} \bar{b}' \quad \text{implies} \quad \bar{a}\bar{b} \approx^{\Delta}_U \bar{a}\bar{b}' \approx^{\Delta}_U \bar{a}'\bar{b}'.$$

Suppose for a contradiction that $\bar{a}\bar{b} \not\simeq^{\mathrm{o}}_{U} \bar{a}'\bar{b}'$. There exists some atomic formula $\varphi(\bar{x}, \bar{y}; \bar{c}) \in \Delta$ with parameters $\bar{c} \subseteq U$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}, \bar{b}; \bar{c}) \leftrightarrow \neg\varphi(\bar{a}', \bar{b}'; \bar{c})\,.$$

But $\bar{b} \approx^{\Delta}_{U\cup\bar{a}} \bar{b}'$ implies that

$$\mathfrak{M} \vDash \varphi(\bar{a}, \bar{b}; \bar{c}) \leftrightarrow \varphi(\bar{a}, \bar{b}'; \bar{c})\,,$$

and $\bar{a} \approx^{\Delta}_{U\cup\bar{b}'} \bar{a}'$ implies that

$$\mathfrak{M} \vDash \varphi(\bar{a}, \bar{b}'; \bar{c}) \leftrightarrow \varphi(\bar{a}', \bar{b}'; \bar{c})\,.$$

Contradiction. The result follows since there are $\binom{n}{i}$ possible ways to shuffle an i-tuple and an $(n-i)$-tuple.

(2) Let $n_0+n_1+n_2 = n$ and $w_1 := \mathrm{eti}^{n_0+n_1}_{\mathrm{o}}(X/\overline{X}), w_2 := \mathrm{eti}^{n_0+n_2}_{\mathrm{o}}(Y/\overline{Y})$. It is sufficient to show that there is no sequence $(\bar{a}^i\bar{c}^i\bar{b}^i)_{i<m}$ of length $m > R(w_1, w_2)$ with $\bar{a}^i \in (X \setminus Y)^{n_1}$, $\bar{b}^i \in (Y \setminus X)^{n_2}$, and $\bar{c}^i \in (X \cap Y)^{n_0}$ such that

$$\bar{a}^i\bar{c}^i\bar{b}^i \not\simeq^{\mathrm{o}}_{\overline{X}\cap\overline{Y}} \bar{a}^k\bar{c}^k\bar{b}^k \qquad \text{for all } i, k < m.$$

Suppose that $m > R(w_1, w_2)$, i.e., $[{}^{m}\!/_{w_2}] \to (3)^2_{w_1}$. There is a subset $I \subseteq [m]$ of indices of size $|I| \geq {}^{m}\!/_{w_2}$ such that

$$\bar{c}^i\bar{b}^i \simeq^{\mathrm{o}}_{\overline{Y}} \bar{c}^k\bar{b}^k \qquad \text{for all } i, k \in I\,.$$

If we colour pairs $\{i, k\} \subseteq I$ of indices, $i < k$, by the $\simeq^{\mathrm{o}}_{\overline{X}}$-class of $\bar{a}^i\bar{c}^k$, then we can find a subsequence $I_0 \subseteq I$ of size $|I_0| \geq 3$ such that

$$\bar{a}^i\bar{c}^k \simeq^{\mathrm{o}}_{\overline{X}} \bar{a}^j\bar{c}^l \qquad \text{for all } i, j, k, l \in I_0 \text{ such that } i < k \text{ and } j < l.$$

W.l.o.g. assume that $0, 1, 2 \in I_0$. It follows that

$$\bar{a}^0\bar{c}^0\bar{b}^0 \simeq^{\mathrm{o}}_{\overline{X}\cap\overline{Y}} \bar{a}^0\bar{c}^2\bar{b}^2 \simeq^{\mathrm{o}}_{\overline{X}\cap\overline{Y}} \bar{a}^1\bar{c}^2\bar{b}^2 \simeq^{\mathrm{o}}_{\overline{X}\cap\overline{Y}} \bar{a}^1\bar{c}^1\bar{b}^1\,.$$

Contradiction. □

Lemma 3.3.4. *Let $\mathfrak{M}$ be a relational structure, $X, U \subseteq M$. Let m be the number of relations of arity greater than 1 and let r be the supremum of their arities.*

$$\mathrm{eti}^n_{\mathrm{o}}(U/X) \leq 2^{m(n+1)^r \mathrm{eti}^{r-1}_{\mathrm{o}}(X/U)}\,.$$

Proof. Let $\bar{a}, \bar{a}' \in U^n$. We have $\bar{a} \simeq^{\mathrm{o}}_{X} \bar{a}'$ iff

$$\mathfrak{M} \vDash \varphi(\bar{a}, \bar{b}) \leftrightarrow \varphi(\bar{a}', \bar{b})$$

for all $\bar{b} \subseteq X$ and for all atomic formulae $\varphi(\bar{x}, \bar{y})$ containing at least one x_i and one y_j. Obviously, we only need to consider tuples $\bar{b}$ of less than r elements. Also note that, if $\bar{b} \simeq^{\mathrm{o}}_U \bar{b}'$, then $\mathfrak{M} \models \varphi(\bar{a}, \bar{b})$ iff $\mathfrak{M} \models \varphi(\bar{a}, \bar{b}')$. Hence, it is sufficient to take one representative of each $\simeq^{\mathrm{o}}_U$-class. Finally, if $\varphi'(\bar{x}, \bar{y})$ is obtained from $\varphi(\bar{x}, \bar{y})$ by a permutation of $\bar{y}$, then $\mathfrak{M} \models \varphi(\bar{a}, \bar{b})$ iff $\mathfrak{M} \models \varphi'(\bar{a}, \bar{b}')$ where $\bar{b}'$ is the corresponding permutation of $\bar{b}$. Thus, we can ignore the ordering of the variables $\bar{y}$. The claim follows since there are at most $m(n+1)^r$ atomic formulae with variables $\bar{x}\bar{y}$ and the number $\simeq^{\mathrm{o}}_U$-classes is $\mathrm{eti}^{r-1}_{\mathrm{o}}(X/U)$. □

Lemma 3.3.5. *Let $X, U \subseteq M$, $\bar{c} \in M^k$, and $n < \omega$.*

$$\mathrm{ti}^n_{\mathrm{o}}(X/U \cup \bar{c}) \leq \mathrm{ti}^{n+k}_{\mathrm{o}}(X \cup \bar{c}/U)\,.$$

Proof. Note that $\bar{a} \not\approx^{\mathrm{o}}_{U \cup \bar{c}} \bar{b}$ implies $\bar{a}\bar{c} \not\approx^{\mathrm{o}}_U \bar{b}\bar{c}$. □

In the definition of partition width we only considered atomic formulae. This is no restriction as the type indices of formulae of higher quantifier rank are bounded by the quantifier-free ones.

Lemma 3.3.6. *Let $\mathfrak{M}$ be a structure, $X \subseteq M$, and $n, k < \omega$.*

(1) $\mathrm{eti}^n_k(X/\overline{X}) \leq \beth_k(\mathrm{eti}^{n+k}_{\mathrm{o}}(X/\overline{X}))$.
(2) $\mathrm{ti}^n_k(X/\overline{X}) \leq \beth_k(\mathrm{ti}^{n+k}_{\mathrm{o}}(X/\overline{X}))$.
(3) $\mathrm{mti}^n_k(X/\overline{X}) \leq \beth_k(\mathrm{mti}^{n+k}_{\mathrm{o}}(X/\overline{X}))$.
(4) $\mathrm{emti}^n_k(X/\overline{X}) \leq \beth_k(\mathrm{emti}^{n+k}_{\mathrm{o}}(X/\overline{X}))$.

Proof. Since the proofs are very similar we only show a strong version of (3). Let $\Delta(k)$ be the fragment of *infinitary* monadic second-order logic consisting of all formulae of quantifier rank at most k. We prove that $\mathrm{mti}^n_{\Delta(k+1)}(X/\overline{X}) \leq 2^{\mathrm{mti}^{n+1}_{\Delta(k)}(X/\overline{X})}$.

For $\bar{A}, \bar{A}' \in \wp(X)^n$ we have

$$\bar{A} \approx^{\Delta(k+1)}_{\overline{X}} \bar{A}' \quad \text{iff} \quad \text{for all } B \text{ there is some } B' \text{ with } \bar{A}B \approx^{\Delta(k)}_{\overline{X}} \bar{A}'B' \text{ and vice versa}\,.$$

Since $\bar{A}B \approx^{\Delta(k)}_{\overline{X}} \bar{A}'B'$ iff $\bar{A}(B \cap X) \approx^{\Delta(k)}_{\overline{X}} \bar{A}'(B' \cap X)$ and $B \setminus X = B' \setminus X$, we only need to consider sets $B \subseteq \overline{X}$. Defining

$$e(\bar{A}) := \left\{ [\bar{A}B] \in \wp(X)^{n+1} / \approx^{\Delta(k)}_{\overline{X}} \;\middle|\; B \subseteq X \right\}$$

we obtain $\bar{A} \approx^{\Delta(k+1)}_{\overline{X}} \bar{A}'$ iff $e(\bar{A}) = e(\bar{A}')$. It follows that

$$\begin{aligned} \mathrm{mti}^n_{\Delta(k+1)}(X/\overline{X}) &= \left| \wp(X)^n \,/\, \approx^{\Delta(k+1)}_{\overline{X}} \right| \\ &\leq \left| \wp\big(\wp(X)^{n+1} \,/\, \approx^{\Delta(k)}_{\overline{X}}\big) \right| = 2^{\mathrm{mti}^{n+1}_{\Delta(k)}(X/\overline{X})}\,. \end{aligned}$$

□

The next result shows that having finite partition width is a finitary condition. This is the reason for the various compactness properties of Section 3.6.

Lemma 3.3.7. *Let $X, U \subseteq M$, $\Delta \subseteq \mathrm{FO}$, and $n < \omega$.*

(1) *Let $\bar{a}, \bar{b} \subseteq M$. If $\bar{a} \not\approx^{\Delta}_{U} \bar{b}$ then there is a finite subset $U_0 \subseteq U$ and a single formula $\varphi \in \Delta$ such that $\bar{a} \not\approx^{\varphi}_{U_0} \bar{b}$. The same holds for $\simeq^{\Delta}_{U}$.*

(2) *If $\mathrm{ti}^n_\Delta(X/U)$ is finite then there are finite subsets $U_0 \subseteq U$ and $\Delta_0 \subseteq \Delta$ such that $X^n/{\approx^{\Delta}_{U}} = X^n/{\approx^{\Delta_0}_{U_0}}$. The same holds for eti^n_Δ and $\simeq^{\Delta}_{U}$.*

(3) *If $\mathrm{eti}^n_\Delta(X/U)$ is finite then*

$$\mathrm{eti}^n_\Delta(X/U) = \sup\{\, \mathrm{eti}^n_{\Delta_0}(X/U) \mid \Delta_0 \subseteq \Delta \text{ finite} \,\}\,.$$

(4) *If $\mathrm{ti}^n_\Delta(X/U)$ is finite then the relation $\approx^{\Delta}_{U}$ is $\mathcal{B}(\Delta)$-definable on X^n. ($\mathcal{B}(\Delta)$ is the boolean closure of Δ.)*

Proof. (1) If $\bar{a} \not\approx^{\Delta}_{U} \bar{b}$ then there is some formula $\varphi(\bar{x}, \bar{c}) \in \Delta$ with $\bar{c} \subseteq U$ such that $\mathfrak{M} \models \varphi(\bar{a}, \bar{c}) \leftrightarrow \neg\varphi(\bar{b}, \bar{c})$. Setting $U_0 := \bar{c}$ we obtain $\bar{a} \not\approx^{\varphi}_{U_0} \bar{b}$.

(2) According to (1) there are finite sets $U_{[\bar{a}][\bar{b}]}$ and formulae $\varphi_{[\bar{a}][\bar{b}]}$, for each pair of distinct classes $[\bar{a}], [\bar{b}] \in X^n/{\approx^{\Delta}_{U}}$, such that $\bar{a} \not\approx^{\varphi_{[\bar{a}][\bar{b}]}}_{U_{[\bar{a}][\bar{b}]}} \bar{b}$. Setting $U_0 := \bigcup_{[\bar{a}]\neq[\bar{b}]} U_{[\bar{a}][\bar{b}]}$ and $\Delta_0 := \{\, \varphi_{[\bar{a}][\bar{b}]} \mid [\bar{a}] \neq [\bar{b}] \,\}$ we obtain

$$\bar{a} \approx^{\Delta}_{U} \bar{b} \quad \text{iff} \quad \bar{a} \approx^{\Delta_0}_{U_0} \bar{b} \qquad \text{for all } \bar{a}, \bar{b} \in X^n\,.$$

(3) immediately follows from (2).

(4) For each pair $[\bar{a}], [\bar{b}] \in X^n/{\approx^{\Delta}_{U}}$ of distinct classes we fix a Δ-formula $\varphi_{[\bar{a}][\bar{b}]}(\bar{x}, \bar{y})$ and parameters $\bar{c}_{[\bar{a}][\bar{b}]}$ such that

$$\mathfrak{M} \models \varphi_{[\bar{a}][\bar{b}]}(\bar{a}, \bar{c}_{[\bar{a}][\bar{b}]}) \leftrightarrow \neg\varphi_{[\bar{a}][\bar{b}]}(\bar{b}, \bar{c}_{[\bar{a}][\bar{b}]})\,.$$

Then we have $\bar{a} \approx^{\Delta}_{U} \bar{a}'$ iff

$$\mathfrak{M} \models \bigwedge_{[\bar{b}]\neq[\bar{b}']} \big(\varphi_{[\bar{b}][\bar{b}']}(\bar{a}, c_{[\bar{b}][\bar{b}']}) \leftrightarrow \varphi_{[\bar{b}][\bar{b}']}(\bar{a}', c_{[\bar{b}][\bar{b}']})\big)\,. \qquad \square$$

Lemma 3.3.8. *Let $\bar{w} \in \omega^\omega$. Let $(X_\nu)_{\nu\in I}$ be an increasing chain of sets X_ν (i.e., $u \leq \nu$ implies $X_u \subseteq X_\nu$) indexed by an arbitrary linear order $(I, \leq)$ such that $\mathrm{eti}^n_0(X_\nu/\overline{X_\nu}) \leq w_n$ for all $n < \omega$.*

$$\mathrm{eti}^n_0\Big(\bigcup_{\nu\in I} X_\nu \,\Big/\, \overline{\bigcup_{\nu\in I} X_\nu}\Big) \leq w_n \quad \text{and} \quad \mathrm{eti}^n_0\Big(\bigcap_{\nu\in I} X_\nu \,\Big/\, \overline{\bigcap_{\nu\in I} X_\nu}\Big) \leq w_n\,.$$

Proof. For the first claim, let $W := \bigcup_{\nu\in I} X_\nu$. Suppose there are $w_n + 1$ tuples $\bar{a}_i \in W^n$, $i \leq w_n$, such that $\bar{a}_i \not\simeq^{0}_{\overline{W}} \bar{a}_k$ for $i \neq k$. There exists some $\nu \in I$ with $\bar{a}_i \subseteq X_\nu$ for all $i \leq w_n$. Hence,

$$\mathrm{eti}^n_0(X_\nu/\overline{X_\nu}) \geq \mathrm{eti}^n_0(X_\nu/\overline{W}) \geq w_n + 1\,.$$

Contradiction.

To prove the second bound, set $W := \bigcap_{v \in I} X_v$. Suppose there are $w_n + 1$ tuples $\bar{a}_i \in W^n$, $i \leq w_n$, such that $\bar{a}_i \not\approx^{\mathrm{o}}_{\overline{W}} \bar{a}_k$ for $i \neq k$. By the preceding lemma, there exist finite sets $U_{ik} \subseteq \overline{W}$, $i \neq k$, such that $\bar{a}_i \not\approx^{\mathrm{o}}_{U_{ik}} \bar{a}_k$ for $i \neq k$. Since $U := \bigcup_{i \neq k} U_{ik}$ is finite there is some $v \in I$ with $U \subseteq \overline{X_v}$. As $\bar{a}_i \subseteq X_v$ for all $i \leq w_n$ it follows that

$$\mathrm{eti}^n_{\mathrm{o}}(X_v/\overline{X_v}) \geq \mathrm{eti}^n_{\mathrm{o}}(X_v/U) \geq w_n + 1 .$$

Contradiction. □

3.4 MSO-FUNCTORS

In this section we investigate the effect the MSO-functors of Section 1.3 have on partition width. First, note that adding unary predicates $\bar{P}$ to a structure does not change the partition width since $\mathrm{etp}_\Delta(\bar{a}/U)$ does not contain formulae of the form Px_i, and $\mathrm{emtp}_\Delta(\bar{A}/U)$ no formulae PX_i.

Lemma 3.4.1. *Let $X, U \subseteq M$. $\mathrm{eti}^n_\Delta(X/U)$ and $\mathrm{emti}^n_\Delta(X/U)$ do not change if we add arbitrarily many unary predicates to $\mathfrak{M}$.*

Corollary 3.4.2. *If $\mathfrak{M}$ is a structure and $\bar{P}$ a sequence of unary predicates then $\mathrm{pwd}_n(\mathfrak{M}, \kappa^{\leq\alpha}) = \mathrm{pwd}_n((\mathfrak{M}, \bar{P}), \kappa^{\leq\alpha})$.*

We have already seen that every structure denoted by a $\Upsilon_{C,\tau}$-term and, hence, every structure of finite partition width can be interpreted in some tree. To prove the converse we need to compare the type indices of one structures interpretable in another one.

Lemma 3.4.3. *Let $\mathcal{I} : \mathfrak{M} \leq_{\mathrm{MSO}_k} \mathfrak{N}$. For all $\bar{A}, \bar{B} \subseteq \wp(N)$, $U \subseteq N$, and $n < \omega$, we have*

$$\bar{A} \approx^{n+k}_U \bar{B} \quad \textit{implies} \quad \mathcal{I}(\bar{A}) \approx^n_{\mathcal{I}(U)} \mathcal{I}(\bar{B}) .$$

Proof. Suppose $\mathcal{I}(\bar{A}) \not\approx^n_{\mathcal{I}(U)} \mathcal{I}(\bar{B})$. There exists an MSO_n-formula $\varphi(\bar{x}, \bar{C})$ with parameters $\bar{C} \subseteq \wp(\mathcal{I}(U))$ such that

$$\mathfrak{M} \models \varphi(\mathcal{I}(\bar{A}), \bar{C}) \wedge \neg\varphi(\mathcal{I}(\bar{B}), \bar{C}) .$$

Choose $\bar{D} \subseteq \wp(U)$ such that $\bar{C} = \mathcal{I}(\bar{D})$. Then

$$\mathfrak{N} \models \varphi^{\mathcal{I}}(\bar{A}, \bar{D}) \wedge \neg\varphi^{\mathcal{I}}(\bar{B}, \bar{D}) .$$

Since $\varphi^{\mathcal{I}} \in \mathrm{MSO}_{n+k}$ we have $\bar{A} \not\approx^{n+k}_U \bar{B}$. □

Corollary 3.4.4. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be structures of finite signature and $\mathcal{I} : \mathfrak{M} \leq_{\mathrm{MSO}_k} \mathfrak{N}$. If $\mathrm{mpwd}_n(\mathfrak{N}, \kappa^{<\alpha})$ is finite for all $n < \omega$ then so is $\mathrm{mpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$. The same holds for $\mathrm{smpwd}_n(\mathfrak{N}, \kappa^{<\alpha})$.*

Proof. Let $(U_v)_v$ be a partition refinement of $\mathfrak{N}$ of finite width. The preceding lemma implies, together with Lemmas 3.3.1 (2) and 3.3.6, that the partition refinement $(\mathcal{I}(U_v))_v$ of $\mathfrak{M}$ also has finite width. □

We are now ready to give a characterisation of the class of structures of finite partition width in terms of interpretations in trees. One direction was already presented in Proposition 3.1.9. The other one is a direct consequence of Lemma 3.2.6 and the preceding corollary.

Proposition 3.4.5. *If $\mathfrak{M} \leq_{\mathrm{MSO}_k} (\kappa^{<\alpha}, \preceq, \bar{P})$ for finitely many unary predicates $\bar{P}$ and some $k < \omega$, then $\mathrm{smpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ is finite for all $n < \omega$.*

The following theorem summarises the various characterisations we have obtained so far.

Theorem 3.4.6. *Let $\mathfrak{M}$ be a structure of finite signature.*

(a) *For each tree $\kappa^{<\alpha}$ the following statements are equivalent:*
   (1) $\mathrm{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is finite for all $n < \omega$.*
   (2) $\mathrm{smpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is finite for all $n < \omega$.*
   (3) $\mathfrak{M} = \mathrm{val}(T)$ *for some $\Upsilon_{C,\tau}$-term $T \subseteq \kappa^{<\alpha}$.*
   (4) $\mathfrak{M} \leq_{\mathrm{MSO}_n} (\kappa^{<\alpha}, \preceq, \bar{P})$ *for finitely many unary predicates $\bar{P}$ and some $n < \omega$.*

(b) *If $\kappa < \aleph_0$ is finite then the following statements are equivalent to those above:*
   (5) $\mathrm{pwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is finite for all $n < \omega$.*
   (6) $\mathrm{mpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$ *is finite for all $n < \omega$.*
   (7) $\mathfrak{M} = \mathrm{val}(T)$ *for some $\Upsilon^{\leq}_{C,\tau}$-term $T \subseteq \kappa^{<\alpha}$.*
   (8) $\mathfrak{M} \leq_{\mathrm{MSO}_n} (\kappa^{<\alpha}, \preceq, (\mathrm{suc}_i)_{i<\kappa}, \bar{P})$ *for finitely many unary predicates $\bar{P}$ and some $n < \omega$.*

Proof. (1) ⇒ (3) Since the arity of $\mathfrak{M}$ is bounded Lemma 3.3.1 (4) implies that there exists a partition refinement $(U_v)_v$ of $\mathfrak{M}$ such that $\mathrm{spwd}_n(U_v)_v$ is finite for all $n < \omega$. Consequently, the claim follows from Proposition 3.2.9.

(3) ⇒ (4) ⇒ (2) follows by Propositions 3.1.9 and 3.4.5.

(2) ⇒ (1) $\mathrm{spwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \mathrm{smpwd}_n(\mathfrak{M}, \kappa^{<\alpha})$.

Analogously, (5) ⇒ (7) ⇒ (8) ⇒ (6) follows from, respectively, Propositions 3.2.9, 3.1.9, and 3.4.5, together with the fact that $\mathrm{pwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \mathrm{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$.

(6) ⇒ (5) is trivial.

(1) ⇒ (5) also follows from $\mathrm{pwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \mathrm{spwd}_n(\mathfrak{M}, \kappa^{<\alpha})$.
(8) ⇒ (4) follows from Lemma 1.3.3. □

We conclude this section by considering iterations and generalised sums. It turns out that the partition width increases only slightly when we take the iteration of a structure.

Proposition 3.4.7. *For every structure $\mathfrak{M}$ we have*

$$\mathrm{pwd}_n \mathfrak{M}^* \leq 2^n \mathrm{pwd}_n \mathfrak{M}\,.$$

Proof. Let $(U_v)_{v\in T}$ be a $2^{<\omega}$-partition refinement of $\mathfrak{M}$ of width $w_n := \mathrm{pwd}_n(U_v)_v$. Let $T_0 \subseteq T$ be the set of leaves. We construct a partition refinement $(V_v)_{v\in S}$ of $\mathfrak{M}^*$ by recursively attaching copies of the refinement $(U_v)_v$ to each of its leaves. Formally, we define

$$S := T \cup (T_0 1)^{<\omega} T_0 (0 \cup 1T)\,,$$

that is, a vertex $v \in T$ is either of the form $v = v_0 1 v_1 \dots v_{n-1} 1 v_n$ or $v = v_0 1 v_1 \dots 1 v_{n-1} 0$ where $v_1, \dots, v_{n-1} \in T_0$ and $v_n \in T$. We let

$$V_{v_0 1 \dots v_{n-1} 1 v_n} := \{\, a_0 \dots a_{n-1} a_n w \mid a_i \in U_{v_i},\ w \in M^{<\omega} \,\}\,,$$
$$V_{v_0 1 \dots v_{n-1} 0} := \{\, a_0 \dots a_{n-1} \mid a_i \in U_{v_i} \,\}\,.$$

Note that the element a_i is unique, for $i < n$, since $U_{v_i} = \{a_i\}$ is a leaf.

It remains to compute a bound on $\mathrm{eti}^n(V_v/\overline{V_v})$.

For vertices $v = v_0 1 \dots v_{n-1} 0$ we have $|V_v| = 1$ and $\mathrm{eti}^n(V_v/\overline{V_v}) = 1$. Otherwise, v is of the form $v_0 1 \dots 1 v_n$ and $V_v = w U_{v_n} M^{<\omega}$ where $\{w\} = U_{v_0} \times \cdots \times U_{v_{n-1}}$. Let

$$X := w U_{v_n}, \qquad X' := w(M \smallsetminus U_{v_n})\,,$$
$$Y := w U_{v_n} M^+, \qquad Z := M^{<\omega} \smallsetminus w M^{<\omega}.$$

Then $V_v = X \uplus Y$ and $\overline{V_v} = X' \cup Z$. For $\bar a, \bar a' \in X^m$ and $\bar b, \bar b' \in Y^n$ we have

$$\bar a \bar b \simeq^{\mathrm{o}}_{X' \cup Z} \bar a' \bar b' \quad \text{iff} \quad \bar a \simeq^{\mathrm{o}}_{X'} \bar a'\,.$$

Considering the various ways an n-tuple can be distributed over X and Y it follows that

$$\mathrm{eti}^n_{\mathrm{o}}(X \uplus Y/X' \cup Z) \leq \sum_{k\leq n} \binom{n}{k} \mathrm{eti}^k_{\mathrm{o}}(X/X')$$
$$\leq 2^n \mathrm{eti}^n_{\mathrm{o}}(X/X') \leq 2^n w_n\,. \qquad \square$$

Lemma 3.4.8. *Let $\mathfrak{I}$ and $\mathfrak{M}_i$, $i \in I$, be structures and $\bar w^{\mathrm{o}}, \bar w^{1} \in \omega^\omega$. If $\mathrm{pwd}_n \mathfrak{I} \leq w^{\mathrm{o}}_n$ and $\mathrm{pwd}_n \mathfrak{M}_i \leq w^{1}_n$, for $n < \omega$, then the partition width of the generalised sum is*

$$\mathrm{pwd}_n \bigcup_{i\in\mathfrak{I}} \mathfrak{M}_i \leq \max\{w^1_n,\ n^n w^{\mathrm{o}}_n(K_n)^n\}\,,$$

where K_n is the maximal number of atomic m-types realised in some $\mathfrak{M}_i$, for $m \leq n$.

Proof. Let $\mathfrak{N} := \bigcup_{i \in \mathfrak{I}} \mathfrak{M}_i$ and fix partition refinements $(U_u)_{u \in S}$ of $\mathfrak{I}$ and $(V^i_v)_{v \in T^i}$ of $\mathfrak{M}_i$.

Let $S_o \subseteq S$ be the set of leaves of S and $h : I \to S_o$ the mapping such that $U_{h(i)} = \{i\}$. Define

$$F := S \cup \{\, h(i)v \mid i \in I,\ v \in T^i \,\},$$

$$W_w := \begin{cases} \bigcup_{i \in U_w} M_i & \text{if } w \in S, \\ V^i_v & \text{if } w = h(i)v. \end{cases}$$

We claim that $(W_w)_{w \in F}$ is a partition refinement of $\mathfrak{N}$ of width

$$\mathrm{pwd}_n(W_w)_w \leq \max\{n^n (K_n)^n \,\mathrm{pwd}_n(U_u)_u,\ \mathrm{pwd}_n(V_v)_v\}.$$

For $w = h(i)v$, we have

$$\begin{aligned} \mathrm{eti}^n_o(W_{h(i)v}/N \setminus W_{h(i)v}) &= \mathrm{eti}^n_o(V^i_v/N \setminus V^i_v) \\ &= \mathrm{eti}^n_o(V^i_v/M_i \setminus V^i_v) \leq w_n. \end{aligned}$$

The case $w \in S$ is more involved. Let $g : N \to I$ be the function defined by $a \in M_{g(a)}$ for $a \in N$. We claim that $\bar{a} \simeq^o_{N \setminus W_w} \bar{b}$, for $\bar{a}, \bar{b} \in W^n_w$, if the following conditions are satisfied:

(1) $\sim$ induces the same partition $J_0 \cup \cdots \cup J_s$ of the indices of $\bar{a}$ and $\bar{b}$.

(2) $g(\bar{a}) \simeq^o_{I \setminus U_w} g(\bar{b})$.

(3) $\mathrm{tp}_o(\bar{a}|_{J_k}) = \mathrm{tp}_o(\bar{b}|_{J_k})$ for all $k \leq s$.

Then it follows that

$$\mathrm{eti}^n_o(W_u/N \setminus W_u) \leq n^n \cdot w^o_n \cdot (K_n)^n.$$

We have $\bar{a} \simeq^o_{N \setminus W_w} \bar{b}$ iff $\mathfrak{N} \models \varphi(\bar{a}; \bar{c}) \leftrightarrow \varphi(\bar{b}; \bar{c})$ for all atomic formulae $\varphi(\bar{x}; \bar{y})$ and all parameters $\bar{c} \subseteq N \setminus W_w$.

Consider the possible relations occurring in φ. If $\varphi = \bar{z} \sim \bar{z}'$, for $\bar{z}, \bar{z}' \subseteq \bar{x} \cup \bar{y}$, then this condition holds by (1). If $\varphi = R^o \bar{z}$, then it follows from (2) and, if $\varphi = R^1 \bar{z}$, then it holds by (3). □

3.5 Pairing functions and grids

Baldwin and Shelah argue in [2] that monadic second-order theories in which a pairing function can be defined are hopelessly complicated and then proceed to classify the other ones. They show that the

models of every stable theory without definable pairing function can be decomposed in a tree-like fashion and that these theories can be interpreted in the theory of a suitable class of trees. Extended to include unstable theories a finitary version of their results would answer the conjecture of Seese cited in the preface.

Note that, by (the proof of) Theorem 1.2.9, a corresponding result for guarded second-order logic does hold. If the tree width of a structure $\mathfrak{M}$ is infinite, then arbitrarily large grids are GSO-definable in $\mathfrak{M}$ and, consequently, the GSO-theory of $\mathfrak{M}$ is undecidable.

It is quite easy to show that the existence of a pairing function implies an infinite partition width while a proof of the converse seems to be quite involved requiring an adaptation of the Excluded Grid Theorem of Robertson and Seymour [64]. So far, only partial results have been obtained.

admitting MSO*-coding*

Definition 3.5.1. A structure $\mathfrak{M}$ *admits* MSO*-coding* if there exists an MSO-formula $\varphi(x, y, z; \bar{X})$ such that, for each natural number $n < \omega$, there are sets $A, B, C \subseteq M$ of size $|A| = |B| = n$ such that, for suitable monadic parameters $\bar{P}$, $\varphi(x, y, z; \bar{P})$ defines a bijection $A \times B \to C$.

admitting strong MSO*-coding*

We say that $\mathfrak{M}$ *admits strong* MSO*-coding* if there are infinite sets A, B, and C and an MSO-formula φ as above.

Lemma 3.5.2. *Let $\mathfrak{M}$ be a structure and $n < \aleph_0$. The following statements are equivalent:*

(1) *There exists an* MSO*-formula $\chi(x, y, z)$ with monadic parameters that defines a bijection $A \times B \to C$ for sets of size $|A| = |B| = n$.*

(2) *There exists an* MSO*-formula $\vartheta(x, y)$ with monadic parameters that defines an $n \times n$ grid.*

(3) *There exist* MSO*-formulae $\varphi(x, y)$ and $\psi(x, y)$ each of which defines an equivalence relation with n classes such that every class of the first one intersects each class of the other one.*

Proof. (1) $\Rightarrow$ (3) Let $f : A \times B \to C$ be the given bijection. We can define two equivalence relations on C by setting

$$\varphi(x, y) := \exists u \exists v \exists z (f(u, z) = x \wedge f(v, z) = y),$$
$$\text{and} \quad \psi(x, y) := \exists u \exists v \exists z (f(z, u) = x \wedge f(z, v) = y).$$

(2) $\Rightarrow$ (1) Fix $n < \aleph_0$ and $C \cong n \times n$ as above. Let $A := n \times \{0\} \subseteq C$ and $B := \{0\} \times n \subseteq C$. We claim that the function $f : A \times B \to C$ defined by $f((i, 0), (0, k)) := (i, k)$ is MSO-definable.

With the help of the parameters

$$H_m := \{ (i, k) \mid i \cong m \pmod 3 \} \subseteq C$$
$$\text{and} \quad V_m := \{ (i, k) \mid k \cong m \pmod 3 \} \subseteq C,$$

for $m < 3$, we can define the successor relations

$$S_0 := \{ ((i,k),(i+1,k)) \mid i < n-1,\ k < n \}$$
$$\text{and} \quad S_1 := \{ ((i,k),(i,k+1)) \mid i < n,\ k < n-1 \}.$$

Then the desired coding function can be defined by

$$f(x,y) = z \quad \text{iff} \quad (x,z) \in (S_1)^* \text{ and } (y,z) \in (S_0)^*.$$

(3) $\Rightarrow$ (2) Let $\sim_0$ and $\sim_1$ be the two equivalences. Fix elements a_{ik}, $i,k < n$, such that

$$a_{ik} \sim_0 a_{ml} \text{ iff } i = m \qquad \text{and} \qquad a_{ik} \sim_1 a_{ml} \text{ iff } k = l.$$

With the help of the parameters

$$P := \{ a_{ii} \mid i < n \} \quad \text{and} \quad Q := \{ a_{i(i+1)} \mid i+1 < n \},$$

we define the relations

$$S_0 := \{ (a_{ik}, a_{(i+1)k}) \mid i,k < n \},$$
$$S_1 := \{ (a_{ik}, a_{i(k+1)}) \mid i,k < n \},$$

by setting

$$S_0xy := x \sim_1 y \wedge \exists u \exists v (Qu \wedge Pv \wedge x \sim_0 u \wedge y \sim_0 v \wedge u \sim_1 v),$$
$$\text{and} \quad S_1xy := x \sim_0 y \wedge \exists u \exists v (Pu \wedge Qv \wedge x \sim_1 u \wedge y \sim_1 v \wedge u \sim_0 v). \ \square$$

Remark. Note that the translation in the preceding lemma is uniform, that is, given $\chi(x,y,z;\bar{Z})$ we can construct a formula $\vartheta(x,y;\bar{Z})$ such that, whenever $\bar{P}$ are parameters such that $\chi(x,y,z;\bar{P})$ defines a bijection $A \times B \to C$ with $|A| = |B| = n$, then we can find parameters $\bar{Q}$ such that $\vartheta(x,y;\bar{Q})$ defines an $n \times n$ grid. Analogous statements hold for the other directions.

It follows that structures admitting MSO-coding are complicated. In particular, their MSO-theory is undecidable.

Theorem 3.5.3. *If $\mathfrak{M}$ is a structure that admits* MSO*-coding then the* MSO*-theory of $\mathfrak{M}$ is undecidable.*

Proof. If $\mathfrak{M}$ admits MSO-coding then there exists a formula $\vartheta(x,y;\bar{Z})$ that, for suitable choice of the parameters $\bar{Z}$, defines arbitrarily large grids. The undecidability follows since these grids can be used to code domino problems. $\square$

We also give a characterisations of MSO-coding in terms of pairing functions.

Lemma 3.5.4. *If $\mathfrak{M}$ admits strong* MSO*-coding then there is an infinite set $A \subseteq M$ and an* MSO*-formula $\varphi(x, y, z; \bar{P})$ defining a pairing function on A for some unary predicates $\bar{P} \subseteq \wp(M)$.*

Proof. Let $f : A \times B \to C$ be an MSO-definable bijection. Note that every function $g : A \to B$ is MSO-definable with the help of the parameter

$$P := \{ f(a, g(a)) \mid a \in A \} \subseteq C$$

since $g(x) = y$ iff $f(x, y) \in P$. By fixing a bijection $g : A \to B$ we obtain a MSO-definable bijection $h : A \times A \to C$ by setting

$$h(x, y) := f(x, g(y)) .$$

Finally, every pairing function $k : A \times A \to A$ can be defined by

$$k(x, y) := z \quad :\text{iff} \quad h(x, z) \in P_0 \text{ and } h(y, z) \in P_1$$

where

$$P_0 := \{ h(x, k(x, y)) \mid x, y \in A \}$$
$$\text{and} \quad P_1 := \{ h(y, k(x, y)) \mid x, y \in A \} . \qquad \square$$

Lemma 3.5.5 (Baldwin and Shelah [2])**.** *Suppose that, for every $n < \omega$, there exist an element $c \in M$, two sequences $(a_i)_{i<n}$ and $(b_i)_{i<n}$, and a formula $\varphi(x, y, z) \in$ FO such that*

- $\mathrm{tp}(a_i b_k) = \mathrm{tp}(a_0 b_0)$ *for all $i, k < n$,*
- $\mathfrak{M} \models \varphi(a_i, b_k, c)$ *iff $i = k = 0$.*

Then there exists an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ which admits strong FO*-coding.*

Proof. We can choose a suitable elementary extension that contains infinite sequences $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ with the above properties and such that, for all $i, k < \omega$, there is an elementary map α_{ik} which interchanges a_0 and a_i, b_0 and b_k, and fixes all the other elements a_j and b_l. Then,

$$\mathfrak{M} \models \varphi(a_j, b_l, \alpha_{ik}(c)) \quad \text{iff} \quad j = i \text{ and } l = k .$$

That is, φ defines a coding of $\bar{a} \times \bar{b}$ into $C := \{ \alpha_{ik}(c) \mid i, k < \omega \}$. $\square$

We conjecture that the property of admitting MSO-coding is equivalent to an infinite partition width.

Conjecture. *A structure $\mathfrak{M}$ with finite signature has finite partition width if and only if it does not admit* MSO*-coding.*

Note that this conjecture fails if we allow infinite signatures. Let $\mathfrak{M} = (\omega \times \omega, (E_n)_{n<\omega})$ where

$$E_n := \left\{ (\langle i, k\rangle, \langle j, l\rangle) \;\middle|\; |i-j| + |k-l| = 1,\ i, j, k, l < n \right\}.$$

Then, $\operatorname{pwd}_1 \mathfrak{M} = \aleph_0$. On the other hand, the MSO-theory of $\mathfrak{M}$ is decidable since each formula contains only finitely many relation symbols and every finite reduct of $\mathfrak{M}$ is the disjoint union of a finite structure and an infinite set.

Since all structures admitting MSO-coding have an undecidable MSO-theory a proof of this conjecture would settle the conjecture of Seese that every class of finite graphs with decidable MSO-theory has finite clique width. The following lemma deals with the easy direction.

cancellative

We call a function $f : A \times B \to C$ *cancellative* if $f(a, b) = f(a', b)$ implies $a = a'$ and $f(a, b) = f(a, b')$ implies $b = b'$.

Proposition 3.5.6. *Let $\mathfrak{M}$ be a τ-structure. If there are unary predicates $\tilde{P}$ and an MSO_k-formula $\varphi(x, y, z; \tilde{P})$ defining a cancellative function $f : A \times B \to C$ then $|A| \leq K$ or $|B| \leq K$ where*

$$K := 3 \cdot \beth_k(N_{k+2} \operatorname{mpwd}_{k+2} \mathfrak{M}) \quad \textit{and} \quad N_k := |MS^k_0(\varnothing)|$$

where MS^k_0 is taken with respect to the signature $\tau \cup \tilde{P}$.

Proof. Let $f : A \times B \to C$ be the given function. Fix a partition refinement $(U_\nu)_{\nu \in T}$ of $\mathfrak{M}$ such that $\operatorname{mpwd}_{k+2}(U_\nu)_\nu$ is minimal and define

$$w_n := \sup \{ \operatorname{mti}^n_k(U_\nu / \overline{U_\nu}) \mid \nu \in T \}.$$

By Lemmas 3.3.1 (2) and 3.3.6 we have

$$w_2 \leq \beth_k(N_{2+k} \operatorname{mpwd}_{2+k} \mathfrak{M}) = {}^{K}\!/_{3}.$$

Suppose, for a contradiction, that $m := |A| = |B| > 3w_2$.

We claim that there exists some vertex $\nu \in T$ such that

$$\tfrac{1}{3}m \leq |U_\nu \cap A| \leq \tfrac{2}{3}m \quad \text{and} \quad |B \setminus U_\nu| > w_2,$$
$$\text{or} \quad \tfrac{1}{3}m \leq |U_\nu \cap B| \leq \tfrac{2}{3}m \quad \text{and} \quad |A \setminus U_\nu| > w_2.$$

Let ν_0 be some vertex with $\frac{1}{3}m \leq |U_{\nu_0} \cap B| \leq \frac{2}{3}m$. If $|A \setminus U_{\nu_0}| \leq w_2$ then there exists some $\nu \geq \nu_0$ such that

$$\tfrac{1}{3}m \leq |U_\nu \cap A| \leq \tfrac{2}{3}m$$
$$\text{and} \quad |B \setminus U_\nu| \geq |B \setminus U_{\nu_0}| \geq {}^{m}\!/_{3} > w_2.$$

Thus, by symmetry we may assume that there exists some $\nu \in T$ satisfying the first condition.

There are at most w_2 elements $b \in B \setminus U_v$ such that $f(a, b) = c$ for some $a \in U_v \cap A$, $c \in U_v \cap C$. Otherwise, there would be tuples $f(a, b) = c$ and $f(a', b') = c'$ with $b \neq b'$ and $\{a\}\{c\} \approx^k_{U_u} \{a'\}\{c'\}$. Then, $f(a', b') = c'$ would imply

$$f(a, b') = c = f(a, b),$$

and by cancellation, we would have $b = b'$ in contradiction to our assumption.

Since $|B \setminus U_v| > w_2$ it follows that there exists some $b \in B \setminus U_v$ such that $f(a, b) \in \overline{U_v}$ for all $a \in U_v \cap A$.

Furthermore, since $|U_v \cap A| \geq \aleph_3 > w_2$ there are two different elements $a, a' \in U_v \cap A$ such that $a \approx^k_{U_v} a'$. This implies $f(a, b) = c$ iff $f(a', b) = c$ for all $c \in \overline{U_v}$. Contradiction. □

Corollary 3.5.7. *If $\mathfrak{M}$ admits* MSO*-coding then* $\mathrm{pwd}_n \mathfrak{M} \geq \aleph_0$ *for some n.*

Corollary 3.5.8. *A group has finite partition width if and only if it is finite.*

In Section 5.9 we will see that the Cayley graph of a group has finite partition width if and only if the group is virtually free.

3.6 Coding and compactness

After having defined the partition width of a structure we can begin to develop a model theory for structures of finite partition width. In the present section we consider the first-order theory of a structure of finite partition width. In particular, we prove that elementary extensions preserve finiteness of partition width and we present a compactness theorem for structures of finite partition width. We start by considering substructures.

Definition 3.6.1. Let $(U_v)_{v \in T}$ be a partition refinement of $\mathfrak{M}$ and $C \subseteq M$. The *restriction* of $(U_v)_v$ to C is the partition refinement

restriction $(U_v)_v|_C$

$$(U_v)_v|_C := (U_v \cap C)_{v \in T_0}$$

where $T_0 := \{ v \in T \mid U_v \cap C \neq \varnothing \}$.

Lemma 3.6.2. *If $(U_v)_v$ is a partition refinement of $\mathfrak{M}$ and $C \subseteq M$ then*

$$\mathrm{pwd}_n(U_v)_v|_C \leq \mathrm{pwd}_n(U_v)_v \qquad \text{for all } n < \omega.$$

Corollary 3.6.3. *If $\mathfrak{M} \subseteq \mathfrak{N}$ then* $\mathrm{pwd}_n(\mathfrak{M}, \kappa^{<\alpha}) \leq \mathrm{pwd}_n(\mathfrak{N}, \kappa^{<\alpha})$ *for all $n < \omega$.*

In order to compute the partition width of structures constructed by model theoretic means we need to code partition refinements by relations.

Definition 3.6.4. (a) Let $(U_v)_{v\in T}$ be a family of sets $U_v \subseteq M$ indexed by a partial order $(T, \preceq)$. A pair $(U, \sqsubseteq)$ of relations $U \subseteq M^{1+n}$ and $\sqsubseteq \subseteq M^{2n}$ *code* $(U_v)_{v\in T}$ if there exists an isomorphism

$(U, \sqsubseteq)$ codes $(U_v)_v$

$$f : (D, \sqsubseteq) \cong (T, \preceq),$$

where $D := \{ \bar{a} \in M^n \mid \bar{a} \sqsubseteq \bar{a} \}$, such that

$$U := \{ (a, \bar{b}) \in M \times D \mid a \in U_{f(\bar{b})} \},$$

and $\quad \bar{a} \sqsubseteq \bar{b}$ implies $\bar{a}, \bar{b} \in D$.

reduced

canonical coding

(b) We call a partition refinement $(U_v)_{v\in T}$ of $\mathfrak{M}$ *reduced* if all non-leaves of T have at least two immediate successors. If $(U_v)_{v\in T}$ is reduced we can define a *canonical coding* of $(U_v)_v$ in the following way. For each $v \in T$ choose leaves $u_0, u_1 \in T$ with $v = u_0 \sqcap u_1$ and set $h(v) := (a_0, a_1)$ where $U_{u_i} = \{a_i\}$, $i < 2$. Let $D := \operatorname{rng} h$. We define

$$\bar{a} \sqsubseteq \bar{b} \quad :\text{iff} \quad \bar{a}, \bar{b} \in D \text{ and } h^{-1}(\bar{a}) \preceq h^{-1}(\bar{b}),$$

$$U := \{ (c, \bar{a}) \mid \bar{a} \in D,\ c \in U_{h^{-1}(\bar{a})} \}.$$

Remark. Note that not every partition refinement $(U_v)_{v\in T}$ of a structure $\mathfrak{M}$ can be coded, since we might have $|T| > |M^n|$ for all $n < \omega$. But we can always obtain a codable partition refinement by removing some vertices $v \in T$ with exactly one immediate successor. The same holds for non-standard partition refinements which will be defined below.

The fact that a relation U codes some partition refinement can be expressed in first-order logic, with the sole exception that it is not possible to state that the components are arranged in a tree. Therefore, we consider partition refinements indexed by non-standard trees.

T^κ_{tree}

Definition 3.6.5. Let T^κ_{tree} be the theory of all trees $(S, \preceq)$ where $S \subseteq \kappa^{<\omega}$ is prefix-closed.

non-standard partition refinement

Definition 3.6.6. A *non-standard $\kappa^{<\omega}$-partition refinement* of a structure $\mathfrak{M}$ is a family $(U_v)_{v\in T}$ of subsets $U_v \subseteq M$ indexed by a model T of T^κ_{tree} satisfying the following conditions:

(1) For all $a \in M$ there exists some $v \in T$ with $U_v = \{a\}$.

(2) If $u \preceq v$, for $u, v \in T$, then $U_u \supseteq U_v$.

(3) If $u, v \in T$ are incomparable then $U_u \cap U_v = \varnothing$.

Note that we do not require the U_v to be nonempty.

The widths $\mathrm{pwd}_n(U_v)_v$ and $\mathrm{spwd}_n(U_v)_v$ of $(U_v)_v$ are defined in the same way as for standard partition refinements.

For a structure $\mathfrak{M}$ we define the *non-standard* [*symmetric*] *partition width* $\mathrm{pwd}_n^{\mathrm{ns}}\,\mathfrak{M}$ [$\mathrm{spwd}_n^{\mathrm{ns}}\,\mathfrak{M}$] of $\mathfrak{M}$ as the minimal partition width of a non-standard $2^{<\omega}$-[$\aleph_0^{<\omega}$-]partition refinement of $\mathfrak{M}$.

non-standard partition width $\mathrm{pwd}_n^{\mathrm{ns}}\,\mathfrak{M}$, $\mathrm{spwd}_n^{\mathrm{ns}}\,\mathfrak{M}$

Lemma 3.6.7. *If $(U_v)_{v\in T}$ is a non-standard $\kappa^{<\omega}$-partition refinement of $\mathfrak{M}$ and $C \subseteq M$ then $(U_v \cap C)_{v\in T}$ is a non-standard $\kappa^{<\omega}$-partition refinement of $\mathfrak{M}|_C$ of width*

$$\mathrm{pwd}_n(U_v \cap C)_{v\in T} \leq \mathrm{pwd}_n(U_v)_v \qquad \text{for all } n < \omega.$$

Proof. The index structure of $(U_v \cap C)_{v\in T}$ is $T \vDash T^{\kappa}_{\mathrm{tree}}$. It remains to check conditions (1)–(3).

(1) If $a \in C \subseteq M$ then there is some $v \in T$ with $U_v = \{a\}$. Hence, $U_v \cap C = \{a\}$.

(2) If $u \preceq v$ then $U_u \supseteq U_v$ which implies $U_u \cap C \supseteq U_v \cap C$.

(3) If u and v are incomparable then $U_u \cap U_v = \varnothing$ which implies $(U_u \cap C) \cap (U_v \cap C) = \varnothing$.

Hence, $(U_v \cap C)_{v\in T}$ is a non-standard partition refinement of $\mathfrak{M}|_C$. The second claim is immediate. □

Corollary 3.6.8. *If $\mathfrak{M} \subseteq \mathfrak{N}$ then $\mathrm{pwd}_n^{\mathrm{ns}}(\mathfrak{M}, \kappa^{<\alpha}) \leq \mathrm{pwd}_n^{\mathrm{ns}}(\mathfrak{N}, \kappa^{<\alpha})$ for all $n < \omega$.*

Lemma 3.6.9. *Let $\mathfrak{M}$ be a τ-structure and $(U, \sqsubseteq)$ a pair of additional relation symbols. For each $\kappa \leq \aleph_0$, there exists an* FO*-theory T^{κ}_{pr} such that $(\mathfrak{M}, U, \sqsubseteq) \vDash T^{\kappa}_{\mathrm{pr}}$ if and only if $(U, \sqsubseteq)$ codes a non-standard $\kappa^{<\omega}$-partition refinement of $\mathfrak{M}$.*

T^{κ}_{pr}

Proof. Let Ψ be the theory obtained from $T^{\kappa}_{\mathrm{tree}}$ by replacing every occurrence of $\preceq$ by $\sqsubseteq$ and relativising every formula to the set $D := \{\, \bar{a} \mid \bar{a} \sqsubseteq \bar{a} \,\}$. Further, let Φ consist of the following formulae which express the properties of a non-standard partition refinement:

$$\forall x \exists \bar{y} \forall z (Uz\bar{y} \leftrightarrow z = x)$$
$$\forall \bar{y} \forall \bar{z} (\bar{y} \sqsubseteq \bar{z} \to \forall x (Ux\bar{z} \to Ux\bar{y}))$$
$$\forall \bar{y} \forall \bar{z} (\bar{y} \not\sqsubseteq \bar{z} \wedge \bar{z} \not\sqsubseteq \bar{y} \to \neg \exists x (Ux\bar{y} \wedge Ux\bar{z}))$$
$$\forall \bar{x} \forall \bar{y} (\bar{x} \sqsubseteq \bar{y} \to \bar{x} \sqsubseteq \bar{x} \wedge \bar{y} \sqsubseteq \bar{y})$$
$$\forall x \forall y (Ux\bar{y} \to \bar{y} \sqsubseteq \bar{y})$$

Let $T^{\kappa}_{\mathrm{pr}} := \Phi \cup \Psi$. We claim that $(\mathfrak{M}, U, \sqsubseteq) \vDash T^{\kappa}_{\mathrm{pr}}$ iff $(U, \sqsubseteq)$ codes a non-standard $\kappa^{<\omega}$-partition refinement of $\mathfrak{M}$.

($\Leftarrow$) is obvious. For ($\Rightarrow$), suppose that $(\mathfrak{M}, U, \sqsubseteq) \models T^{\kappa}_{\mathrm{pr}}$. We define

$$T := \{\, \bar{a} \in M^n \mid \bar{a} \sqsubseteq \bar{a} \,\},$$

and $\quad U_{\bar{a}} := \{\, b \in M \mid (b, \bar{a}) \in U \,\}, \qquad$ for $\bar{a} \in T$.

Then $(T, \sqsubseteq) \models T^{\kappa}_{\mathrm{tree}}$, $\bar{a} \sqsubseteq \bar{b}$ implies $\bar{a}, \bar{b} \in D$, and $(U_{\bar{a}})_{\bar{a} \in T}$ forms the desired non-standard $\kappa^{<\omega}$-partition refinement coded by $(U, \sqsubseteq)$. □

Lemma 3.6.10. *Let $\mathfrak{M}$ be a τ-structure and $(U, \sqsubseteq)$ a pair of additional relation symbols.*

$\Pi^2_{\bar{w}}$ (1) *For every sequence $\bar{w} \in \omega^\omega$ there is a set of sentences $\Pi^2_{\bar{w}} \subseteq \mathrm{FO}$ such that $(\mathfrak{M}, U, \sqsubseteq) \models \Pi^2_{\bar{w}}$ if and only if $(U, \sqsubseteq)$ codes a non-standard $2^{<\omega}$-partition refinement $(U_\nu)_\nu$ of $\mathfrak{M}$ with $\mathrm{pwd}_n(U_\nu)_\nu \leq w_n$ for all $n < \omega$.*

$\Pi^\omega_{\bar{w}}$ (2) *For every sequence $\bar{w} \in \omega^\omega$ there is a set of sentences $\Pi^\omega_{\bar{w}} \subseteq \mathrm{FO}$ such that $(\mathfrak{M}, U, \sqsubseteq) \models \Pi^\omega_{\bar{w}}$ if and only if $(U, \sqsubseteq)$ codes a non-standard $\aleph_0^{<\omega}$-partition refinement $(U_\nu)_\nu$ of $\mathfrak{M}$ with $\mathrm{spwd}_n(U_\nu)_\nu \leq w_n$ for all $n < \omega$.*

Proof. (1) Since $(\mathfrak{M}, U, \sqsubseteq) \models T^2_{\mathrm{pr}}$ iff $(U, \sqsubseteq)$ codes a non-standard $2^{<\omega}$-partition refinement of $\mathfrak{M}$, it remains to express that the partition width is bounded.

According to Lemma 3.3.7 (3) it is sufficient to do so for all finite subsets $\tau_0 \subseteq \tau$. We construct formulae $\varphi^{\tau_0}_{n,m}$ expressing that the n-ary partition width of the τ_0-reduct is at most m. Then we can set

$$\Pi^2_{\bar{w}} := T^2_{\mathrm{pr}} \cup \{\, \varphi^{\tau_0}_{n,w_n} \mid n < \omega,\ \tau_0 \subseteq \tau \text{ finite} \,\}.$$

Let r be the maximal arity of relations in τ_0. For $\bar{a}, \bar{b} \in X$, we have

$$\bar{a} \simeq^0_{\overline{X}} \bar{b} \quad \text{iff} \quad \bar{a} \simeq^0_{\bar{c}} \bar{b} \quad \text{for all } \bar{c} \in \overline{X}^r.$$

Consequently, we can express that $\bar{x} \simeq^0_{\overline{X}} \bar{y}$ by the formula

$$\psi(\bar{x}, \bar{y}; X) := (\forall \bar{z}. \bigwedge_{i<r} \neg X z_i)\big[\mathrm{etp}_{\tau_0}(\bar{x}/\bar{z}) = \mathrm{etp}_{\tau_0}(\bar{y}/\bar{z})\big]$$

where $\bar{z}$ is an r-tuple. Finally, we set

$$\varphi^{\tau_0}_{n,m} := \big(\forall \bar{y}. \bar{y} \sqsubseteq \bar{y}\big)\big(\exists \bar{x}^0 \dots \bar{x}^{m-1}. \bigwedge_{i<n, j<m} U x^j_i \bar{y}\big) \big(\forall \bar{x}'. \bigwedge_{i<n} U x'_i \bar{y}\big) \bigvee_{j<m} \psi(\bar{x}', \bar{x}^j; U_\bar{y})$$

where the $\bar{x}^j$, $\bar{x}'$, and $\bar{y}$ are n-tuples, and $U_\bar{y}$ indicates that every atom Xz in ψ should be replaced by $Uz\bar{y}$.

(2) As above we construct formulae $\varphi^{\tau_0}_{n,m}$ expressing that the n-ary symmetric partition width of the τ_0-reduct is at most m, and set

$$\Pi^{\omega}_{\bar{w}} := T^{\aleph_0}_{\mathrm{pr}} \cup \{\, \varphi^{\tau_0}_{n,w_n} \mid n < \omega,\ \tau_0 \subseteq \tau \text{ finite} \,\}.$$

Let r be the maximal arity of relations in τ_0. The formula

$$\eta(\bar{y}_0, \bar{y}_1) := \bar{y}_0 \sqsubset \bar{y}_1 \wedge \neg\exists \bar{z}(\bar{y}_0 \sqsubset \bar{z} \sqsubset \bar{y}_1)$$

defines the successor relation of the partial order $\sqsubseteq$. For tuples $\bar{x}^0, \dots, \bar{x}^m$ contained in $U_\bar{y}$ the formula

$$\vartheta(z; \bar{y}, \bar{x}^0, \dots, \bar{x}^m) := \forall \bar{y}'\Big(\big(\eta(\bar{y}, \bar{y}') \wedge Uz\bar{y}'\big) \to \neg \bigwedge_{i<n, j\leq m} Ux^j_i\bar{y}'\Big)$$

states that the element z is no member of any component $U_\bar{y}'$ containing some of the $\bar{x}^j$.

We have to express that there is no sequence $\bar{a}^0, \dots, \bar{a}^m$ of $m+1$ tuples of pairwise distinct types over all components that do not contain any of the $\bar{a}^i$. This can be done by defining

$$\begin{aligned}\varphi^{\tau_0}_{n,m} := \forall \bar{y}\neg\Big(\exists \bar{x}^0 \dots \bar{x}^m.\ & \bigwedge_{i<n, j<m} Ux^j_i\bar{y}\Big) \\ & \bigwedge_{j\neq k}\Big(\exists \bar{z}.\ \bigwedge_{i<r} \vartheta(z_i; \bar{y}, \bar{x}^0, \dots, \bar{x}^m)\Big) \\ & \big[\mathrm{etp}_0(\bar{x}^j/\bar{z}) \neq \mathrm{etp}_0(\bar{x}^k/\bar{z})\big].\end{aligned}$$

□

Having established our main tool we first apply it to show that the non-standard partition width of a structure is determined by the non-standard partition widths of its finite substructures. This generalises the analogous result for the clique width of countable graphs by Courcelle [27].

Proposition 3.6.11. *Let $\mathfrak{M}$ be a relational structure and $\bar{w} \in \omega^\omega$.*

(1) $\mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M} \leq w_n$, *for all $n < \omega$, if and only if all finite substructures of $\mathfrak{M}$ have a non-standard $2^{<\omega}$-partition refinement of width at most $\bar{w}$.*

(2) $\mathrm{spwd}^{\mathrm{ns}}_n \mathfrak{M} \leq w_n$, *for all $n < \omega$, if and only if all finite substructures of $\mathfrak{M}$ have a non-standard $\aleph_0^{<\omega}$-partition refinement of width at most $\bar{w}$.*

Proof. One direction immediately follows from Corollary 3.6.8. For the other one, set $\Phi := \Delta \cup \Pi$ where Δ is the atomic diagram of $\mathfrak{M}$ and Π is either $\Pi^2_{\bar{w}}$ or $\Pi^{\omega}_{\bar{w}}$.

If Φ has a model $(\mathfrak{N}, U, \sqsubseteq)$ then there is a non-standard partition refinement $(U_\nu)_\nu$ of $\mathfrak{N}$ of width $\bar{w}$. The restriction $(U_\nu \cap M)_\nu$ of $(U_\nu)_\nu$ to M yields the desired refinement of $\mathfrak{M}$.

To prove that Φ is consistent let $\Phi_0 \subseteq \Phi$ be finite. Then there is a finite set $A \subseteq M$ such that $\Phi_0 \subseteq \Delta_0 \cup \Pi$ where Δ_0 is the atomic diagram of $\mathfrak{M}|_A$. Let $(U_v)_v$ be a reduced partition refinement of $\mathfrak{M}|_A$ of width $\bar{w}$, and let $(U, \sqsubseteq)$ be relations coding it. Then $(\mathfrak{M}|_A, U, \sqsubseteq) \models \Phi_0$. □

Of course, we are interested in a standard partition refinement. Unfortunately, the width of a non-standard partition refinement may increase when we transform it into a standard one.

Example (Courcelle [27]). Let $\mathfrak{G}$ be the graph with universe $V := [2] \times \omega$ and edge relation

$$E := \{ (\langle b, k\rangle, \langle 1, n\rangle) \mid k < n,\ b < 2 \}.$$

Then $\mathrm{pwd}_1 \mathfrak{G}_0 = \mathrm{pwd}_1^{\mathrm{ns}} \mathfrak{G}_0 = \mathrm{pwd}_1^{\mathrm{ns}} \mathfrak{G} = 1$ for every finite induced subgraph $\mathfrak{G}_0 \subseteq \mathfrak{G}$ but $\mathrm{pwd}_1 \mathfrak{G} = 2$.

To compute $\mathrm{pwd}_1 \mathfrak{G}_0$ and $\mathrm{pwd}_1^{\mathrm{ns}} \mathfrak{G}_0$ it is sufficient to consider the case that $\mathfrak{G}_0 = \mathfrak{G}|_{[2]\times[n]}$. A partition refinement of width 1 is given by $(U_v)_{v\in T}$ where $T := 0^{<2n}1^{<2}$ and

$$\begin{aligned}
U_{0^{2k}} &:= [2] \times [n-k], \\
U_{0^{2k}1} &:= \{\langle 0, n-k-1\rangle\}, \\
U_{0^{2k}0} &:= [2] \times [n-k-1] \cup \{\langle 1, n-k-1\rangle\}, \\
U_{0^{2k}01} &:= \{\langle 1, n-k-1\rangle\}.
\end{aligned}$$

For $\mathrm{pwd}_1^{\mathrm{ns}} \mathfrak{G}$ we use as index structure the tree T of all sequences $w : I \to [2]$ where I is a prefix of $\omega+\zeta$. Then we can define analogously

$$\begin{aligned}
U_{0^n} &:= [2] \times \omega, && \text{for } n < \omega, \\
U_{0^{\omega+\omega^*-2k}} &:= [2] \times [k], \\
U_{0^{\omega+\omega^*-2k}1} &:= \{\langle 0, k-1\rangle\}, \\
U_{0^{\omega+\omega^*-2k}0} &:= [2] \times [k-1] \cup \{\langle 1, k-1\rangle\}, \\
U_{0^{\omega+\omega^*-2k}01} &:= \{\langle 1, k-1\rangle\}, \\
\text{and} \qquad U_v &:= \varnothing, && \text{for all other indices } v.
\end{aligned}$$

Suppose that there exists a partition refinement $(U_v)_v$ of $\mathfrak{G}$ of width 1. By symmetry, we may assume that $U_0 \cap [b] \times \omega$ is infinite for some $b < 2$.

If $\langle b, n\rangle \in U_0$ and $k > n$ then $\langle 1-b, k\rangle \notin U_1$ since there exists some $n' > k$ with $\langle b, n'\rangle \in U_0$ and $\langle b, n\rangle \not\approx^0_{\langle 1-b,k\rangle} \langle b, n'\rangle$. Similarly, $\langle b, k\rangle \notin U_1$ for $k > n$ since $\langle b, n\rangle \not\approx^0_{\langle b,k\rangle} \langle b, n'\rangle$ for all $n' > k$. Hence, $U_1 \subseteq [2] \times [m]$ for some $m < \omega$.

Fix some element $\langle c, k\rangle \in U_1$. There are elements $\langle 0, n_0\rangle, \langle 1, n_1\rangle \in U_0$ with $n_0, n_1 > k$. But $\langle 0, n_0\rangle \not\approx^0_{\langle c,k\rangle} \langle 1, n_1\rangle$ contradicts our assumption that $\mathrm{eti}^1_0(U_0/U_1) = 1$.

Proposition 3.6.12. *Let $\mathfrak{M}$ be a structure with m relations of arity greater than 1 and let r be the maximum of their arities.*

(1) *If $(U_v)_v$ is a non-standard $2^{<\omega}$-partition refinement $(U_v)_v$ of $\mathfrak{M}$ of width $w_n := \mathrm{pwd}_n(U_v)_v$ then*

$$\mathrm{pwd}_n \mathfrak{M} \leq 2^{m(n+1)^r w_{r-1} 2^{mr^r w_{r-1}+r-1}}.$$

(2) *If $(U_v)_v$ is a non-standard $\aleph_0^{<\omega}$-partition refinement $(U_v)_v$ of $\mathfrak{M}$ of width $w_n := \mathrm{pwd}_n(U_v)_v$ then*

$$\mathrm{spwd}_n \mathfrak{M} \leq 2^{m(n+1)^r w_{r-1} 2^{mr^r w_{r-1}+r-1}}.$$

(3) *If $\mathfrak{M}$ is a transition system then we can improve the bounds to $[\mathrm{s}]\mathrm{pwd}_1 \mathfrak{M} \leq w_1 4^{mw_1}$, and for undirected graphs $\mathfrak{G} = (V, E)$ we have $[\mathrm{s}]\mathrm{pwd}_1 \mathfrak{G} \leq w_1 2^{w_1}$.*

Proof. Since both cases are similar we only prove (1). Let $(U_v)_{v\in T}$ be a non-standard $2^{<\omega}$-partition refinement of $\mathfrak{M}$. By induction on α, we define

- a strictly decreasing sequence $T_\alpha \subseteq T$ of subsets of T;
- an increasing sequence of trees S_α; and
- a partial partition refinement $(V_v)_{v\in S_\alpha}$

such that $u \in T_\alpha$ and $u \preceq v$ imply $v \in T_\alpha$ and we can partition T_α into sets T_α^β satisfying the following conditions:

- $u, v \in T_\alpha$ belong to the same component T_α^β iff $u \sqcap v \in T_\alpha$.
- For every maximal path $C \subseteq S_\alpha$ such that $W := \bigcap_{v\in C} V_v$ contains at least 2 elements, there exists some β with $\bigcup_{v\in T_\alpha^\beta} U_v = W$ and, vice versa, for every component T_α^β there exists such a chain $C \subseteq S_\alpha$.

Intuitively, S_α is the part of T we have already converted and T_α is the part that still has to be transformed into a standard refinement.

Let S_0 be the standard part of T, set $T_0 := T \setminus S_0$, and let $V_v := U_v$ for $v \in S_0$. If α is a limit we set $S_\alpha := \bigcup_{\beta<\alpha} S_\beta$ and $T_\alpha := \bigcap_{\beta<\alpha} T_\beta$.

Suppose that $\alpha = \beta + 1$. Fix a maximal chain $C \subseteq S_\beta$ such that $W := \bigcap_{v\in C} V_v$ contains at least 2 elements. If such a chain does not exist then $(V_v)_{v\in S_\beta}$ is already a partition refinement of $\mathfrak{M}$ (after adding some singletons as leaves if necessary) and we are done.

If there is some $v_0 \in T_\beta$ such that $U_v = W$ then let T' consists of all $u \in T_\beta$ with $v_0 \preceq u$. We add the standard part of T' to S_β above C and remove from T_β this part and all other elements v with $v \sqcap v_0 \in T_\beta$ (the elements below v_0). Set $V_u := U_u$ for the new elements $u \in S_{\beta+1} \setminus S_\beta$.

If such a vertex v_0 does not exist, let $T' \subseteq T_\beta$ be the set of all $v \in T_\beta$ such that $U_v \subseteq W$. Then, by assumption, $\bigcup_{v\in T'} U_v = W$. Fix

a maximal chain $I \subseteq T'$. Note that, for every $v \in T'$ and all $u \in I$ we have $u \sqcap v \in I$. Since I is a linear order there exists a partition refinement $(H_v)_{v\in F}$ of $(I, \preceq)$ of width 1 where each component is some interval $H_v \subseteq I$. We add the tree F to S_β above C, define

$$V_v := \bigcup_{w\in H_v} U_w \setminus \bigcup \{\, U_w \mid w \in I,\ w > u \text{ for all } u \in H_v \,\},$$

for $v \in F$, and set $T_{\beta+1} := T_\beta \setminus I$.

Since $T_\alpha \supset T_\beta$ for $\alpha < \beta$, the construction must stop after at most $|T|^+$ steps with some partition refinement $(V_v)_{v\in S}$.

The components V_v are of the form X or $X \setminus Y$ where X and Y are either components U_w, for some $w \in T$, or of the form $\bigcup_{w\in C} U_w$, for some chain $C \subseteq T$. By Lemma 3.3.8, we have $\mathrm{eti}^n_{\mathrm{o}}(X/\overline{X}) \leq w_n$ in both cases. It follows, by Lemmas 3.3.3 and 3.3.4, that

$$\begin{aligned}\mathrm{eti}^n_{\mathrm{o}}(Y \cup \overline{X}/X \setminus Y) &\leq 2^n \mathrm{eti}^n_{\mathrm{o}}(Y/\overline{Y}) 2^{m(n+1)^r \mathrm{eti}^{r-1}_{\mathrm{o}}(X/\overline{X})} \\ &\leq 2^n w_n 2^{m(n+1)^r w_{r-1}},\end{aligned}$$

where m is the number of relations of arity greater than 1, and r is the maximum of their arities. Therefore,

$$\mathrm{eti}^n_{\mathrm{o}}(X \setminus Y/Y \cup \overline{X}) \leq 2^{m(n+1)^r w_{r-1} 2^{mr^r w_{r-1} + r - 1}}.$$

If $\mathfrak{M}$ is a transition system then Lemma 3.3.2 implies that

$$\begin{aligned}\mathrm{eti}^1_{\mathrm{o}}(X \setminus Y/Y \cup \overline{X}) &\leq \mathrm{eti}^1_{\mathrm{o}}(U_w/\overline{U_w}) \mathrm{eti}^1_{\mathrm{o}}(\overline{U_u}/U_u) \\ &\leq w_1 4^{m w_1}.\end{aligned}$$

□

Corollary 3.6.13. (1) *If there exists a sequence $\bar{w} \in \omega^\omega$ such that $\mathrm{pwd}_n \mathfrak{A} \leq w_n$, $n < \omega$, for every finite substructure $\mathfrak{A} \subseteq \mathfrak{M}$ then $\mathrm{pwd}_n \mathfrak{M} \leq \aleph_{\mathrm{o}}$ for $n < \omega$.*

(2) *If there exists a sequence $\bar{w} \in \omega^\omega$ such that $\mathrm{spwd}_n \mathfrak{A} \leq w_n$, $n < \omega$, for every finite substructure $\mathfrak{A} \subseteq \mathfrak{M}$ then $\mathrm{spwd}_n \mathfrak{M} \leq \aleph_{\mathrm{o}}$ for $n < \omega$.*

A direct consequence of Proposition 3.6.11 is the fact that having a finite partition width is a property of first-order theories.

Theorem 3.6.14. *If $\mathfrak{M}$ is of finite non-standard partition width and $\mathfrak{M} \equiv_{\mathrm{FO}} \mathfrak{N}$ then*

$$\mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M} = \mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{N} \quad \textit{and} \quad \mathrm{spwd}^{\mathrm{ns}}_n \mathfrak{M} = \mathrm{spwd}^{\mathrm{ns}}_n \mathfrak{N}$$

for all $n < \omega$.

Proof. Let $w_i := \mathrm{pwd}_i \mathfrak{M}$, for $i < \omega$. W.l.o.g. assume that the signature is finite. Since there are only finitely many structures of size n there exists an FO-formula $\psi^n_{i,k}(x_0, \ldots, x_{n-1})$ stating that $\mathrm{pwd}_i \mathfrak{M}|_{\bar{x}} \leq k$.

$\mathfrak{M} \models \forall \bar{x} \psi^n_{i,w_i}(\bar{x})$ implies $\mathfrak{N} \models \forall \bar{x} \psi^n_{i,w_i}(\bar{x})$. By Proposition 3.6.11 it follows that $\mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{N} \leq \mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M}$ for $n < \omega$. The claim follows by symmetry.

In the same way we can show that the non-standard symmetric partition widths are equal. □

Corollary 3.6.15. *If $\mathfrak{M} \equiv_{\mathrm{FO}} \mathfrak{N}$ and $\mathfrak{M}$ is of finite* [*symmetric*] *partition width then so is $\mathfrak{N}$.*

Corollary 3.6.16. *Let τ be a finite signature. The class of τ-structures of finite* [*symmetric*] *partition width is $\mathrm{L}_{\omega_1\omega}$-definable.*

Proof. There are only finitely many τ-structures $\mathfrak{M}$ of cardinality n with $\mathrm{pwd}_i \mathfrak{M} \leq k$, and we can construct an FO-formula $\varphi^n_{i,k}$ which defines the class of these structures. Consequently, the class of structures of finite partition width is axiomatised by

$$\bigwedge_{i<\omega} \bigvee_{k<\omega} \bigwedge_{n<\omega} \forall x_0 \cdots \forall x_{n-1} \psi^n_{i,k}(\bar{x})$$

where $\psi^n_{i,k}(\bar{x})$ is the relativisation of $\varphi^n_{i,k}$ to the set $\bar{x}$. □

For the non-standard partition width we are able to prove that for every structure $\mathfrak{M}$ such that $\mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M}$ is finite there exists a non-standard partition refinement of exactly this width.

Proposition 3.6.17. *Let $\mathfrak{M}$ be a structure.*

(1) *There exists a non-standard $2^{<\omega}$-partition refinement $(U_v)_v$ with $\mathrm{pwd}_n(U_v)_v = \mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M}$ for all $n < \omega$.*

(2) *There exists a non-standard $\aleph_0^{<\omega}$-partition refinement $(U_v)_v$ with $\mathrm{spwd}_n(U_v)_v = \mathrm{spwd}^{\mathrm{ns}}_n \mathfrak{M}$ for all $n < \omega$.*

Proof. Since the proofs are nearly identical, we prove only (1). Let $w_n := \mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M}$, and let Δ be the atomic diagram of $\mathfrak{M}$. If $(\mathfrak{N}, U, \sqsubseteq) \models \Phi := \Delta \cup \Pi^2_{\bar{w}}$ then $\mathfrak{M} \subseteq \mathfrak{N}$ and $(U, \sqsubseteq)$ codes a non-standard partition refinement of $\mathfrak{N}$ of width $\bar{w}$ which induces one of $\mathfrak{M}$ of the same width.

To show that Φ is consistent let $\Phi_0 \subseteq \Phi$ be finite. There exists some $k < \omega$ such that Φ_0 does not contain any formula of the form $\varphi^{\tau_0}_{n,m}$ for $n \geq k$. Let $(U, \sqsubseteq)$ code a non-standard partition refinement $(U_v)_v$ of $\mathfrak{M}$ such that

$$\mathrm{pwd}_n(U_v)_v = \mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M} \qquad \text{for all } n < k\,.$$

Then $(\mathfrak{M}, U, \sqsubseteq) \models \Phi_0$. □

Let $\mathfrak{M} \preceq \mathfrak{N}$. Every non-standard partition refinement $(U_v)_{v \in T}$ of $\mathfrak{N}$ induces a corresponding refinement $(U_v \cap M)_{v \in T}$ of $\mathfrak{M}$, that is, each

partition refinement of $\mathfrak{N}$ can be obtained by extending one of $\mathfrak{M}$. The following proposition states the converse: every non-standard partition refinement of $\mathfrak{M}$ can be extended to one of $\mathfrak{N}$.

Proposition 3.6.18. *Let $(U_v)_{v\in T}$ be a non-standard $2^{<\omega}$-partition refinement of $\mathfrak{M}$. For every $\mathfrak{N} \succeq \mathfrak{M}$ there exists an elementary extension $S \succeq T$ and a non-standard $2^{<\omega}$-partition refinement $(V_v)_{v\in S}$ of $\mathfrak{N}$ of the same width such that $V_{h(v)} \supseteq U_v$ for all $v \in T$ where $h : T \to S$ is the corresponding elementary embedding.*

Proof. W.l.o.g. we may assume that $|M| \geq \aleph_0$. Set $w_n := \mathrm{pwd}_n(U_v)_v$. Let $(U, \sqsubseteq)$ be relations coding $(U_v)_v$. Let $\Delta_{\mathfrak{N}}$ be the elementary diagram of $\mathfrak{N}$, Ξ the elementary diagram of $(M, \sqsubseteq)$, and set

$$\Gamma := \{\, Pa \mid a \in N \,\} \cup \{\, Ua\bar{b} \mid a, \bar{b} \subseteq M \,\}.$$

By modifying $\Pi^2_{\bar{w}}$ we can obtain a set of formulae expressing that $(U \cap (P \times M^n), \sqsubseteq)$ codes a non-standard partition refinement of P. Let Π^P be this set.

We have to show that $\Psi := \Xi \cup \Gamma \cup \Pi^P \cup \Delta_{\mathfrak{N}}$ has a model $(\mathfrak{N}', P, V, \sqsubseteq')$. Then there exists an elementary embedding $h : (T, \sqsubseteq) \preceq (S, \sqsubseteq')$ where

$$S := \{\, \bar{a} \in (N')^n \mid \bar{a} \sqsubseteq \bar{a} \,\},$$

and $(V_{\bar{a}})_{\bar{a}\in S}$ with $V_{\bar{a}} := \{\, b \in N \mid (b, \bar{a}) \in V \,\}$ is a non-standard partition refinement of $\mathfrak{N}$ with $U_v \subseteq V_{h(v)}$.

Let $\Psi_0 \subseteq \Psi$ be a finite subset. Then $\Psi_0 \subseteq \Xi_0 \cup \Gamma_0 \cup \Pi^P \cup \Delta_0$ for some finite sets $\Xi_0 \subseteq \Xi$, $\Gamma_0 \subseteq \Gamma$, and $\Delta_0 \subseteq \Delta_{\mathfrak{N}}$. Let $A \subseteq N$ be the finite set of elements mentioned in $\Xi_0 \cup \Gamma_0 \cup \Delta_0$, and set $M_0 := A \cap M$, $N_0 := A \setminus M$. Let $\bar{a}$ be an enumeration of N_0. There exists a tuple $\bar{b} \subseteq M$ such that $\mathrm{tp}(\bar{b}/M_0) = \mathrm{tp}(\bar{a}/M_0)$. Then $(\mathfrak{M}, M_0 \cup \bar{b}, U, \sqsubseteq) \models \Psi_0$. □

We conclude this section by considering two versions of a compactness theorem for structures of finite non-standard partition width. After proving a version where the non-standard partition width is bounded by a given constant we show that compactness fails if we only demand that the partition width is finite.

Theorem 3.6.19 (Compactness). *Let $\bar{w} \in \omega^\omega$. A set $\Phi \subseteq \mathrm{FO}$ of sentences has a model $\mathfrak{M}$ with $\mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M} \leq w_n$ for $n < \omega$ if and only if every finite subset $\Phi_0 \subseteq \Phi$ has such a model. The same holds for $\mathrm{spwd}^{\mathrm{ns}}_n \mathfrak{M}$.*

Proof. Φ has a model $\mathfrak{M}$ of width $\mathrm{pwd}^{\mathrm{ns}}_n \mathfrak{M} \leq w_n$ if and only if $\Phi \cup \Pi^2_{\bar{w}}$ is consistent. Since all finite subsets of $\Phi \cup \Pi^2_{\bar{w}}$ are consistent, so is the whole set. □

Corollary 3.6.20. *A set $\Phi \subseteq \mathrm{FO}$ of sentences has a model of finite partition width if and only if there exists a sequence $\bar{w} \in \omega^\omega$ such that every finite subset $\Phi_0 \subseteq \Phi$ has a model $\mathfrak{M}$ with $\mathrm{pwd}_n \mathfrak{M} \leq w_n$ for $n < \omega$. The same holds for the symmetric partition width.*

We have seen that a group has a finite partition width if and only if it is finite. Using this result we can show that certain theorems of model theory fail if we restrict the class of structures to those of finite partition width. Note that, by the preceding theorem, if, instead, we consider only models of partition width less than some given finite bound, then the situation is completely different.

Lemma 3.6.21. *There is an* FO*-sentence* φ_{fin} *such that, for each* $n < \aleph_0$, φ_{fin} *has a model of cardinality* n *that is of finite partition width, but* φ_{fin} *has no infinite model of finite partition width.*

Proof. Let φ_{fin} be the conjunction of the group axioms in signature $\tau := \{\cdot\}$. Since a group has a finite partition width if and only if it is finite the claim follows. □

Theorem 3.6.22. *When restricted to models of finite partition width, first-order logic does not have any of the following properties:*

(1) *the compactness property;*

(2) *Beth's definability property;*

(3) *Craig's interpolation property.*

Proof. (1) A counterexample is

$$\Phi := \{\varphi_{\text{fin}}\} \cup \{\, \varphi_n \mid n < \omega \,\}$$

where φ_n states that there are at least n different elements. Φ has no model of finite partition width but each finite subset of Φ has one.

(2) Let α be a sentence stating that $<$ is a discrete linear order with minimal and maximal element and that s is the successor function mapping the last element to the first one. Let β be the group axiom for $+$ and define

$$\chi := \forall xy(x + 0 = x \wedge x + sy = s(x + y)) .$$

If $\mathfrak{M} := (M, +, \leq, s, 0)$ is a model of $\psi := \alpha \wedge \beta \wedge \chi$ of finite partition width then $|M| < \aleph_0$ and, hence, $(M, +) \cong (\mathbb{Z}_n, +)$ for some $n \in \mathbb{N}$. Thus, ψ is an implicit definition of $+$ in $(M, \leq, s, 0)$. But there cannot be an explicit one since, otherwise, the set of even positions were definable by

$$\varphi(x) := \exists y(x = y + y) .$$

(3) follows from (2). □

3.7 The independence property

Proposition 3.5.6 can be used to link the concept of partition width with the model theoretic notion of VC-dimension or, equivalently, the independence property.

independence property

Definition 3.7.1. Let T be a first-order theory. An FO-formula $\varphi(\bar{x}, \bar{y})$ has the *independence property* (w.r.t. T) if there exists a model $\mathfrak{M}$ of T containing sequences $(\bar{a}_I)_{I \subseteq \omega}$ and $(\bar{b}_i)_{i<\omega}$ such that

$$\mathfrak{M} \models \varphi(\bar{a}_I, \bar{b}_i) \quad \text{iff} \quad i \in I\,.$$

on singletons

We say that a structure $\mathfrak{M}$ has the independence property if there exists a formula φ that has the independence property w.r.t. $\mathrm{Th}(\mathfrak{M})$. If $\bar{a}_I$ and $\bar{b}_i$ are singletons we say that $\mathfrak{M}$ has the *independence property on singletons*.

Theorem 3.7.2 (II, 4.11 of [71])**.** *Let T be a first-order theory. The following statements are equivalent:*

(1) *No formula $\varphi(x; \bar{y})$ has the independence property w.r.t. T.*

(2) *No formula $\varphi(\bar{x}; \bar{y})$ has the independence property w.r.t. T.*

(3) *For all formulae $\varphi(x; \bar{y})$ and all $n < \omega$ there is some $k < \omega$ such that $|S^1_\varphi(A)| < 2^{k/n}$ for all sets A of size k.*

(4) *For all $\varphi(\bar{x}; \bar{y})$ there is some $n < \omega$ such that $|S^m_\varphi(A)| < |A|^n$ for all finite sets A with at least two elements.*

In [2] it is shown that the independence property and the independence property on singletons coincide if we allow monadic parameters.

Lemma 3.7.3 (Baldwin and Shelah [2])**.** *Let $\mathfrak{M}$ be a structure and $\varphi(x, y_0 \dots y_n)$ a formula with the independence property w.r.t.* $\mathrm{Th}(\mathfrak{M})$*. Then there exists an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$, a set $P \subseteq N$, and a formula $\psi(x, y_0 \dots y_{n-1})$ that has the independence property w.r.t.* $\mathrm{Th}(\mathfrak{N}, P)$.

Corollary 3.7.4. *Let $\mathfrak{M}$ have the independence property. There exists an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ and unary predicates $\bar{P}$ such that $(\mathfrak{N}, \bar{P})$ has the independence property on singletons.*

It immediately follows that the independence property implies MSO-coding.

Lemma 3.7.5. *Let $\mathfrak{M}$ have the independence property on singletons. There exists an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ that admits strong* FO*-coding.*

Proof. Choose an elementary extension $\mathfrak{N}$ that contains sequences $(a_I)_{I\subseteq\omega}$ and $(b_i)_{i\in\omega}$ such that, for some formula $\varphi(x, y)$, we have

$$\mathfrak{N} \models \varphi(a_I, b_i) \quad \text{iff} \quad i \in I\,.$$

Fix disjoint infinite sets X, $Y \subseteq B := \{\, b_i \mid i < \omega \,\}$, and define a function $f : X \times Y \to M$ by $f(b_i, b_j) := a_{\{i,j\}}$. For $x \in X$, $y \in Y$, and $z \in Z := f(X, Y)$ we have

$$f(x, y) = z \quad \text{iff} \quad \mathfrak{M} \models \varphi(z, x) \wedge \varphi(z, y)\,.$$

Hence, f is an FO-definable bijection $X \times Y \to Z$. □

Together with the results above it follows that no structure with the independence property has finite partition width. This slightly extends a result of Parigot [58] who showed that trees do not have the independence property.

Proposition 3.7.6. *If $\mathfrak{M}$ is a structure with the independence property then* $\mathrm{pwd}_n\,\mathfrak{M} \geq \aleph_0$ *for some n.*

Proof. If $\mathfrak{M}$ has the independence property then there exists an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ and unary predicates $\bar{P}$ such that $(\mathfrak{N}, \bar{P})$ has the independence property on singletons. Hence, there exists an elementary extension $(\mathfrak{N}', \bar{P}')$ which admits strong FO-coding. If $\mathfrak{M}$ where of finite partition width, then so would be $\mathfrak{N}$, $(\mathfrak{N}, \bar{P})$, and $(\mathfrak{N}', \bar{P}')$. The latter contradicts Corollary 3.5.7. □

3.8 Indiscernible sequences

In the present section we consider sets A and B such that $\mathrm{ti}^m_\Delta(A/B)$ is large. We will construct sequences $a_i \in A$ and $b_i \in B$, $i \in I$, such that the bijection $b_i \mapsto a_i$ is FO-definable. This property might by useful when defining grids. Once we have managed to define the rows $(a^s_i)_s$ of a grid, we can obtain the columns by connecting each pair of rows by such a bijection.

In the following let Δ be a finite set of formulae $\varphi(\bar{x}; \bar{y})$ where we distinguish between free variables $\bar{x}$ and parameters $\bar{y}$. Accordingly we write

$$\bar{a} \approx^\Delta_U \bar{b} \quad :\text{iff} \quad \mathfrak{M} \models \varphi(\bar{a}; \bar{c}) \leftrightarrow \varphi(\bar{b}; \bar{c}) \ \text{ for all } \varphi \in \Delta,\ \bar{c} \in U^n.$$

W.l.o.g. we assume that $|\bar{x}| = m$ and $|\bar{y}| = n$ for all $\varphi(\bar{x}; \bar{y}) \in \Delta$.

Definition 3.8.1. Let $\varphi(\bar{x}; \bar{y}) \in \Delta$ and $\sigma \in \{=, \neq, <, >, \leq, \geq\}$. Two sequences $\bar{a}^s \in M^m$, $s \in I$, and $\bar{b}^s \in M^n$, $s \in I$, are of *φ-type σ* if *φ-type σ*

$$\mathfrak{M} \models \varphi(\bar{a}^s; \bar{b}^t) \quad \text{iff} \quad s\ \sigma\ t\,.$$

Mirroring the proof of Ramsey's theorem we first construct two sequences of φ-type σ.

Lemma 3.8.2. *Let $\sim$ an equivalence relation with $k > 1$ classes on the set of subsets $X \subseteq [m]$ of size $|X| = 2$. There exists a subset $I \subseteq [m]$ of size*

$$|I| \geq \log_k(m(k-1)+1) > \log_k m$$

such that $\{i,k\} \sim \{i,l\}$ for all $i, k, l \in I$ with $i < k, l$.

Proof. We construct two sequences $(I_s)_s$ and $(J_s)_s$ of subsets of $[m]$ such that every element of J_s is greater than all elements of I_s and

$$\{a_i, a_k\} \sim \{a_i, a_l\} \quad \text{for all } i \in I_s,\ k, l \in J_s\,.$$

Let $I_0 := \varnothing$ and $J_0 := [m]$. Suppose that I_s and J_s are already defined. If $J_s = \varnothing$ then we stop and set $I := I_s$. Otherwise, let i be the minimal element of J_s and set $I_{s+1} := I_s \cup \{i\}$. Let $\approx$ be the equivalence relation on $J_s \setminus \{i\}$ defined by

$$k \approx l \quad :\text{iff} \quad \{i,k\} \sim \{i,l\}\,.$$

By the Pigeon Hole Principle, $J_s \setminus \{i\}$ contains a $\approx$-class J_{s+1} of size at least $(|J_s| - 1)/k$. This concludes the construction.

The set I we have obtained has the size $|I| = \max\{\, s+1 \mid J_s \neq \varnothing \,\}$. The claim follows if we prove that

$$|J_s| \geq k^{-s}\left(m - \frac{k^s - 1}{k-1}\right)$$

since

$$\begin{aligned} k^{-s}\left(m - \frac{k^s-1}{k-1}\right) > 0 \quad &\text{iff} \quad k^s < m(k-1)+1 \\ &\text{iff} \quad s < \left\lceil \log_k(m(k-1)+1) \right\rceil. \end{aligned}$$

We have $|J_0| = m$ and, by induction,

$$\begin{aligned} |J_s| &\geq k^{-1}(|J_{s-1}| - 1) \\ &\geq k^{-1}k^{-(s-1)}\left(m - \frac{k^{s-1}-1}{k-1}\right) - \frac{1}{k} \\ &= k^{-s}\left(m - \frac{k^{s-1}-1}{k-1} - k^{s-1}\right) \\ &= k^{-s}\left(m - \frac{k^{s-1} - 1 + k^s - k^{s-1}}{k-1}\right) \\ &= k^{-s}\left(m - \frac{k^s - 1}{k-1}\right) \end{aligned}$$

as desired. □

Lemma 3.8.3. *Let Δ be a finite set of formulae and let $\bar{a}^i \in A^m$, $i \in I$, be a sequence such that $\bar{a}^i \not\approx_B^\Delta \bar{a}^k$ for $i \neq k$. There exist a formula $\varphi(\bar{x};\bar{y}) \in \Delta$, a subset $J \subseteq I$ of size $|J| \geq |\Delta|^{-1}(\log_2|I| + 1)$, and a sequence $\bar{b}^i \in B^n$, $i \in J$, such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^i;\bar{b}^l) \leftrightarrow \neg\varphi(\bar{a}^l;\bar{b}^l) \qquad \text{for all } i, l \in J,\ l < i\,.$$

Proof. By induction on s, we construct a decreasing sequence of sets $I_s \subseteq I$, an index $i_s \in I_{s-1}$, a formula $\varphi_{i_s} \in \Delta$, and tuples $\bar{b}^{i_s} \in B^n$ such that the above condition is satisfied for $\varphi := \varphi_{i_s}$, $l := i_s$, and $i \in I_s$. Then the set $J' := \{i_0, i_1, \dots\}$ contains a subset $J \subseteq J'$ of size $|J| \geq |J'|/|\Delta|$ such that $\varphi_i = \varphi_k$ for all $i, k \in J$.

Let $I_{-1} := I$. Assume that I_s is already defined. To choose I_{s+1}, i_{s+1}, $\varphi_{i_{s+1}}$, and $\bar{b}^{i_{s+1}}$ we consider the following cases.

If $I_s = \varnothing$ then we stop. If $I_s = \{i\}$ then we set $i_{s+1} := i$, choose arbitrary $\bar{b}^i \in B^n$ and $\varphi_i \in \Delta$, and set $I_{s+1} := \varnothing$ thereby stopping in the next step. Otherwise, I_s contains at least two different elements $i, k \in I_s$. Since $\bar{a}^i \not\approx_B^\Delta \bar{a}^k$ there is some $\bar{b} \in B^n$ and some $\varphi \in \Delta$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}^i;\bar{b}) \leftrightarrow \neg\varphi(\bar{a}^k;\bar{b}).$$

Hence, the sets

$$I_s^+ := \{\, i \in I_s \mid \mathfrak{M} \vDash \varphi(\bar{a}^i;\bar{b}) \,\}$$
$$\text{and} \quad I_s^- := \{\, i \in I_s \mid \mathfrak{M} \nvDash \varphi(\bar{a}^i;\bar{b}) \,\}$$

are both nonempty. If $|I_s^+| \geq |I_s^-|$ we set $I_{s+1} := I_s^+$ and choose some $i_{s+1} \in I_s^-$. Otherwise, $I_{s+1} := I_s^-$ and $i_{s+1} \in I_s^+$.

Since $|I_{s+1}| \geq \frac{1}{2}|I_s|$ it follows that $|I_{s-1}| \geq 2^{-i}|I|$, i.e., $I_{s-1} \neq \varnothing$ for $i \leq \log_2|I|$. Hence, the above procedure can be carried out for at least $\log_2|I| + 1$ steps. □

Lemma 3.8.4. *Let $(\bar{a}^i)_{i\in I}$ and $(\bar{b}^i)_{i\in I}$ be sequences such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^i;\bar{b}^l) \leftrightarrow \neg\varphi(\bar{a}^l;\bar{b}^l) \qquad \text{for all } i, l \in J,\ l < i\,.$$

There exists a set $J \subseteq I$ of indices of size $|J| > \log_2|I|$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}^i;\bar{b}^l) \leftrightarrow \varphi(\bar{a}^j;\bar{b}^l) \qquad \text{for all } i, j, l \in J,\ i, j < l\,.$$

Proof. We obtain the desired subset J if we reverse the ordering of I and apply Lemma 3.8.2 to the relation

$$\{i,k\} \sim \{j,l\} \quad :\text{iff} \quad \mathfrak{M} \vDash \varphi(\bar{a}^i;\bar{b}^k) \leftrightarrow \varphi(\bar{a}^j;\bar{b}^l)\,. \qquad \square$$

Proposition 3.8.5. *Let Δ be a finite set of formulae and let $\bar{a}^s \in A^m$, $s \in I$, be a sequence such that $\bar{a}^s \not\approx_B^\Delta \bar{a}^t$ for $s \neq t$.*

There exist a formula $\varphi(\bar{x};\bar{y}) \in \Delta$, a relation $\sigma \in \{=, \neq, \leq, >\}$, a subset $J \subseteq I$ of size

$$|J| > \frac{1}{4} \log_2 \frac{\log_2 |I| + 1}{|\Delta|},$$

and a sequence $\bar{b}^s \in B^n$, $s \in J$, such that $(\bar{a}^s)_{s\in J}$ and $(\bar{b}^s)_{s\in J}$ are of φ-type σ.

Proof. Combining the preceding lemmas we find a sequence $\bar{b}^s \in B^n$, $s \in J$, such that

$$\mathfrak{M} \models \varphi(\bar{a}^s;\bar{b}^t) \leftrightarrow \neg\varphi(\bar{a}^t;\bar{b}^t) \qquad \text{for all } s, t \in J,\ t < s,$$

and $$\mathfrak{M} \models \varphi(\bar{a}^s;\bar{b}^t) \leftrightarrow \varphi(\bar{a}^v;\bar{b}^t) \qquad \text{for all } s, v, t \in J,\ s, v < t.$$

Hence, we can partition $J = J_= \cup J_\neq \cup J_\leq \cup J_>$ into sets such that the restriction of $(\bar{a}^s)_s$ and $(\bar{b}^s)_s$ to J_σ is of φ-type σ. At least one of these sets J_σ has the required size. □

Having constructed sequences $(\bar{a}^s)_s$ and $(\bar{b}^s)_s$ of φ-type σ we show how to define a bijection $b^s_k \mapsto a^s_i$ for some i and k. First, we consider the simpler case of sequences of singletons.

Lemma 3.8.6. *Let $(a^s)_{s\in I}$ and $(b^s)_{s\in I}$ be sequences of φ-type σ. There exists a formula $\chi(x, y)$ with monadic parameters $A := \{ a^s \mid s \in I \}$ and $B := \{ b^s \mid s \in I \}$ such that*

$$\mathfrak{M} \models \chi(a^s, b^t) \quad \text{iff} \quad s = t.$$

Proof. If $\sigma \in \{=, \neq\}$ we can set $\chi := \varphi$ and $\chi := \neg\varphi$, respectively. Suppose that $\sigma = \leq$. Since we can reverse the order of I and replace φ by $\neg\varphi$ the other cases follow by symmetry.

We can define the ordering on A by the formula

$$\vartheta(x, y) := (\forall z \in B)(\varphi(y, z) \to \varphi(x, z)).$$

Hence, we obtain the desired formula χ by saying that x is the maximal element in A such that $\varphi(x, y)$ holds.

$$\chi(x, y) := \varphi(x, y) \wedge \neg(\exists z \in A)\big(x \neq z \wedge \vartheta(x, z) \wedge \varphi(z, y)\big). \qquad \square$$

For the general case we need some technical preparations that are finitary versions of results of Shelah [72].

Definition 3.8.7. (a) We say that sequences $\bar{s}$ and $\bar{t}$ have the same *order type* if $s_i < s_k \Leftrightarrow t_i < t_k$ for all i and k.

order type

(b) Let $\varphi(\bar{x})$ be an FO-formula with $n := |\bar{x}|$. A sequence $(\bar{a}^s)_{s\in I}$ of n-tuples is *φ-indiscernible* iff

φ-indiscernible

$$\mathfrak{M} \models \varphi(a^{s_0}_0, \dots, a^{s_{n-1}}_{n-1}) \leftrightarrow \varphi(a^{t_0}_0, \dots, a^{t_{n-1}}_{n-1})$$

for all sequences $\bar{s}, \bar{t} \in I^n$ of the same order type.

Lemma 3.8.8. *Let $\varphi(\bar{x})$ be a formula with $|x| = n$. Any sequence $\bar{a}^s \in M^n$, $s \in I$, of length $|I| \to (m)^n_{2^{n^n}}$ contains a φ-indiscernible subsequence of length m.*

Proof. For each sequence $s_0 < \cdots < s_{n-1}$ of n distinct elements $s_i \in I$ and every function $g : [n] \to [n]$, we record whether

$$\mathfrak{M} \vDash \varphi(a_0^{s_{g(0)}}, \dots a_{n-1}^{s_{g(n-1)}})$$

holds or not. Since $|I| \to (m)^n_{2^{n^n}}$ there exists a subset $J \subseteq I$ of size $|J| \geq m$ such that all increasing sequences $\bar{s} \in J^n$ have the same colour. Consequently, $(\bar{a}^s)_{s \in J}$ is φ-indiscernible. □

Definition 3.8.9. Let $(\bar{a}^s)_{s \in I}$ be a finite φ-indiscernible sequence. Let $I_- \subseteq I$ consists of the first n elements of I and $I_+ \subseteq I$ of the last n. Set $I_0 := I \setminus (I_- \cup I_+)$. I_-, I_0, I_+

(a) Let $\asymp \subseteq [n] \times [n]$ be the equivalence relation defined by $i \asymp k$ iff there exists an FO-formula $\chi(x, y)$ with first-order parameters $\bar{a}^s$, for $s \in I_- \cup I_+$, and monadic parameters $A_i := \{ a_i^s \mid s \in I \}$, for $i < n$, such that $i \asymp k$

$$\mathfrak{M} \vDash \chi(a_i^s, a_k^t) \quad \text{iff} \quad s = t \qquad \text{for all } s,\ t \in I_0 .$$

(b) $\asymp$ partitions $[n]$ into classes $N_0 \cup \cdots \cup N_{m-1}$. We set $\bar{a}_i^s := \bar{a}^s|_{N_i}$, i.e., $\bar{a}^s = \bar{a}_0^s \dots \bar{a}_{m-1}^s$.

(c) To simplify notation we set $\varphi[s_0, \dots, s_{m-1}] := \varphi(\bar{a}_0^{s_0}, \dots, \bar{a}_{m-1}^{s_{m-1}})$.

Remark. $\asymp$ is obviously an equivalence relation.

Lemma 3.8.10. *For each $\asymp$-class $N = \{k_0, \dots, k_r\}$ there exists a formula $\psi(\bar{x})$ with first-order parameters $\bar{a}^s$, $s \in I_- \cup I_+$, and monadic parameters $A_i := \{ a_i^s \mid s \in I \}$, $i < n$, such that*

$$\mathfrak{M} \vDash \psi(\bar{b}) \quad \text{iff} \quad \bar{b} = \bar{a}^s|_N \quad \textit{for some } s \in I .$$

Proof. By definition, there exist formulae $\chi_i(x, y)$, for $i < r$, such that

$$\mathfrak{M} \vDash \chi_i(a_{k_i}^s, a_{k_r}^t) \quad \text{iff} \quad s = t .$$

We define

$$\psi(\bar{x}) := \bigwedge_{i \leq r} A_{k_i} x_i \wedge \bigwedge_{i < r} \chi_i(x_i, x_r) .$$ □

We construct the bijection $b_k^s \mapsto a_j^s$ by showing that, when considering an φ-indiscernible subsequence of $(\bar{a}^s \bar{b}^s)_s$, there are indices $j < m$ and $k < n$ such that $j \asymp m + k$.

Proposition 3.8.11. *Let $(\bar{a}^s)_{s\in I}$ be a φ-indiscernible sequence. For all $\bar{s}, \bar{t} \in I^m$, we have that*

$$\mathfrak{M} \vDash \varphi[s_0, \dots, s_{m-1}] \leftrightarrow \varphi[t_0, \dots, t_{m-1}] .$$

The proof is split into the following two lemmas. Let $\psi_i(\bar{x})$ be the formula from Lemma 3.8.10 defining equality for N_i.

Lemma 3.8.12. $\mathfrak{M} \vDash \varphi[s_0, \dots, s_{m-1}] \leftrightarrow \varphi[t_0, \dots, t_{m-1}]$ *for all $\bar{s}, \bar{t} \in I^m$ such that $|\{s_0, \dots, s_{m-1}\}| = |\{t_0, \dots, t_{m-1}\}| = m$.*

Proof. It is sufficient to prove the claim for transpositions. Suppose otherwise. Then, by symmetry, we have

$$\mathfrak{M} \vDash \varphi[s_0, \dots, s_i, s_{i+1}, \dots, s_{m-1}] \wedge \neg\varphi[s_0, \dots, s_{i+1}, s_i, \dots, s_{m-1}]$$

for some $\bar{s} \in I^m$ such that no s_j lies in the interval (s_i, s_{i+1}). By indiscernibility, we further may assume that $s_j \in I_- \cup I_+$ for $j < i$ or $j > i+1$. Define

$$\varphi'(x, y) := \exists \bar{x}' \exists \bar{y}' \big(\psi_i(x\bar{x}') \wedge \psi_{i+1}(y\bar{y}') \\ \wedge \varphi(\bar{a}_0^{s_0}, \dots, \bar{a}_{i-1}^{s_{i-1}}, x\bar{x}', y\bar{y}', \bar{a}_{i+2}^{s_{i+2}}, \dots, \bar{a}_{m-1}^{s_{m-1}}) \big)$$

Let $j \in N_i$ and $k \in N_{i+1}$ be the minimal elements of their respective $\asymp$-class. For $s, t \in I_0$, $s \neq t$, it follows that

$$\mathfrak{M} \vDash \varphi'(a_j^s, a_k^t) \quad \text{iff} \quad s < t .$$

Hence, the sequences $(a_j^s)_s$ and $(a_k^s)_s$ are of φ'-type σ, for $\sigma \in \{\leq, <\}$, and, by Lemma 3.8.6, we have $j \asymp k$. Contradiction. □

Lemma 3.8.13. *Let $\bar{s}, \bar{t} \in I^m$ be sequences such that, for some permutation σ,*

$$s_{\sigma 0} \leq \cdots \leq s_{\sigma(i-1)} < s_{\sigma i} = \cdots = s_{\sigma k} < s_{\sigma(k+1)} \leq \cdots \leq s_{\sigma(m+1)}$$

and $s_{\sigma k} < t_{\sigma k} < s_{\sigma(k+1)}$ and $t_{\sigma j} = s_{\sigma j}$ for $j \neq k$. Then

$$\mathfrak{M} \vDash \varphi[s_0, \dots, s_{m-1}] \leftrightarrow \varphi[t_0, \dots, t_{m-1}] .$$

Proof. We prove the claim by induction on $k - i$. Suppose otherwise. By permuting $[n]$ we may assume that

$$s_0 \leq \cdots \leq s_{i-1} < s_i = \cdots = s_k < s_{k+1} \leq \cdots \leq s_{m-1}$$

and $s_k < t_k < s_{k+1}$. Let

$$\varphi'(\bar{x}_i, \dots, \bar{x}_k) := \varphi(\bar{a}_0^{s_0}, \dots, \bar{a}_{i-1}^{s_{i-1}}, \bar{x}_i, \dots, \bar{x}_k, \bar{a}_{k+1}^{s_{k+1}}, \dots, \bar{a}_{m-1}^{s_{m-1}}) .$$

By symmetry, we may assume that

$$\mathfrak{M} \vDash \varphi'[s_i, \dots, s_{k-1}, s_k] \wedge \neg\varphi'[s_i, \dots, s_{k-1}, t_k]\,.$$

By indiscernibility, the first part implies that $\mathfrak{M} \vDash \varphi'[v, \dots, v]$ for all $s_{i-1} < v < s_{k+1}$, while the second part implies, by induction hypothesis and the preceding lemma, that $\mathfrak{M} \vDash \neg\varphi'[v_i, \dots, v_k]$ for all $v_i, \dots, v_k \in (s_{i-1}, s_{k+1})$ such that $|\{v_i, \dots, v_k\}| > 1$.

Again we may assume that $s_0, \dots, s_{i-1} \in I_-$ and $s_{k+1}, \dots, s_{m-1} \in I_+$. It follows that

$$\mathfrak{M} \vDash \varphi'[v_i, \dots, v_k] \quad \text{iff} \quad v_i = \cdots = v_k$$

for all $v_i, \dots, v_k \in I_0$. Define

$$\chi(x, y) := \exists\bar{x}'\exists\bar{y}'\exists\bar{z}_{i+1}\cdots\exists\bar{z}_{k-1}(\psi_i(x\bar{x}') \wedge \psi_k(y\bar{y}') \wedge \bigwedge_{i<j<k}\psi_j(\bar{z}_j)$$
$$\wedge\, \varphi(\bar{a}_0^{s_0}, \dots, \bar{a}_{i-1}^{s_{i-1}}, x\bar{x}', \bar{z}_{i+1}, \dots, \bar{z}_{k-1}, y\bar{y}', \bar{a}_{k+1}^{s_{k+1}}, \dots, \bar{a}_{m-1}^{s_{m-1}})\,.$$

Let $j_0 \in N_i$ and $j_1 \in N_k$ be the minimal elements of their respective $\asymp$-class. It follows that $\mathfrak{M} \vDash \chi(a^s_{j_0}, a^t_{j_1})$ iff $s = t$, for $s, t \in I_0$. Hence, $j_0 \asymp j_1$. Contradiction. □

With all these preparations we are finally able to construct a formula that defines a bijection $\bar{b}^s \mapsto \bar{a}^s$.

Proposition 3.8.14. *Let $A, B \subseteq M$ be disjoint sets. If*

$$\mathrm{ti}^n_\Delta(A/B) \geq L := 2^{16^K|\Delta|} \quad \textit{for} \quad K := R(l + 2m + 2n)^{m+n}_{2^{(m+n)^{m+n}}},$$

then there exist sequences $a^s \in A$, $s < l$, and $b^s \in B$, $s < l$, and some FO*-formula $\varphi(x, y; \bar{Z})$ such that, for some $\bar{P} \subseteq \wp(A \cup B)$,*

$$\mathfrak{M} \vDash \varphi(a^s, b^t; \bar{P}) \quad \text{iff} \quad s = t\,.$$

Proof. Fix a set of representatives $\bar{a}^s \in A^m$, $s < L$, of the $\approx^\Delta_B$-classes of A^m. By Proposition 3.8.5 there exists a subset $J \subseteq [L]$ and a sequence $\bar{b}^s \in B^n$, $s \in J$, of length

$$|J| > \frac{1}{4}\log_2\frac{\log_2 L + 1}{|\Delta|} > K\,,$$

some formula $\vartheta(\bar{x}; \bar{y}) \in \Delta$, and a relation $\sigma \in \{=, \neq, \leq, >\}$ such that

$$\mathfrak{M} \vDash \vartheta(\bar{a}_i, \bar{b}_k) \quad \text{iff} \quad i\ \sigma\ k\,.$$

Since $|J| > K \to (l+2m+2n)^{m+n}_{2^{(m+n)^{m+n}}}$, Lemma 3.8.8 implies that there exists a subset $I \subseteq J$ of size $|I| \geq l + 2(m+n)$ such that the sequences $(\bar{a}_i\bar{b}_i)_{i\in I}$ is ϑ-indiscernible. Since

$$\mathfrak{M} \vDash \vartheta(\bar{a}^s; \bar{b}^s) \leftrightarrow \neg\vartheta(\bar{a}^t; \bar{b}^s) \qquad \text{for } t > s\,,$$

Proposition 3.8.11 implies that there are indices $j < m$ and $k < n$ with $j \asymp m + k$. For $s \in I_- \cup I_+$, let $P_i^s := \{a_i^s\}$ and $Q_i^s := \{b_i^s\}$. Further, set $A_i := \{ a_i^s \mid s \in I \}$ and $B_i := \{ b_i^s \mid s \in I \}$. There exists a formula $\chi(x, y)$ with monadic parameters $\bar{P}$, $\bar{Q}$, $\bar{A}$, and $\bar{B}$ such that

$$\mathfrak{M} \models \chi(a_i^s, b_k^t) \quad \text{iff} \quad s = t \qquad \text{for all } s,\ t \in I_0 .$$

Hence, the $(a_i^s)_{s \in I_0}$ and $(b_k^s)_{s \in I_0}$ are the desired sequences of length $|I_0| = |I| - 2(m + n) \geq l$. □

We have constructed an FO-definable bijection $b^s \mapsto a^s$. The next result shows that, if the set of parameters B is well-ordered by some formula, then so is the sequence $(a^s)_s$.

Lemma 3.8.15. *Let $\Delta \subseteq$ FO be finite, $A \subseteq M^n$, and $B \subseteq M$ such that*

$$\bar{a} \not\approx_B^\Delta \bar{b} \qquad \text{for all } \bar{a}, \bar{b} \in A,\ \bar{a} \neq \bar{b} .$$

If there is an FO-formula $\varphi(x, y, \bar{P})$ well-ordering B with monadic parameters $\bar{P}$ then there is such a formula with parameters $\bar{P}$ and B which well-orders A.

Proof. Fix an enumeration $\vartheta_i(\bar{x}; \bar{y})$, $i < m$, of Δ where we assume w.l.o.g. that $r := |\bar{y}|$ is the same for all i. We order $B^r \times [m]$ lexicographically. For $\bar{a}, \bar{b} \in A$ let $(\bar{c}, i)$ be the minimal element in $B^r \times [m]$ such that

$$\mathfrak{M} \models \vartheta_i(\bar{a}; \bar{c}) \leftrightarrow \neg\vartheta_i(\bar{b}; \bar{c}) .$$

We define $\bar{a} < \bar{b}$ iff

$$\mathfrak{M} \models \neg\vartheta_i(\bar{a}; \bar{c}) \wedge \vartheta_i(\bar{b}; \bar{c}) .$$

Clearly, this relation is definable from φ and B. It also is obviously irreflexive and and antisymmetric. Suppose that A contains elements with $\bar{a}_1 < \bar{a}_2 < \bar{a}_3$ and $\bar{a}_3 < \bar{a}_1$, and let $(\bar{c}_1, i_1), (\bar{c}_2, i_2), (\bar{c}_3, i_3) \in B^r \times [m]$ witness these facts.

We consider the following cases. If $(\bar{c}_3, i_3) < (\bar{c}_1, i_1)$ then

$$\mathfrak{M} \models \vartheta_{i_3}(\bar{a}_1; \bar{c}_3) \leftrightarrow \vartheta_{i_3}(\bar{a}_2; \bar{c}_3) ,$$

and, since $\bar{a}_3 < \bar{a}_1$, we have

$$\mathfrak{M} \models \vartheta_{i_3}(\bar{a}_2; \bar{c}_3) \wedge \neg\vartheta_{i_3}(\bar{a}_3; \bar{c}_3) .$$

Since $\bar{a}_2 < \bar{a}_3$ it follows that $(\bar{c}_2, i_2) < (\bar{c}_3, i_3)$. Hence,

$$\mathfrak{M} \models \vartheta_{i_2}(\bar{a}_1; \bar{c}_2) \leftrightarrow \vartheta_{i_2}(\bar{a}_3; \bar{c}_2) .$$

Since $\bar{a}_2 < \bar{a}_3$ we have

$$\mathfrak{M} \vDash \vartheta_{i_2}(\bar{a}_1; \bar{c}_2) \wedge \neg\vartheta_{i_2}(\bar{a}_2; \bar{c}_2)\,.$$

But because of $(\bar{c}_2, i_2) < (\bar{c}_1, i_1)$ this implies $\bar{a}_2 < \bar{a}_1$. Contradiction.

If $(\bar{c}_3, i_3) = (\bar{c}_1, i_1)$ then $\bar{a}_1 < \bar{a}_2$ implies

$$\mathfrak{M} \vDash \neg\vartheta_{i_3}(\bar{a}_1; \bar{c}_3) \wedge \vartheta_{i_3}(\bar{a}_2; \bar{c}_3)\,.$$

Thus, $(\bar{c}_3, i_3)$ does not witness the fact that $\bar{a}_3 < \bar{a}_1$. Contradiction.

If $(\bar{c}_3, i_3) > (\bar{c}_1, i_1)$ then

$$\mathfrak{M} \vDash \vartheta_{i_1}(\bar{a}_1; \bar{c}_1) \leftrightarrow \vartheta_{i_1}(\bar{a}_3; \bar{c}_1)\,,$$

and from $\bar{a}_1 < \bar{a}_2$ it follows that

$$\mathfrak{M} \vDash \neg\vartheta_{i_1}(\bar{a}_3; \bar{c}_1) \wedge \vartheta_{i_1}(\bar{a}_2; \bar{c}_1)\,.$$

Therefore, $\bar{a}_2 < \bar{a}_3$ implies $(\bar{c}_2, i_2) < (\bar{c}_1, i_1)$. Hence,

$$\mathfrak{M} \vDash \vartheta_{i_2}(\bar{a}_1; \bar{c}_2) \leftrightarrow \vartheta_{i_2}(\bar{a}_2; \bar{c}_2)\,.$$

Since $\bar{a}_2 < \bar{a}_3$ we have

$$\mathfrak{M} \vDash \neg\vartheta_{i_2}(\bar{a}_1; \bar{c}_2) \wedge \vartheta_{i_2}(\bar{a}_3; \bar{c}_2)\,.$$

But because of $(\bar{c}_2, i_2) < (\bar{c}_3, i_3)$ this implies $\bar{a}_1 < \bar{a}_3$. Contradiction. □

3.9 Tangles

To show that a given structure has a certain partition width it suffices to construct a suitable partition refinement. The proof of the opposite statement seems much more difficult since it is defined by an universal condition. In this section we will derive an equivalent existential one.

By the compactness results established above it is sufficient to only consider *finite* structures. In the remainder of this section we fix a finite structure $\mathfrak{M}$.

The following observation yields a necessary condition for large partition width, but not a sufficient one.

Lemma 3.9.1. *If* $\mathrm{pwd}_n \mathfrak{M} > k$ *then* $\mathfrak{M}$ *contains some subset* $C \subseteq M$ *such that, for all* $X \subset C$*,*

$$\mathrm{eti}^n_0(X/M \setminus X) > k \quad \textit{or} \quad \mathrm{eti}^n_0(C \setminus X/M \setminus (C \setminus X)) > k\,.$$

Proof. Suppose that, for every $C \subseteq M$, there is some $X \subset C$ with

$$\operatorname{eti}^n_o(X/M \setminus X) \leq k \quad \text{and} \quad \operatorname{eti}^n_o(C \setminus X/M \setminus (C \setminus X)) \leq k.$$

We construct a partition refinement $(U_v)_{v \in T}$ of $\mathfrak{M}$ by induction on v such that $\operatorname{eti}^n_o(U_v/\overline{U_v}) \leq k$ for all $v \in T$. Set $U_\varepsilon := M$. Assume that U_v is already defined. By assumption there is some $X \subset U_v$ with

$$\operatorname{eti}^n_o(X/M \setminus X) \leq k \quad \text{and} \quad \operatorname{eti}^n_o(U_v \setminus X/M \setminus (U_v \setminus X)) \leq k.$$

We set $U_{v0} := X$ and $U_{v1} := U_v \setminus X$. □

A partition refinement $(U_v)_{v \in T}$ induces a system $(U_v \cup \overline{U_v})_{v \in T}$ of partitions of M. This observation motivates the following definition.

cut

Definition 3.9.2. A *cut* of $X \subseteq M$ of *n-order* k is a pair (A, B) of subsets with $A \cup B = X$ such that

$$\operatorname{eti}^n_o(A/B) \leq k \quad \text{and} \quad \operatorname{eti}^n_o(B/A) \leq k.$$

$C^n_k(X)$

Let $C^n_k(X)$ be the set of all cuts of X of n-order k. We abbreviate

trivial cut

$C^n_k(M)$ as C^n_k. A cut (A, B) is called *trivial* if $A = \varnothing$ or $B = \varnothing$.

Transferring results of Robertson and Seymour [65], we will prove that a finite structure $\mathfrak{M}$ has large partition width if and only if there exists a *tangle*. Intuitively, a tangle is a system of cuts (A, B) where the set B is 'complicated'. The axioms listed below read as follows:

- If we can cut M into two parts, then at least one of them should be complicated.
- Small sets are simple.
- The union of disjoint simple sets is simple.

tangle

Definition 3.9.3. A subset $T \subseteq C^n_k$ is called a *tangle* if it satisfies the following conditions:

(T1) $(A, B) \in T$ iff $(B, A) \notin T$ for all $(A, B) \in C^n_k$.

(T2) $|B| > 1$ for all $(A, B) \in T$.

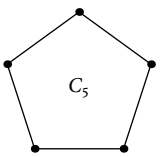
(T3) If $(A_0, B_0), (A_1, B_1) \in C^n_k \setminus T$ are cuts with $B_0 \cap B_1 = \varnothing$ then $(A_0 \cap A_1, B_0 \cup B_1) \notin T$.

pre-tangle

$T \subseteq C^n_k$ is called a *pre-tangle* if it satisfies (T2), (T3), and

(T1′) $(A, B) \in T$ or $(B, A) \in T$ for all $(A, B) \in C^n_k$.

Note that $\varnothing$ is a tangle if and only if $C^n_k(M)$ is empty.



Example. Consider the cycle C_5. The set $T \subseteq C^1_2$ defined by

$$T := \{ (A, B) \in C^1_2 \mid |A| \leq 1 \}$$

is a tangle since, if (A, B) is a cut of order at most 2, then either $|A| \leq 1$ or $|B| \leq 1$.

The order of a cut (A, B) was defined symmetrically in A and B. On the other hand, the width of a partition refinement $(U_v)_v$ only depends on $\mathrm{eti}^n_o(U_v/\overline{U_v})$ and not on $\mathrm{eti}^n_o(\overline{U_v}/U_v)$. In order to compare these notions we define a variant of partition width where both type indices are bounded.

bidirectional partition width $\mathrm{bpwd}_n(U_v)_v$

Definition 3.9.4. (a) The *bidirectional partition width* $\mathrm{bpwd}_n(U_v)_v$ of a partition refinement $(U_v)_v$ is the least number k such that each cut $(U_v, \overline{U_v})$ is of n-order at most k. $\mathrm{bpwd}_n \mathfrak{M}$ is defined in the usual way using $\mathrm{bpwd}_n(U_v)_v$ instead of $\mathrm{pwd}_n(U_v)_v$.

(b) Let $\mathfrak{A} \subseteq \mathfrak{M}$, $W \subseteq M$, and $(U_v)_{v\in T}$ be a partition refinement of $\mathfrak{A}$. The *bidirectional width of* $(U_v)_v$ *over* W is the minimal sequence $\bar{w} \in \omega^\omega$ such that

partition width over W

$$\mathrm{eti}^n_o\big(U_v \,/\, (A \cup W) \setminus U_v\big),\ \mathrm{eti}^n_o(A \setminus U_v \,/\, U_v \cup W) \leq w_n$$

for all $v \in T$ and $n < \omega$.

The bidirectional partition width $\mathrm{bpwd}_n(A/W)$ of a subset $A \subseteq M$ over $W \subseteq M$ is defined the usual way.

The relation between the usual and the bidirectional version of partition width follows from Lemmas 3.3.2 and 3.3.4.

Lemma 3.9.5. *Let $\mathfrak{M}$ be a structure with m relations of arity greater than 1 and no relations of arity greater than r. For every partition refinement $(U_v)_v$ of $\mathfrak{M}$ we have*

$$\mathrm{pwd}_n(U_v)_v \leq \mathrm{bpwd}_n(U_v)_v \leq 2^{m(n+1)^r \mathrm{pwd}_{r-1}(U_v)_v}.$$

If $\mathfrak{M}$ is a transition system then

$$\mathrm{pwd}_1(U_v)_v \leq \mathrm{bpwd}_1(U_v)_v \leq 4^{m\,\mathrm{pwd}_1(U_v)_v}.$$

We are going to prove that $C^n_k(M)$ contains a tangle of n-order k if and only if $\mathrm{bpwd}_n \mathfrak{M} > k$. The following observation shows that tangles are indeed an obstruction to a small partition width. It is also helpful in constructing tangles.

Lemma 3.9.6. *Let $\mathfrak{M}$ be a finite structure. If T is a tangle of n-order k then $(A, B) \notin T$ for every cut (A, B) with $\mathrm{bpwd}_n(B/A) \leq k$.*

Proof. We prove the claim by induction on $|B|$. If $|B| \leq 1$ then $(A, B) \notin T$ by (T2). For $|B| > 1$, consider a partition refinement $(U_v)_v$ of B with $\mathrm{bpwd}_n((U_v)_v/A) \leq k$. By induction $(\overline{U_{v0}}, U_{v0}), (\overline{U_{v1}}, U_{v1}) \notin T$. Hence, (T3) implies $(A, B) \notin T$. □

Below it will be shown that every pre-tangle contains a tangle. Therefore, it is sufficient to prove the above claim for pre-tangles.

Proposition 3.9.7. *Let $\mathfrak{M}$ be a finite structure and $k > 0$. C^n_k contains a pre-tangle if and only if* $\mathrm{bpwd}_n \mathfrak{M} > k$.

Proof. ($\Rightarrow$) Let $T \subseteq C^n_k$ be a pre-tangle. Suppose that there exists a partition refinement $(U_\nu)_\nu$ with $\mathrm{bpwd}_n(U_\nu)_\nu \leq k$. $(M, \varnothing) \notin T$ implies $(\varnothing, U_\varepsilon) \in T$. Furthermore, if $(\overline{U_\nu}, U_\nu) \in T$ then $(\overline{U_{\nu 0}}, U_{\nu 0}) \in T$ or $(\overline{U_{\nu 1}}, U_{\nu 1}) \in T$ since both belong to C^n_k and $U_\nu = U_{\nu 0} \cup U_{\nu 1}$. By induction on ν, it follows that there is a leaf ν with $(\overline{U_\nu}, U_\nu) \in T$. But $|U_\nu| = 1$. Contradiction.

($\Leftarrow$) Let $T := \{ (A, B) \in C^n_k \mid \mathrm{bpwd}_n(B/A) > k \}$. We claim that T is a pre-tangle.

(T1′) If $(A, B) \notin T$ and $(B, A) \notin T$ for some cut $(A, B) \in C^n_k$ then $\mathrm{bpwd}_n(B/A) \leq k$ and $\mathrm{bpwd}_n(A/B) \leq k$ and, hence, $\mathrm{bpwd}_n \mathfrak{M} \leq k$. Contradiction.

(T2) If $|B| \leq 1$ then $\mathrm{bpwd}_n(B/A) = 1 \leq k$. Hence, $(A, B) \notin T$.

(T3) If there are cuts $(A_0, B_0), (A_1, B_1) \in C^n_k$ with $B_0 \cap B_1 = \varnothing$ and $\mathrm{bpwd}_n(B_i/A_i) \leq k$, $i < 2$, then $\mathrm{bpwd}_n(B_0 \cup B_1/A_0 \cap A_1) \leq k$. □

In order to transform a pre-tangle T into a tangle we have to remove cuts (A, B) from T where both, $(A, B), (B, A) \in T$. We split the proof into two parts. First, we show that certain subsets $S \subseteq T$ can be removed from T. The second step consists in proving that, for all cuts (A, B) such that $(A, B), (B, A) \in T$, we find such a subset $S \subseteq T$ with $(A, B) \in S$.

Lemma 3.9.8. *Let T be a pre-tangle of n-order k, and let $S \subseteq T$ be a set of cuts satisfying the following conditions:*

(a) *If $(A, B) \in S$ then $(B, A) \in T \setminus S$.*

(b) *If there are cuts $(A, B) \in S$ and $(C, D) \notin T \setminus S$ with $D \subseteq A$ then $(A \cap C, B \cup D) \notin T \setminus S$.*

Then $T \setminus S$ is a pre-tangle of n-order k.

Proof. If T satisfies (T1′) then so does $T \setminus S$ by (a). Clearly, (T2) also holds for $T \setminus S$. If (T3) fails then there are cuts $(A, B), (C, D) \in C^n_k$ with $B \cap D = \varnothing$ such that $(A, B), (C, D) \notin T \setminus S$ but $(A \cap C, B \cup D) \in T \setminus S$. Since T satisfies (T3) we have $(A, B) \in T$ or $(C, D) \in T$. By symmetry we may assume the former. (b) implies that $(A \cap C, B \cup D) \notin T \setminus S$. Contradiction. □

Lemma 3.9.9. *Let T be a pre-tangle of n-order k. If there is a cut such that $(A, B), (B, A) \in T$ then there is a set $S \subseteq T$ satisfying the conditions of the previous lemma such that $(A, B) \in S$.*

Proof. We define an increasing sequence $S_0 \subseteq S_1 \subseteq \cdots$ of sets $S_i \subseteq T$ such that

- each S_i satisfies condition (a);

- the limit $S := \bigcup_i S_i$ satisfies (a) and (b);
- if $(C, D) \in S_i$ then $B \subseteq D$.

Let $S_0 := \{(A, B)\}$. Suppose that S_i is already defined. If condition (b) is satisfied then set $S_{i+1} := S_i$. Otherwise, there are cuts $(C, D) \in S_i$ and $(E, F) \notin T \setminus S_i$ with $F \subseteq C$ and $(C \cap E, D \cup F) \in T \setminus S_i$. Since $B \subseteq D$, i.e., $B \nsubseteq F, C$, we have $(E, F), (D \cup F, C \cap E) \notin S_i$.

$(C, D) \in S_i$ implies

$$(D, C) = ((D \cup F) \cap E,\ (C \cap E) \cup F) \in T \setminus S_i$$

by (a). Hence, if $(D \cup F, C \cap E) \notin T$, then $(E, F) \in T$ by (T3). Together with $(E, F) \notin T \setminus S_i$ this implies $(E, F) \in S_i$. Contradiction.

Consequently, $(D \cup F, C \cap E) \in T \setminus S_i$ and we can set

$$S_{i+1} := S_i \cup \{(C \cap E, D \cup F)\}. \qquad \square$$

Proposition 3.9.10. *Every pre-tangle of n-order k contains a tangle of n-order k.*

Proof. Let T be a pre-tangle of n-order k. If there is no cut with (A, B), $(B, A) \in T$ then T is a tangle. Otherwise, there is some subset $S \subseteq T$ with $(A, B) \in S$ such that $T \setminus S$ is a pre-tangle of n-order k. Iterating this step we obtain a tangle $T' \subseteq T$ of n-order k. $\square$

Theorem 3.9.11. *Let $\mathfrak{M}$ be a finite structure. C^n_k contains a tangle if and only if* $\mathrm{bpwd}_n \mathfrak{M} > k$.

Let us mention two alternate versions of the axiom (T3).

Lemma 3.9.12. *A subset $T \subseteq C^n_k$ is a tangle of n-order k iff it satisfies* (T1), (T2), *and*

(T3′) *If (A_0, B_0), $(A_1, B_1) \in T$ are cuts with $A_0 \cap A_1 = \varnothing$ and $(A_0 \cup A_1, B_0 \cap B_1) \in C^n_k$ then $(A_0 \cup A_1, B_0 \cap B_1) \in T$.*

Proof. This is just the dual version of (T3): If $(A_i, B_i) \in T$ then $(B_i, A_i) \notin T$ and hence $(B_0 \cap B_1, A_0 \cup A_1) \notin T$. The converse is analogous. $\square$

Lemma 3.9.13. *A subset $T \subseteq C^n_k$ is a tangle of n-order k iff it satisfies* (T1), (T2), *and*

(T3″) *There is no partition $A_0 \uplus A_1 \uplus A_2 = M$ of M such that $(A_i, \overline{A_i}) \in T$ for all $i < 3$.*

Proof. Both (T3′) and (T3″) state that if the cuts $(A_0, A_1 \cup A_2)$ and $(A_1, A_0 \cup A_2)$ are contained in T then also $(A_0 \cup A_1, A_2) \in T$ or $(A_0 \cup A_1, A_2) \notin C^n_k$. This is equivalent to $(A_2, A_0 \cup A_1) \notin T$. $\square$

We conclude this section by investigating how to find tangles for substructures and extensions.

restriction $T|_X$

Definition 3.9.14. For $X \subseteq M$ and $T \subseteq C^n_k(M)$. The *restriction* $T|_X$ of T to X is the set

$$\left\{ (A, B) \in C^n_k(X) \;\middle|\; (A \cup \overline{X}, B) \in T \text{ or } (A, B \cup \overline{X}) \in T \right\}.$$

From the definition, it is not obvious that the restriction of a tangle is again a tangle. For transition systems, will show that the restriction $T|_C$ contains a tangle of a somewhat smaller order. Again, it is sufficient to prove the claim for pre-tangles.

Lemma 3.9.15. *Let $\mathfrak{M}$ be a finite transition system. If $(A, B) \in C^1_k$ and $(C, D) \in C^1_m$ are cuts where $k, m > 1$ then $(A \cap C, B \cup D) \in C^1_{km}$.*

Proof. By Lemma 3.3.2 we have

$$\begin{aligned} \operatorname{eti}^1_0(B \cup D/A \cap C) &\leq \operatorname{eti}^1_0(B/A \cap C) + \operatorname{eti}^1_0(D/A \cap C) \\ &\leq \operatorname{eti}^1_0(B/A) + \operatorname{eti}^1_0(D/C) \leq k + m \end{aligned}$$

and $\quad \operatorname{eti}^1_0(A \cap C/B \cup D) \leq \operatorname{eti}^1_0(A/B) \cdot \operatorname{eti}^1_0(C/D) \leq km\,.$

Since $km \geq k + m$, for $k, m > 1$, the result follows. □

Lemma 3.9.16. *Let $\mathfrak{M}$ be a finite transition system. If $T \subseteq C^1_{km}(M)$ is a pre-tangle with $k, m > 1$ and (C, D) is a cut in $C^1_m(M) \setminus T$ then $T' := T|_C \cap C^1_k(C)$ is a pre-tangle of $\mathfrak{M}|_C$ of 1-order k.*

Proof. If $(A, B) \in C^1_k(C)$ then $(A \cup D, B) \in C^1_{km}(M)$, by the preceding lemma. It follows that, for all $(A, B) \in C^1_k(C)$,

$$(A \cup D, B) \in T \quad \text{or} \quad (B, A \cup D) \in T\,.$$

(T1′) If $(A, B) \notin T|_C$ then $(A \cup D, B) \notin T$. Hence, $(B, A \cup D) \in T$ which implies $(B, A) \in T|_C$.

(T2) Let $(A, B) \in C^1_k(C)$ with $|B| \leq 1$. Then $(A \cup D, B) \notin T$. Since $(C, D) \notin T$ it follows from (T3) that $(A, B \cup D) \notin T$. Hence, $(A, B) \notin T|_C$.

(T3) If (A_0, B_0), $(A_1, B_1) \notin T|_C$ then $(A_i \cup D, B_i) \notin T$, $i < 2$. By (T3) we have $((A_0 \cup D) \cap (A_1 \cup D), B_0 \cup B_1) \notin T$. Since $(C, D) \notin T$ it follows that $(A_0 \cap A_1, B_0 \cup B_1 \cup D) \notin T$. Together, this implies that $(A_0 \cap A_1, B_0 \cup B_1) \notin T|_C$. □

Finally, we show that every tangle of a substructure can be extended to a tangle of the entire structure.

Lemma 3.9.17. *Let $X \subseteq M$. If $T \subseteq C^n_k(X)$ is a tangle of n-order k then so is the set*

$$T' := \{ (A \cup D_0, B \cup D_1) \in C^n_k(M) \mid (A, B) \in T \text{ and } D_0 \uplus D_1 = M \setminus X \}\,.$$

Proof. (T1) Let $(A, B) \in C_k^n(M)$. We have

$$\begin{aligned}(A, B) \in T' \quad &\text{iff} \quad (A \cap X, B \cap X) \in T \\ &\text{iff} \quad (B \cap X, A \cap X) \notin T \quad \text{iff} \quad (B, A) \notin T'.\end{aligned}$$

(T2) If there is a cut $(A, B) \in T'$ then $(A \cap X, B \cap X) \in T$ which implies $1 < |B \cap X| \leq |B|$.

(T3) If (A_0, B_0), $(A_1, B_1) \in C_k^n(M) \setminus T'$ with $B_0 \cap B_1 = \varnothing$ then $(A_i \cap X, B_i \cap X) \in C_k^n(X) \setminus T$, $i < 2$. Hence,

$$\big((A_0 \cap X) \cap (A_1 \cap X),\ (B_0 \cap X) \cup (B_1 \cap X)\big) \notin T$$

which implies $(A_0 \cap A_1, B_0 \cup B_1) \notin T'$. □

Corollary 3.9.18. *If $\mathfrak{M}_0 \subseteq \mathfrak{M}_1$ and $T_0 \subseteq C_k^n(M_0)$ is a tangle of n-order k then there is some tangle $T_1 \subseteq C_k^n(M_1)$ of n-order k with $T_0 \subseteq T_1|_{M_0}$.*

4 Transition systems and Graphs

We have seen in Section 3.3 that the type equivalences $\simeq^{\mathrm{o}}_U$ behave much more nicely if the structure in question is of arity at most two. Therefore, we turn to a detailed investigation of the special case of transition systems. Furthermore, it seems prudent to first tackle difficult questions in this simpler setting. The main part of the following sections is dedicated to the development of technical tools which ease the transfer of proofs concerning tree width to partition width. In particular, we hope to obtain in this way an analogue to the Excluded Grid Theorem of Robertson and Seymour. Unfortunately, the version we will actually be able to prove in Section 4.5 is rather weak.

4.1 Separations and sunflowers

In this and the following sections we fix a finite transition system $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \tilde{P})$, and we set $\alpha := 4^{|A|}$. Note that, according to Lemma 3.3.2, we have $\mathrm{eti}^1_{\mathrm{o}}(A/B) \leq \alpha^{\mathrm{eti}^1_{\mathrm{o}}(B/A)}$ for all $A, B \subseteq M$.

Every component F_ν of a tree decomposition $(F_\nu)_\nu$ (except for the leaves) constitutes a separator of the underlying graph (V, E). That is, there exist sets A and B with $A \cup B = V$ and $A \cap B = F_\nu$ such that no element of $A \setminus B$ is adjacent to one of $B \setminus A$.

We are looking for a variant of the notion of a cut that exposes similar behaviour. That is, to each cut (A, B) we want to associate a set C that is responsible for the complexity of (A, B). Since, in our case, this complexity is caused not by edges but by the number of types, we choose a set C containing representatives of each realised external type.

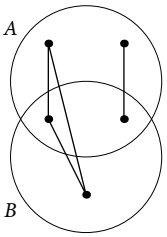
Definition 4.1.1. A *separation* of a set $X \subseteq M$ is a pair (A, B) of sets such that $X = A \cup B$ and *separation*

- for every $a \in A \setminus B$ there is some $c \in A \cap B$ with $a \simeq^{\mathrm{o}}_{B\setminus A} c$ and
- for every $b \in B \setminus A$ there is some $c \in A \cap B$ with $b \simeq^{\mathrm{o}}_{A\setminus B} c$.

The *order* of a separation (A, B) is the cardinality of the intersection *order*

$S_k(X)$
trivial separation

$|A \cap B|$. By $S_k(X)$ we denote the set of all separations of X of order at most k. A separation (A, B) is called *trivial* if $A \subseteq B$ or $B \subseteq A$.



Each cut can be transformed into a separation and, conversely, every separation induces a cut.

Lemma 4.1.2. *Let $\mathfrak{M}$ be a structure and $X \subseteq M$ a subset. For every cut $(A, B) \in C^n_k(X)$ there exists a separation $(A', B') \in S_{2kn}(X)$ with $A \subseteq A'$ and $B \subseteq B'$.*

Proof. Fix representatives $\bar{a}^i$, $i < r$, of the $\simeq^{\mathrm{o}}_B$-classes $[\bar{a}^i] \in A^n/\simeq^{\mathrm{o}}_B$, and representatives $\bar{b}^i$, $i < s$, of $B^n/\simeq^{\mathrm{o}}_A$. Setting $C := \bigcup_{i<r} \bar{a}^i \cup \bigcup_{i<s} \bar{b}^i$ we obtain a separation $(A \cup C, B \cup C)$ of order $|C| \leq 2kn$. □

Lemma 4.1.3. *Let $\mathfrak{M}$ be a transition system. For every separation $(A, B) \in S_k(X)$ there exists a cut $(A', B') \in C^1_{\alpha^k}(X)$ with $A' \subseteq A$ and $B' \subseteq B$.*

Proof. Let $B' := B \setminus A$. Then $\mathrm{eti}^1_{\mathrm{o}}(A/B') \leq |A \cap B| \leq k$ and

$$\mathrm{eti}^1_{\mathrm{o}}(B'/A) \leq \alpha^{\mathrm{eti}^1_{\mathrm{o}}(A/B')} \leq \alpha^k .$$

Thus, $(B', A) \in C^1_{\alpha^k}(X)$ is the desired cut. □

If X is a separator of a graph $\mathfrak{G}$ then $\mathfrak{G} \setminus X$ splits up into at least two connected components. Sometimes, it is important to distinguish between all of them. Hence, we need a variant of a separation that splits the structure into more than two sets.

sunflower

Definition 4.1.4. (a) A family $(A_i)_{i<n}$ of sets $A_i \subseteq V$ forms a *sunflower with core C* if

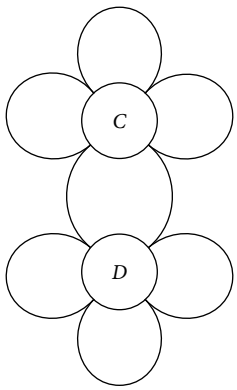
- $A_i \cap A_k = C$, for all $i \neq k$, and
- $\big(\bigcup_{i\in I} A_i,\ \bigcup_{i\notin I} A_i\big)$ is a separation for every $I \subseteq [n]$.

The *domain* of $(A_i)_i$ is the set $\bigcup_{i<n} A_i$. Sets of the form $A_i \setminus C$ are called *petals*.

petal
refining a sunflower
maximal sunflower

(b) A sunflower $(A_i)_i$ *refines* the sunflower $(B_i)_i$ if it has the same core and domain and, for every A_i, there is some set B_k such that $A_i \subseteq B_k$. A sunflower $(A_i)_i$ is called *maximal* if it has no proper refinement.

A tree decomposition of a graph $\mathfrak{G}$ consists of a family of separators of $\mathfrak{G}$ arranged in a tree-like fashion. We would like to define a similar decomposition using sunflowers. First we show how to extend a sunflower $(B_i)_i$ contained in a petal of another sunflower $(A_i)_i$ to the whole structure in such a way that $(A_i)_i$ is contained in a petal of the new sunflower.



Lemma 4.1.5. *Let $(A_i)_{i<m}$ and $(B_i)_{i<n}$ be sunflowers with core C and D, respectively, such that the domain of $(B_i)_i$ is A_0. If $C \subseteq B_0 \setminus D$, then the sequence $(B'_i)_{i<n}$ defined by*

$$B'_i := \begin{cases} B_0 \cup A_1 \cup \dots \cup A_{m-1} & \text{if } i = 0\,, \\ B_i & \text{otherwise}\,. \end{cases}$$

is a sunflower with core D and domain $\bigcup_{i<m} A_i$.

Proof. Let $A_* := A_1 \cup \dots \cup A_{m-1}$ and $B_* := B_1 \cup \dots \cup B_{n-1}$. We have to show that, for every $a \in A_*$, there is some $c \in D$ such that $a \simeq^0_{B_* \setminus D} c$ and, conversely, that, for every $b \in B_*$, there is some $c \in D$ such that $b \simeq^0_{(B_0 \cup A_*) \setminus D} c$.

Let $a \in A_*$. There is some $b \in C$ such that $a \simeq^0_{A_0 \setminus C} b$, and there is some $c \in D$ with $b \simeq^0_{B_* \setminus D} c$. Since $B_* \cap C = \varnothing$ and, hence, $B_* \setminus D \subseteq A_0 \setminus C$ it follows that $a \simeq^0_{B_* \setminus D} c$.

Now, let $b \in B_*$. Then $b \simeq^0_{B_0 \setminus D} c$ for some $c \in D$. Suppose that $b \not\simeq^0_{(B_0 \cup A_*) \setminus D} c$. Then there is some $a \in A_*$ such that $\operatorname{etp}_0(b/a) \neq \operatorname{etp}_0(c/a)$. But there is some $a' \in C \subseteq B_0 \setminus D$ with $a \simeq^0_{A_0 \setminus C} a'$. Since $b \in B_* \subseteq A_0 \setminus C$ and $c \in D \subseteq A_0 \setminus C$ it follows that $\operatorname{etp}_0(b/a') \neq \operatorname{etp}_0(c/a')$ which implies that $b \not\simeq^0_{B_0 \setminus D} c$. Contradiction. □

Lemma 4.1.6. *Let (A, B) be a separation and $(A_i)_{i<n}$ a sunflower with domain A and core C. Let $D := A \cap B$. There exists a set $Z \supseteq C \cup D$ of size $|Z| \leq (n|C| + 1)|D|$ such that the sequence*

$$(A_0 \cup Z, \dots, A_{n-1} \cup Z, B \cup Z)$$

forms a sunflower with core Z.

Proof. For every $i < n$, fix a set $Z_i \subseteq A_i$ of representatives of the $\simeq^0_{(A \setminus A_i) \cup (B \setminus D)}$-classes realised in $A_i \setminus C$. Set $Z := \bigcup_{i<n} Z_i \cup D$. Since

$$\begin{aligned} &\operatorname{eti}^1_0\big(A_i \setminus C \,\big/\, (A \setminus A_i) \cup (B \setminus D)\big) \\ &\leq \operatorname{eti}^1_0(A_i \setminus C / A \setminus A_i) \cdot \operatorname{eti}^1_0(A_i \setminus C / B \setminus D) \\ &\leq |C| \cdot |D| \end{aligned}$$

it follows that $|Z| \leq (n|C| + 1)|D|$. Hence, it remains to show that the sequence $(F_i)_{i<n+1}$ with

$$F_i := \begin{cases} A_i \cup Z & \text{if } i < n\,, \\ B \cup Z & \text{if } i = n\,, \end{cases}$$

forms a sunflower with core Z. Let $I \subseteq [n+1]$ and $a \in \bigcup_{i \notin I} F_i$. We have to find some element $c \in Z$ with $a \simeq^0_{\bigcup_{i \in I} F_i \setminus Z} c$.

First, suppose that $n \notin I$. If $a \in A$ then there is some $c \in C \subseteq Z$ such that $a \simeq^{o}_{\bigcup_{i\in I} A_i \setminus C} c$ since $(A_i)_i$ is a sunflower. For $a \in B$ there exists some $c \in D \subseteq Z$ with $a \simeq^{o}_{A\setminus D} c$.

Now, suppose that $n \in I$. Let $a \in A_i$. If $a \in Z$ then we are done. Otherwise, there is some $c \in Z_i \subseteq Z$ such that $a \simeq^{o}_{(A\setminus A_i)\cup(B\setminus D)} c$. □

If we already have a sunflower $(B_i)_i$, how can we construct a sunflower $(A_i)_i$ of some petal of $(B_i)_i$ such that the situation of Lemma 4.1.5 is realised? We have to ensure that the core of $(B_i)_i$ is contained in either the core of $(A_i)_i$ or in some of its petals.

Lemma 4.1.7. *Let $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \tilde{P})$ be a finite transition system, $Z \subseteq M$, and $n < \omega$. There exists an extension $\mathfrak{M}^+ = (M^+, (E^+_\lambda)_{\lambda\in\Lambda}, \tilde{P}^+)$ of $\mathfrak{M}$ of size*

$$|M^+ \setminus M| \leq (n+1)\binom{|Z|}{2}$$

such that every sunflower $(A_i)_i$ with domain M^+ and core C of size $|C| \leq n$ satisfies $Z \subseteq A_i$ for some i.

Proof. Fix an edge relation E_λ. For every pair of distinct elements $u, v \in Z$, let $X(u, v)$ be a set of $n+1$ new elements. We obtain $\mathfrak{M}^+$ by adding all elements of these sets $X(u, v)$ to M and by creating E_λ-edges (x, u) and (x, v) for every $x \in X(u, v)$. Then $|M^+ \setminus M| = (n+1)\binom{|Z|}{2}$ is of the right size.

Let $(A_i)_i$ be a sunflower with domain M^+ and core C of size $|C| \leq n$. Suppose there are two vertices $u, v \in Z$ such that $u \in A_i \setminus C$ and $v \in A_k \setminus C$ for $i \neq k$. There is at least one vertex $w \in X(u, v) \setminus C$. By symmetry we may assume that $w \notin A_k$. There exists some vertex $v' \in C$ such that $v' \simeq^{o}_{w} v$. Thus, $v' \in \{u, v\} \cap C = \varnothing$. Contradiction. □

Digression: separation-free sets

In the remainder of this section we will investigate sets without non-trivial separations. This absence of separations can be considered as a notion of connectedness. Instead of separations we could also study sets without cuts. This latter notion, investigated in Section 4.3, will turn out to be more useful for our purposes. Therefore, the results below will not be used in other parts of the thesis. For simplicity, we only consider finite undirected graphs.

separation free

Definition 4.1.8. A set $X \subseteq M$ is called *separation free* if $S_1(X)$ contains no non-trivial separation.

Lemma 4.1.9. *Let $\mathfrak{G} = (V, E)$ be an undirected graph. If $(A, B) \in S_1(X)$ with $A \cap B = \{c\}$ then either $(a, c) \in E$ for all $a \in X \setminus \{c\}$ or $(a, c) \notin E$ for all such a.*

Proof. By symmetry we may assume that there exists some element $a \in A \setminus B$ with $(a, c) \in E$. Since $b \simeq^{\circ}_{A \setminus B} c$ for all $b \in B$ it follows that $(a, b) \in E$ for all $b \in B$. Hence, $a \simeq^{\circ}_{B \setminus A} c$ implies $(c, b) \in E$ for all $b \in B$. Furthermore, because of $a' \simeq^{\circ}_{B \setminus A} c$ for all $a' \in A$ we have $(a', b) \in E$ for all $a' \in A$. Together with $b \simeq^{\circ}_{A \setminus B} c$ for all $b \in B$ it follows that $(a', c) \in E$ for all $a' \in A$. □

Lemma 4.1.10. *Let X be a set and v some element with $\mathrm{eti}^1_{\circ}(X/v) > 1$. If X is separation free so is $X \cup \{v\}$.*

Proof. Suppose that there exists a non-trivial separation (A, B) of $X \cup \{v\}$ of order $|A \cap B| = 1$. W.l.o.g. assume that $v \in B$.

If $v \notin A$ then $(A, B \setminus \{v\})$ is a separation of X of order 1. By assumption it has to be trivial, i.e., $B = \{u, v\}$ for some $u \in X$. We can interchange u and v and the result

$$\big((A \setminus \{u\}) \cup \{v\},\ B\big)$$

is still a separation of order 1.

Hence, we may assume that $A \cap B = \{v\}$. It follows that either $(w, v) \in E$ for all $w \in X$, or $(w, v) \notin E$ for all $w \in X$ in contradiction to $\mathrm{eti}^1_{\circ}(X/v) > 1$. □

We can also improve Lemma 4.1.5 for separations of order 1.

Lemma 4.1.11. *Let (A, B) be a separation of order 1 and (A_0, A_1) a separation of A of order 1. Then*

$$\big(A_0 \setminus (B \setminus A_1),\ A_1 \cup B\big)$$

is also a separation of order 1.

Proof. Let $A \cap B = \{u\}$ and $A_0 \cap A_1 = \{v\}$. Either $(x, u) \in E$ for all $x \neq u$ or $(x, u) \notin E$ for all such x. W.l.o.g. assume the former. Then $(x, y) \in E$ for all $x \in B \setminus \{u\}$ and $y \in A \setminus \{u\}$. In particular, $(x, v) \in E$ for all $x \in B$. Furthermore, since $(v, u) \in E$ and $u \in A$, it follows that $(x, v) \in E$ for all $x \in A \setminus \{v\}$. Thus, we have $(x, v) \in E$ for all $x \neq v$ as desired. □

After having defined a notion of connectedness we can study decompositions into connected components. In our context such a decomposition corresponds to a maximal sunflower with a core of size 1.

Lemma 4.1.12. *Let $(A_i)_i$ be a sunflower with core C and domain X. If (B_0, B_1) is a separation of A_0 with $B_0 \cap B_1 = \{c\}$ and $c \notin C$, then either $(a, c) \in E$ for all $a \in X \setminus \{c\}$ or $(a, c) \notin E$ for all such a.*

Proof. We have $a \simeq^{\mathrm{o}}_{c} b$ for all $a, b \in B_k \setminus \{c\}$. Furthermore, if $a \in X \setminus A_{\mathrm{o}}$ and $b \in A_{\mathrm{o}} \setminus \{c\}$ then there is some $a' \in C$ with $a \simeq^{\mathrm{o}}_{A_{\mathrm{o}} \setminus C} a'$. Since $c \notin C$ this implies that $a \simeq^{\mathrm{o}}_{c} a' \simeq^{\mathrm{o}}_{c} b$. □

Lemma 4.1.13. *Let $(A_i)_{i<n}$ be a sunflower with core C of size $|C| = 1$. If (B_{o}, B_1) is a non-trivial separation of A_{o} of order 1 then*

$$\big((B_{\mathrm{o}} \setminus B_1) \cup C,\ B_1 \cup C,\ A_1, \dots, A_{n-1}\big)$$

is a refinement of $(A_i)_{i<n}$.

Proof. Let X be the domain of $(A_i)_i$, $C = \{c\}$ and $B_{\mathrm{o}} \cap B_1 = \{d\}$. If $c = d$ then we are done. Otherwise, by Lemma 4.1.12, we have $a \simeq^{\mathrm{o}}_{c} b$ for all $a, b \in X \setminus \{c\}$ and $a \simeq^{\mathrm{o}}_{d} b$ for all $a, b \in X \setminus \{d\}$. W.l.o.g. assume that $(a, c) \in E$ for all $a \neq c$. Then $(d, c) \in E$ which implies that $(d, a) \in E$ for all $a \neq d$. The result follows. □

Corollary 4.1.14. *If $(A_i)_i$ is a maximal sunflower with a core of size 1 then every set A_k is separation free.*

We can improve this result to allow larger cores if we require them to be 1-cut free (see Definition 4.3.1).

Lemma 4.1.15. *If $(A_i)_i$ is a sunflower with 1-cut-free core C and separation-free domain X, then every set A_i is separation free.*

Proof. Suppose there is a non-trivial separation (B_{o}, B_1) of A_i of order 1. If $C \not\subseteq B_{\mathrm{o}}$ and $C \not\subseteq B_1$ then $\big(B_{\mathrm{o}} \cap C,\ (B_1 \setminus B_{\mathrm{o}}) \cap C\big)$ would be a cut of C of order 1. Hence, we may assume that $C \subseteq B_{\mathrm{o}}$. But then Lemma 4.1.5 implies that $(B_{\mathrm{o}} \cup (X \setminus A_i), B_1)$ is a non-trivial separation of X of order 1. Contradiction. □

The next lemma provides another way to obtain something like a decomposition into connected components.

Lemma 4.1.16. *Let $\mathfrak{G}$ be an undirected graph such that $S_2(V)$ does not contain a non-trivial separation, and let X_i, $i < n$, be a family of disjoint separation-free sets such that there is no separation-free set $Y \supset X_k$, for some $k < n$, disjoint from X_i, for $i \neq k$. Then the set $Z := V \setminus \bigcup_{i<n} X_i$ is separation free.*

Proof. Since X_i is maximal Lemma 4.1.10 implies that $\mathrm{eti}^1_{\mathrm{o}}(X_i/v) = 1$ for every $v \in Z$. If $|Z| \leq 1$ we are done. Otherwise, fix distinct elements $v, v' \in Z$. W.l.o.g. we may assume that $(u, v) \in E$ for all $u \in X_i$.

There is a non-trivial separation (A, B) of $X_i \cup \{v, v'\}$ of order 1. By possibly interchanging v with v' and A with B we may assume that $v' \in B \setminus A$. Let $A \cap B = \{c\}$. If $v \in A \setminus B$ then $(v, c) \in E$ implies $(v, v') \in E$ which in turn implies $(c, v') \in E$. Similarly, if $v = c$ then $(a, c) \in E$, for $a \in A \setminus B$, implies $(a, v') \in E$ and $(c, v') \in E$. Since

$\mathrm{eti}^1_0(X_i/v') = 1$ it follows that $(v', u) \in E$, for all $u \in X_i$, and, therefore, $v \simeq^0_{X_i} v'$.

It remains to consider the case that $v \in B \setminus A$. Let $a \in A \setminus B$. $(a, v) \in E$ and $v \simeq^0_{A \setminus B} v'$ imply that $(a, v') \in E$. As above it follows that $(v', u) \in E$, for all $u \in X_i$, and, therefore, $v \simeq^0_{X_i} v'$.

We have show that $v \simeq^0_{X_0 \cup \cdots \cup X_{k-1}} v'$ for all $v, v' \in Z$. Suppose that there exists a non-trivial separation (A, B) of Z of order 1. W.l.o.g. assume that $(a, b) \in E$, for all $a \in A \setminus B$ and $b \in B \setminus A$. If there exists no component X_i with $(u, v) \notin E$, for $u \in X_i$ and $v \in Z$, then $\big(A \cup \bigcup_i X_i, B\big)$ would form a non-trivial separation of $\mathfrak{G}$ of order 1. Contradiction. Hence, there exists such a component X_i and we can choose some element $c \in X_i$. But then $\big(A \cup \bigcup_i X_i, B \cup \{c\}\big)$ forms a non-trivial separation of $\mathfrak{G}$ of order 2. Again a contradiction. □

Lemma 4.1.17. *For every set X that is not separation free, there exists a separation (A, B) of order 1 with $|A| \geq |B|$ such that A is separation free or $|A| \leq 2|B| + 1$.*

Proof. Let (A, B) be a separation of X of order 1 with $|A| \geq |B|$ such that $|A| - |B| \geq 0$ is minimal. Suppose that A is not separation free and $|A| > 2|B| + 1$. Then there is a separation (A_0, A_1) of A of order 1 with $|A_0| \geq |A_1|$. By Lemma 4.1.11, $(A_0 \setminus (B \setminus A_1),\ A_1 \cup B)$ is also a separation of order 1. Since

$$|A_0 \setminus (B \setminus A_1)| - |A_1 \cup B| < |A| - |B|$$

and

$$\begin{aligned} &|A_1 \cup B| - |A_0 \setminus (B \setminus A_1)| \\ &\leq \tfrac{1}{2}(|A| + 1) + \tfrac{1}{2}(|A| - 2) - \Big(\tfrac{1}{2}(|A| + 1) - 1\Big) \\ &= |A| - \tfrac{1}{2}|A| < |A| - |B| \end{aligned}$$

this contradicts the minimality of $|A| - |B|$. □

4.2 Tree covers

The notion of a sunflower can be used to define a decomposition of a structure that has similar properties as a tree decomposition. Since every component F_v of a tree decomposition $(F_v)_v$ is a separator of the graph in question, we define a *tree cover* to be a family $(F_v)_v$ where every component F_v is the core of a sunflower.

As in the previous section we assume that $\mathfrak{M} = (M, (E_\lambda)_{\lambda \in \Lambda}, \bar{P})$ is a finite transition system, and we set $\alpha := 4^{|A|}$.

Definition 4.2.1. A *tree cover* of $\mathfrak{M}$ is a family $(F_v)_{v \in I}$ of subsets *tree cover*

$F_v \subseteq M$ indexed by an undirected ternary tree I such that, if $v \in I$ and $I_0, \dots, I_s$ are the connected components of $I \setminus v$, then the sequence

$$\big(F_v \cup \bigcup_{u \in I_0} F_u, \dots, F_v \cup \bigcup_{u \in I_s} F_u\big)$$

forms a sunflower with core F_v and domain M.

width The number $\sup \{ |F_v| \mid v \in I \}$ is called the *width* of the tree cover $(F_v)_{v \in I}$.

The following easy observation suggests that tree covers have properties quite similar to tree decompositions. This eases the transfer of proofs concerning tree width to ones about partition width.

Lemma 4.2.2. *If $(F_v)_v$ is a tree cover then the set $\{ v \in I \mid a \in F_v \}$ is connected for every $a \in M$.*

Proof. Otherwise, there would be vertices $u, v \in I$ with $a \in F_u \cap F_v$ and some vertex $w \in I$ on the path from u to v such that $a \notin F_w$. If $(A_i)_i$ is the sunflower with core F_w witnessing that $(F_v)_v$ is a tree cover then there are two distinct petals with $a \in A_i \setminus F_w$ and $a \in A_k \setminus F_w$. But this is impossible. □

Of course, in order to use tree covers to prove statements about partition width we have to show that partition refinements and tree covers are related.

Lemma 4.2.3. *Let $\mathfrak{M}$ be a finite transition system.*

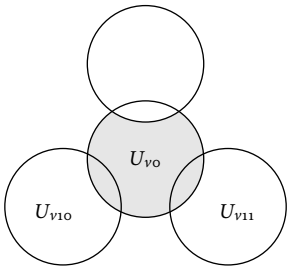
(a) *If $\mathfrak{M}$ has a tree cover of width k then it has a partition refinement $(U_v)_{v \in T}$ of width $\operatorname{pwd}_1(U_v)_v \leq \alpha^{2k}$.*

(b) *If there exists a partition refinement $(U_v)_{v \in T}$ of $\mathfrak{M}$ of width $k := \operatorname{pwd}_1(U_v)_v$ then $\mathfrak{M}$ has a tree cover of width at most $\alpha^k + k$.*

Proof. (a) Let $(F_v)_{v \in I}$ be a tree cover of $\mathfrak{M}$. Fix some leaf of I and let $T \subseteq 2^{<\omega}$ be the directed tree obtained from I if we take this leaf as root. By induction on $|v|$, we define a partition refinement $(U_v)_{v \in T}$ of $\mathfrak{M}$ of width $\operatorname{pwd}_1(U_v)_v \leq \alpha^{2k}$ such that

$$\Big(\bigcup_{w \geq v} F_w\Big) \setminus F_v \subseteq U_{h(v)} \subseteq \bigcup_{w \geq v} F_w \qquad \text{for } v \in T,$$

where $h : 2^{<\omega} \to 2^{<\omega}$ is the homomorphism defined by $h(c) := 1c$ for $c \in [2]$.



We start with $U_\varepsilon := M$. Suppose that U_v is already defined, set $u := h^{-1}(v)$, and let

$$U_{v0} := U_v \cap F_u, \qquad U_{v10} := \bigcup_{w \geq u0} F_w \setminus F_u,$$
$$U_{v1} := U_v \setminus F_u, \qquad U_{v11} := \bigcup_{w \geq u1} F_w \setminus F_u.$$

Then, for $c < 2$, it follows that

$$\begin{aligned}
&\operatorname{eti}^1_o(\overline{U_{v1c}}/U_{v1c})\\
&\le \operatorname{eti}^1_o(F_u/U_{v1c}) + \operatorname{eti}^1_o\Big(\bigcup_{w\supseteq u(1-c)} F_w \cup \bigcup_{w\not\supseteq u} F_w \;/\; U_{v1c}\Big)\\
&\le |F_u| + |F_u| \le 2k\,,
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{eti}^1_o(\overline{U_{v1}}/U_{v1})\\
&\le \operatorname{eti}^1_o(F_u/U_{v1}) + \operatorname{eti}^1_o\Big(\bigcup_{w\not\supseteq u} F_w \;/\; U_{v1}\Big)\\
&\le |F_u| + |F_u| \le 2k\,,
\end{aligned}$$

which implies $\operatorname{eti}^1_o(U_{v1x}/\overline{U_{v1x}}) \le \alpha^{2k}$ for $x \in \{\varepsilon, 0, 1\}$.

(b) Let $(U_v)_{v\in T}$ be a partition refinement of width $\underline{\operatorname{pwd}}_1(U_v)_v = k$. By induction on $|v|$ we define sets $A_v \subseteq U_v$ and $B_v \subseteq \overline{U_v}$ such that

$$\begin{aligned}
&\operatorname{eti}^1_o(U_v/\overline{U_v}) = \operatorname{eti}^1_o(A_v/\overline{U_v})\\
\text{and}\quad &\operatorname{eti}^1_o(\overline{U_v}/U_v) = \operatorname{eti}^1_o(B_v/U_v)\,.
\end{aligned}$$

Let $A_\varepsilon := B_\varepsilon := \varnothing$. Suppose that A_v and B_v are already defined. For $c \in [2]$, choose a set A_{vc} of representatives of $U_{vc}/\simeq^o_{\overline{U_{vc}}}$ such that $A_v \cap U_{vc} \subseteq A_{vc}$. Further, choose sets $B_{vc} \subseteq B_v \cup A_{v(1-c)}$ of representatives of $\overline{U_{vc}}/\simeq^o_{U_{vc}}$.

The tree cover $(F_v)_{v\in T}$ obtained by setting

$$F_\varepsilon := A_0 \cup A_1 \quad\text{and}\quad F_v := A_{v0} \cup A_{v1} \cup B_v \quad \text{for } v \neq \varnothing\,,$$

has a width of $|F_v| = |A_{v0}| + |A_{v1}| + |B_v| \le 2k + \alpha^k$. □

To give an example that shows how proofs about tree width can be transformed into ones for partition width, we develop criteria for a transition system to have a tree cover of a certain width. The proofs are mere translations of results of Robertson and Seymour [64].

Definition 4.2.4. Let $\mathfrak{M}$ be a finite transition system.

(a) $\mathfrak{M}$ is said to *admit* (k, n)*-separations* if $\mathfrak{M}$ has a separation $(A, B) \in S_k(M)$ such that *admitting* (k, n)*-separations*

$$|A \setminus B|,\ |B \setminus A| \le (1 - n^{-1})|M|\,.$$

(b) $\mathfrak{M}$ *strongly admits* (k, n)*-separations* if for every $X \subseteq M$, there is a separation $(A, B) \in S_k(M)$ such that *strongly admitting* (k, n)*-separations*

$$|(A \setminus B) \cap X|,\ |(B \setminus A) \cap X| \le (1 - n^{-1})|X|\,.$$

The following conditions are sufficient for a class of transition systems to only contain structures strongly admitting (k, n)-separations.

Lemma 4.2.5. *Let $\mathcal{K}$ be a class of finite transition systems such that*

(1) *every structure in $\mathcal{K}$ admits (k, n)-separations;*

(2) *$\mathcal{K}$ is closed under (induced) substructures and addition of new elements which are connected to exactly one other vertex.*

Then every structure $\mathfrak{M} \in \mathcal{K}$ strongly admits $(2k, n)$-separations.

Proof. Fix an edge relation E_λ of $\mathfrak{M}$ and a number $m \in \mathbb{N}$ such that there is no integer z with

$$(1 - n^{-1})|X| < z \leq (1 - n^{-1})|X| + m^{-1}\big((1 - n^{-1})|M| + k\big).$$

Let $X \subseteq M$. For each vertex $v \in X$, let N_v be a set of m new vertices. We add all the vertices in N_v to $\mathfrak{M}$ and connect them by an E_λ-edge to v. The structure $\mathfrak{M}'$ obtained in this way is in $\mathcal{K}$ and, hence, there is a separation $(A', B') \in S_k(M')$ of $\mathfrak{M}'$ with

$$|A' \setminus B'|,\ |B' \setminus A'| \leq (1 - n^{-1})|M'|\,.$$

Let $A := A' \cap M$ and $B := B' \cap M$. If $v \in X \cap (A \setminus B)$ then $N_v \subseteq A'$. Hence,

$$\begin{aligned} m \cdot |(A \setminus B) \cap X| \leq |A'| &\leq (1 - n^{-1})|M'| + k \\ &\leq (1 - n^{-1})(m|X| + M) + k\,. \\ \Rightarrow\ |(A \setminus B) \cap X| &\leq (1 - n^{-1})|X| + m^{-1}\big((1 - n^{-1})|M| + k\big)\,. \end{aligned}$$

By choice of m it follows that $|(A \setminus B) \cap X| \leq (1 - n^{-1})|X|$.

The bound $|(B \setminus A) \cap X| \leq (1 - n^{-1})|X|$ is obtained in the same way.

For each $c \in (A' \cap B') \setminus M$ choose elements $a \in A \setminus B$ and $b \in B \setminus A$, if such elements exist, such that $a \simeq^{\circ}_{B \setminus A} c$ and $b \simeq^{\circ}_{A \setminus B} c$. Let D be the set of these elements. Then $|D| \leq 2|(A' \cap B') \setminus M|$ and $(A \cup D, B \cup D)$ is the desired separation of order

$$\begin{aligned} |(A \cap B) \cup D| &\leq |A' \cap B' \cap M| + 2|(A' \cap B') \setminus M| \\ &\leq 2|A' \cap B'| \leq 2k\,. \end{aligned}$$

□

For structures strongly admitting (k, n)-separations we can construct a tree cover of bounded width.

Proposition 4.2.6. *Let $\mathcal{K}$ be a class of finite transition systems that is closed under (induced) substructures such that each $\mathfrak{M} \in \mathcal{K}$ strongly admits (k, n)-separations. Then every $\mathfrak{M} \in \mathcal{K}$ has a tree cover of width $K := k(n + 1) + 1$.*

Proof. By induction on $|M|$ we show that, for each $X \subseteq M$ of size $|X| \leq kn + 1$, $\mathfrak{M}$ has a tree cover $(F_v)_{v \in T}$ of width K such that $X \subseteq F_v$ for some $v \in T$.

If $|M| \leq K$ the result is immediate. Thus, we may assume that $|M| > K$ and $|X| = kn + 1$. By definition, there is a separation $(A, B) \in S_k$ of $\mathfrak{M}$ such that

$$|(A \setminus B) \cap X|,\ |(B \setminus A) \cap X| \leq (1 - n^{-1})|X| .$$

Since $|A \cap B \cap X| \leq |A \cap B| \leq k$ it follows that

$$\begin{aligned} |A \cap X| &= |(A \setminus B) \cap X| + |A \cap B \cap X| \\ &\leq (1 - n^{-1})|X| + k < |X| . \end{aligned}$$

In particular this implies that $A \neq M$. Let $\mathfrak{M}_1 := \mathfrak{M}|_A$ and

$$X_1 := \big((A \setminus B) \cap X\big) \cup (A \cap B) .$$

Then $\mathfrak{M}_1 \in \mathcal{K}$, $|M_1| < |M|$, and $|X_1| \leq (1 - n^{-1})|X| + k \leq kn + 1$. Hence, by induction hypothesis, $\mathfrak{M}_1$ has a tree cover $(F^1_v)_{v \in T_1}$ of width at most K where $X_1 \subseteq F^1_{v_1}$ for some $v_1 \in T_1$. Analogously, we can define $\mathfrak{M}_2 := \mathfrak{M}|_B$ and X_2, and obtain a tree cover $(F^2_v)_{v \in T_2}$ and a vertex $v_2 \in T_2$ with $X_2 \subseteq F^2_{v_2}$.

Let r be a new vertex not in T_1 or T_2 and let T be the tree obtained from the disjoint union of T_1, T_2, and r by adding the edges (r, v_1) and (r, v_2). For $v \in T$, we define

$$F_v := \begin{cases} X \cup (A \cap B) & \text{if } v = r , \\ F^1_v & \text{if } v \in T_1 , \\ F^2_v & \text{if } v \in T_2 . \end{cases}$$

Then $|F_r| \leq K$ and $(F_v)_{v \in T}$ is the desired tree cover of $\mathfrak{M}$. □

4.3 Cut-free components

In this section we try to develop a suitable notion of connectedness. Recall that a graph is k-connected if every separator is of size at least k. Hence, we can try to investigate transition systems without non-trivial separations of size k. This was done at the end of Section 4.1. Another, perhaps less obvious, approach consists in using cuts instead of separations. It turns out that the concept of cut freeness provides a natural analogue to the notion of connectedness.

We continue to assume that $\mathfrak{M} = (M, (E_\lambda)_{\lambda \in \Lambda}, \tilde{P})$ is a finite transition system and $\alpha := 4^{|\Lambda|}$.

4.3.1 Cut-free sets

Definition 4.3.1. A nonempty set $X \subseteq M$ is *k-cut free* if every cut *k-cut free*

$(A, B) \in C^1_k(X)$ satisfies $|A| < k$ or $|B| < k$. For $k = 1$ we simply call X *cut free*.

Remark. The property of being k-cut free is obviously MSO-definable.

The cut free graph P_4.

Example. There are no 1-cut-free undirected graphs with 2 or 3 vertices, and P_4 is the only one with 4 vertices.

We start by deriving simple properties of cut-free sets showing that these sets have a similar behaviour as connected ones. First, we consider the question whether the union of cut-free sets is also cut free.

Lemma 4.3.2. *Let X be a set and v some element with $\mathrm{eti}^1_0(X/v) > 1$. If X is 1-cut-free set then so is $X \cup \{v\}$.*

Proof. Let $(A, B) \in C^1_1(X \cup \{v\})$. Since X is 1-cut free one of the sets $A \cap X$ and $B \cap X$ is empty. By symmetry, we may assume that $X \subseteq A$. If $v \in B$ then $\mathrm{eti}^1_0(A/B) \geq \mathrm{eti}^1_0(X/v) > 1$. Contradiction. Thus, $v \in A$ and $B = \varnothing$. □

Lemma 4.3.3. *Let (A_0, B_0) and (A_1, B_1) be cuts of order 1. If $A_0 \cap A_1 \neq \varnothing$ then $(A_0 \cup A_1, B_0 \cap B_1)$ is also a cut of order 1.*

Proof. Let $b, b' \in B_0 \cap B_1$. Then $b \simeq^0_{A_0} b'$ and $b \simeq^0_{A_1} b'$ which implies $b \simeq^0_{A_0 \cup A_1} b'$. On the other hand, if $b \in A_i$, $b' \in A_k$, and $c \in A_0 \cap A_1$, then $b \simeq^0_{B_i} c$ and $c \simeq^0_{B_k} b'$ implies $b \simeq^0_{B_0 \cap B_1} c \simeq^0_{B_0 \cap B_1} b'$. □

Lemma 4.3.4. *Let X and Y be 1-cut-free sets.*

(a) *If $X \cap Y \neq \varnothing$ then $X \cup Y$ is 1-cut free.*

(b) *If $X \cap Y = \varnothing$ then $X \cup Y$ is 1-cut free iff $(X, Y) \notin C^1_1(X \cup Y)$.*

Proof. (a) Let $(A, B) \in C^1_1(X \cup Y)$. Since X is 1-cut free, one of the sets $A \cap X$ and $B \cap X$ is empty. By symmetry, assume $X \subseteq A$. Analogously, one of the sets $A \cap Y$ and $B \cap Y$ is empty. Since $X \cap Y \neq \varnothing$ and $X \subseteq A$ it follows that $Y \subseteq A$, i.e., $B = \varnothing$.

(b) Trivially, if $(X, Y) \in C^1_1(X \cup Y)$ then $X \cup Y$ is not 1-cut free. To show the other direction we may, by symmetry, assume that $\mathrm{eti}^1_0(X/Y) > 1$. Then there is some element $v \in Y$ with $\mathrm{eti}^1_0(X/v) > 1$. Thus $X \cup \{v\}$ is 1-cut free and so is $(X \cup \{v\}) \cup Y = X \cup Y$ by (a). □

In a similar way to connected components, every structure admits a unique partition into maximal cut-free sets.

Lemma 4.3.5. *If X and Y are maximal 1-cut-free sets with $X \neq Y$ then $X \cap Y = \varnothing$ and $(X, Y) \in C^1_1(X \cup Y)$.*

Proof. Otherwise, $X \cup Y$ would be 1-cut free, by Lemma 4.3.4, in contradiction to the maximality of X and Y. □

It follows that we have a well-defined notion of a cut-free component.

Definition 4.3.6. Let $\mathfrak{M}$ be a transition system and $X \subseteq M$. A *cut-free component* of X is a maximal 1-cut-free subset of X. A cut-free component of size 1 is called *trivial*.

cut-free component
trivial

Once we have decomposed a transition system into its cut-free components we can construct a new structure whose elements are the components of the first one. If our notion of connectedness is well-behaved then the cut-free components of this new structure should be trivial.

Definition 4.3.7. For every transition system $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \tilde{P})$ with cut-free components X_i, $i \in I$, we define the structure

$\mathfrak{M}_{\mathrm{cfc}}$

$$\mathfrak{M}_{\mathrm{cfc}} = \left(M^{\mathrm{cfc}}, (E_\lambda^{\mathrm{cfc}})_{\lambda\in\Lambda}, \tilde{P}^{\mathrm{cfc}}\right)$$

with $M^{\mathrm{cfc}} := \{ X_i \mid i \in I \}$,

$E_\lambda^{\mathrm{cfc}} := \{ (X_i, X_k) \mid X_i \times X_k \subseteq E_\lambda \}$,

and $P_n^{\mathrm{cfc}} := \{ X_i \mid X_i \cap P_n \neq \varnothing \}$.

Lemma 4.3.8. *Let $\mathfrak{M}$ be a transition system. The structure $\mathfrak{M}_{\mathrm{cfc}}$ has only trivial cut-free components.*

Proof. Let Y be a cut-free component of $\mathfrak{M}_{\mathrm{cfc}}$. We claim that $\bigcup Y$ is 1-cut free. Otherwise, there is a non-trivial cut $(A, B) \in C_1^1(\bigcup Y)$. Since $(A \cap X_i,\ B \cap X_i) \in C_1^1(X_i)$ it follows that $X_i \subseteq A$ or $X_i \subseteq B$ for all $X_i \in Y$. Thus,

$$\left(\{ X_i \mid X_i \subseteq A \},\ \{ X_i \mid X_i \subseteq B \}\right) \in C_1^1(Y).$$

Contradiction. □

Recall that a partial partition refinement is a partition refinement where we drop the condition that the leaves are singletons. The next result shows that, when constructing a partition refinement of a transition system, it is sufficient to construct separate refinements for each cut-free component.

Lemma 4.3.9. *Every (not necessarily finite) transition system $\mathfrak{M}$ has a partial partition refinement $(U_v)_v$ of width 1 such that every leaf of $(U_v)_v$ is a cut-free component of $\mathfrak{M}$.*

Proof. We define $(U_v)_v$ by induction on $|v|$. Let $U_\varepsilon := M$. Suppose that U_v is already defined. If U_v is 1-cut free then we are done. Otherwise, there exists a cut $(U_{v0}, U_{v1}) \in C_1^1(U_v)$ with $U_{v0}, U_{v1} \neq \varnothing$. Finally, if $|v|$ is a limit, we can set $U_v := \bigcap_{u\prec v} U_u$. □

Corollary 4.3.10. *Let $\mathfrak{M}$ be a transition system. If $\mathfrak{M}$ has only trivial cut-free components then* $\mathrm{pwd}_1\, \mathfrak{M} = 1$.

4.3.2 The internal structure of a cut-free set

After having shown that the notion of a cut-free component is well-behaved, we now turn to an investigation of the internal structure of a cut-free set. In particular, we are interested in finding something akin to a spanning tree. As a first step, we show that every pair of vertices in a cut-free set is connected by a path where edges of a given type do not appear.

type of an edge $\tau(u, v)$

Definition 4.3.11. Let $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \bar{P})$ be a transition system. The *type* of an edge $(u, v) \in M \times M$ is the set

$$\tau(u, v) := \{ E_\lambda xy \mid (u, v) \in E_\lambda \} \cup \{ E_\lambda yx \mid (v, u) \in E_\lambda \} .$$

Lemma 4.3.12. *Let $\mathfrak{M}$ be a finite transition system and $X \subseteq M$ a 1-cut free set. For every pair $x, y \in X$ of distinct vertices and each edge type σ there exists a path from x to y every edge of which is not of type σ.*

Proof. Let $F := \{ (u, v) \in X^2 \mid \tau(u, v) \neq \sigma \}$. If there is no path from x to y in F then there exists a partition $X = A \uplus B$ with $A \times B \cap F = \varnothing$. Hence, every edge between A and B is of type σ and, consequently, $(A, B) \in C^1_1(X)$. Since $x \in A$ and $y \in B$ this cut is non-trivial. Contradiction. □

In the general case with several directed edge relations we obtain a configuration that only vaguely resembles a spanning tree. We prove that we can enumerate each cut-free component such that all intermediate sets remain cut free. For undirected graphs with only one edge relation the situation is much simpler and we will really obtain a pair of trees.

Lemma 4.3.13. *Let $\mathfrak{M}$ be a 1-cut-free transition system. For every 1-cut-free set $X \subseteq M$ there exists a strictly increasing sequence $(A_\beta)_{\beta\leq\alpha}$ of 1-cut-free sets starting with $A_0 = X$ and ending in $A_\alpha = M$ such that*

- $|A_{\beta+1} \setminus A_\beta| \leq 3$, *for all* $\beta < \alpha$,
- $A_\delta = \bigcup_{\beta<\delta} A_\beta$ *if* $\delta \leq \alpha$ *is a limit.*

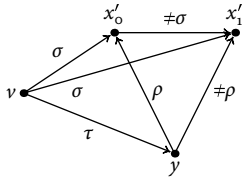
Proof. We define the sets A_β by induction on β. If δ is a limit and A_β is 1-cut free for $\beta < \delta$, then $A_\delta := \bigcup_{\beta<\delta} A_\beta$ is also 1-cut free. Hence, it remains to consider the successor step.

Suppose that A_β is already defined. If $\mathrm{eti}^1_0(A_\beta/x) > 1$ for some element $x \in M \setminus A_\beta$, then we can set $A_{\beta+1} := A_\beta \cup \{x\}$. Otherwise, $(A_\beta, M \setminus A_\beta) \notin C^1_1(V)$ implies that $\mathrm{eti}^1_0(M \setminus A_\beta/A_\beta) > 1$. Fix an arbitrary element $v \in A_\beta$. We will construct some set $C \subseteq M \setminus A_\beta$ of size $|C| \leq 3$ such that $C \cup \{v\}$ is 1-cut free. Then we can set $A_{\beta+1} := A_\beta \cup C$.

For every edge type σ, set

$$W_\sigma := \{\, x \in M \setminus A_\beta \mid \tau(v,x) = \sigma \,\},$$

and let X_i^σ, $i \in I_\sigma$, be the family of cut-free components of W_σ.

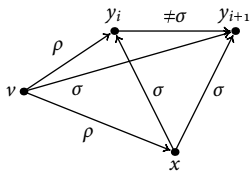


If, for some types $\sigma \neq \tau$, there are components X_i^σ and X_k^τ with $\mathrm{eti}_0^1(X_i/Y_k) > 1$, then we can find elements $x_0, x_1 \in X_i^\sigma$ and $y \in X_k^\tau$ such that $\tau(y,x_0) \neq \tau(y,x_1)$. By Lemma 4.3.12 there exists a path from x_0 to x_1 where every edge is not of type σ. Following that path we find an edge (x'_0, x'_1) with

$$\tau(x'_0, x'_1) \neq \sigma \quad \text{and} \quad \tau(y,x'_0) = \tau(y,x_0) \neq \tau(y,x'_1).$$

It follows that the set $\{v, x'_0, x'_1, y\}$ is 1-cut free.

Now, consider the case that $(X_i^\sigma, X_k^\tau) \in C_1^1(X_i^\sigma \cup X_k^\tau)$ for all components X_i^σ and X_k^τ. Then, every set X_i^σ is also a cut-free component of $M \setminus A_\beta$. By Lemma 4.3.9, there exists a partial partition refinement $(U_v)_v$ of $M \setminus A_\beta$ where every leaf is one of the X_i^σ. Let σ be the type of edges (x,y) with $x \in U_0$ and $y \in U_1$. Since $(U_c \cup A_\beta, U_{1-c}) \notin C_1^1(M)$, for $c < 2$, it follows that there are $x \in U_0$ and $y \in U_1$ such that $\tau(v,y) \neq \sigma$ and $\tau(x,v) \neq \sigma$. If $\tau(v,x) \neq \tau(v,y)$ then we can set $C := \{x,y\}$.



Hence, we may assume that there is some edge type $\rho \neq \sigma$ such that $\tau(v,x) = \rho$ and $\tau(v,y) = \rho$ for all $x \in U_0$ with $\tau(x,v) \neq \sigma$ and all $y \in U_1$ with $\tau(v,y) \neq \sigma$. Since $\mathrm{eti}_0^1(M \setminus A_\beta/v) > 1$ we can find some $z \in M \setminus A_\beta$ with $\tau(v,z) \neq \rho$. By symmetry, we may assume that $z \in U_1$. Hence, $\tau(v,z) = \sigma$. Since M is 1-cut free there exists a path $y_0, \dots, y_m$ from $v = y_0$ to $z = y_m$ all edges of which are not of type σ. W.l.o.g. we may assume that $y_i \notin A_\beta$ for $i > 0$. This implies that $y_i \in U_1$ for $i > 0$ since the path cannot cross the cut between U_0 and U_1 as all edges from U_0 to U_1 are of type σ. By assumption, $\tau(v,y_1) \neq \sigma$ implies $\tau(v,y_1) = \rho$. Therefore, we find some edge (y_i, y_{i+1}) of the path such that $\tau(v,y_i) = \rho$ and $\tau(v,y_{i+1}) = \sigma$. Finally, fix some element $x \in U_0$ with $\tau(v,x) = \rho$. The desired set is $C := \{x, y_i, y_{i+1}\}$. □

The following immediate corollary again underlines the similarity between the notions of cut freeness and connectedness.

Corollary 4.3.14. *For all 1-cut-free sets $X \subseteq Z$ and each cardinal $\kappa \leq |Z \setminus X|$ there there exists a cut-free set $X \subseteq Y \subseteq Z$ of size $\kappa \leq |Y \setminus X| \leq \kappa + 3$.*

We also obtain a relationship between cut freeness and the notion of cographs. These are graphs that do not contain P_4 as induced subgraph.

Corollary 4.3.15. *Let $\mathfrak{G} = (V,E)$ be a 1-cut-free undirected graph and $x \in V$. There exists a set $C \subseteq V$ containing x of size $|C| = 4$ such that $\mathfrak{G}|_C \cong P_4$.*

Proof. By the preceding lemma we can construct an increasing sequence $(A_i)_i$ of 1-cut-free sets with $A_0 = \{x\}$ and $|A_1 \setminus A_0| \leq 3$. Since there are no 1-cut-free undirected graphs of size 2 or 3 it follows that $|A_1| = 4$ and, hence, $\mathfrak{G}|_{A_1} \cong P_4$. □

Corollary 4.3.16. *An undirected graph $\mathfrak{G} = (V, E)$ has a non-trivial cut-free component if and only if it contains P_4 as induced subgraph.*

For undirected graphs with only one edge relation, we can improve the above lemma by constructing some kind of a spanning tree.

Definition 4.3.17. Let $\mathfrak{G} = (V, E)$ be a finite undirected graph and $X \subseteq V$.

spanning tree
(a) A *spanning tree* for X is a pair (T_0, T_1) of undirected trees with set of vertices X such that $(u, v) \notin E$ for all edges (u, v) of T_0 and $(u, v) \in E$ for all edges (u, v) of T_1.

spanning forest
(b) A *spanning forest* of X is a pair (F_0, F_1) of forests consisting of one spanning tree for each cut-free component of X.

girth
(c) The *girth* of a spanning forest (F_0, F_1) is the maximal distance in F_0 of two vertices that are adjacent in F_1.

Lemma 4.3.18. *Let $\mathfrak{G} = (V, E)$ be a finite undirected graph and $X \subseteq V$. There exists a spanning forest (F_0, F_1) of girth 3 for X.*

Proof. W.l.o.g. we may assume that X is 1-cut free. Let $X_0 \subseteq X$ be a maximal set such that there exists a spanning tree (F_0, F_1) of girth 3 for X_0. We have to show that $X_0 = X$.

Suppose otherwise. First, we consider the case that there exists some $v \in X \setminus X_0$ with $\mathrm{eti}^1_0(X_0/v) > 1$. There are elements $y_0, y_1 \in X_0$ with $(x, y_0) \notin E$ and $(x, y_1) \in E$. Following the unique F_1-path from y_0 to y_1 we find some edge (z_0, z_1) with $(x, z_0) \notin E$ and $(x, z_1) \in E$. Consequently, we can extend (F_0, F_1) to a spanning tree for $X_0 \cup \{v\}$ by adding the edge (x, z_0) to F_0 and the edges (x, z_1) and (z_0, z_1) to F_1. Contradiction.

It follows that $\mathrm{eti}^1_0(X_0/v) = 1$ for all $v \in X \setminus X_0$. Since X is 1-cut free, Lemma 4.3.13 implies that there exists a set $C \subseteq X \setminus X_0$ of size $|C| \leq 3$ such that $X_0 \cup C$ is 1-cut free. By the above arguments, we have $|C| > 1$ and, since there are no 1-cut-free undirected graphs of size 2 or 3, it follows that $|C| = 3$. Therefore, fixing an arbitrary vertex $w \in X_0$, we obtain a set $C \cup \{w\}$ that induces an subgraph isomorphic to P_4. Adding the edges of this subgraph to F_0 and F_1 yields a spanning tree for $X_0 \cup C$. Again a contradiction. □

A tree is a connected graph that becomes unconnected if we remove any vertex that is not a leaf. A similar result holds for cut-free transition systems.

Lemma 4.3.19. *Let $\mathfrak{M}$ be 1-cut free, and let C, $W \subseteq M$ be disjoint nonempty sets such that, for every $a \in W$, there is a cut $(A(a), B(a)) \in C_1^1(M \setminus \{a\})$ with $C \subseteq A(a)$. Suppose that the cut $(A(a), B(a))$ is chosen such that $B(a)$ is minimal.*

(a) *If we define the order $\sqsubseteq$ on W by*

$$a \sqsubseteq b \quad :\text{iff} \quad B(a) \supseteq B(b)\,,$$

then $(W, \sqsubseteq)$ forms a tree.

(b) *There is some element $a \in W$ such that $|B(a) \cap W| \leq 1$.*

Proof. (a) Let $a, b \in W$. By definition, $\sqsubseteq$ is reflexive and transitive. Since $A(a) \cap A(b) \supseteq C \neq \varnothing$, Lemma 4.3.3 implies that the cut $(A(a) \cup A(b), B(a) \cap B(b))$ is of order 1. To show that $\sqsubseteq$ is antisymmetric we prove that $B(a) = B(b)$ implies $a = b$. Since $B(a) = B(b)$ implies $a \in A(b)$ and $b \in B(a)$ we otherwise would have

$$\big(A(a) \cup A(b),\ B(a) \cap B(b)\big) \in C_1^1(M)\,,$$

which contradicts the 1-cut freeness of $\mathfrak{M}$.

Thus, $\sqsubseteq$ is a partial order. To prove that it is also a tree we consider three cases.

($b \in A(a)$) From

$$\big((A(a) \cup A(b)) \setminus \{a\},\ B(a) \cap B(b)\big) \in C_1^1(M \setminus \{a\})$$

and the minimality of $B(a)$ it follows that $B(a) \cap B(b) = B(a)$. Hence, $B(a) \subseteq B(b)$, i.e., $b \sqsubseteq a$.

($a \in A(b)$) By symmetry, we have $a \sqsubseteq b$.

($a \in B(b)$ and $b \in B(a)$) In this case we have $B(a) \cap B(b) = \varnothing$ since, otherwise,

$$\big(B(a) \cup B(b),\ A(a) \cap A(b)\big) \in C_1^1(M)\,,$$

would contradict the 1-cut freeness of $\mathfrak{M}$.

If $a \sqsubseteq c$ and $b \sqsubseteq c$ then $B(c) \subseteq B(a)$ and $B(c) \subseteq B(b)$. Hence, $B(a) \cap B(b) \neq \varnothing$ and the above results imply that $a \in A(b)$ or $b \in A(a)$. It follows that $a \sqsubseteq b$ or $b \sqsubseteq a$ as desired.

(b) Choose some $a \in W$ such that $B(a)$ is minimal. We claim that $|B(a) \cap W| \leq 1$.

By (a) it follows that $a \notin A(b)$ for all $b \in W$. Thus, $a \in B(b)$ for all $b \in W \setminus \{a\}$. If $b, b' \in B(a) \cap W$ then $B(b) \subseteq A(a) \cup \{a\}$, and $b' \notin B(b) \subseteq A(a) \cup \{a\}$ implies $b' \in A(b)$. Finally, it follows that $B(b) \subseteq B(b')$. By symmetry we obtain that $B(b') \subseteq B(b)$. Therefore, it follows that $b = b'$. □

4.3.3 Systems of disjoint components

In the remainder of this section we show that in a transition system either there exists a family of large disjoint cut-free sets or we can construct a partial partition refinement of small width. A similar theorem was used by Robertson and Seymour in [64] to construct a grid in a graph of large tree width: If the tree width is large then there exists a family of large disjoint connected sets which can be used as rows (or columns) of the desired grid.

$d(\mathcal{A})$

Definition 4.3.20. For every family $\mathcal{A} = \{A_0, \dots, A_n\}$ of disjoint 1-cut-free sets with $|A_0| \leq \cdots \leq |A_n|$ we define

$$d(\mathcal{A}) := \big(|A_0|, \dots, |A_n|\big).$$

maximal family of 1-cut-free sets

Such a family is called *maximal* if there is no $\mathcal{B}$ with $|\mathcal{B}| = |\mathcal{A}|$ and $d(\mathcal{A}) <_{\text{lex}} d(\mathcal{B})$.

Note that by maximality we refer not to the size of the family but to the cardinality of its components. If $\mathcal{A}$ is a maximal family then all connected components in the complement $\overline{\bigcup \mathcal{A}}$ are small, and the following lemma shows that the type index is bounded by

$$\operatorname{eti}^1_0\big(\bigcup \mathcal{A} \,/\, \overline{\bigcup \mathcal{A}}\big) \leq |\mathcal{A}|.$$

Lemma 4.3.21. *Let $\mathfrak{M}$ be a transition system. If $\mathcal{A}$ is a maximal family of 1-cut-free sets and B a cut-free component of $M \setminus \bigcup \mathcal{A}$, then $(A, B) \in C^1_1(A \cup B)$ for all $A \in \mathcal{A}$.*

Proof. Otherwise, by Lemma 4.3.4, the set $A \cup B$ would be 1-cut free and the family

$$(\mathcal{A} \setminus \{A\}) \cup \{A \cup B\}$$

would contradict the maximality of $d(\mathcal{A})$. □

Suppose that $\mathcal{A}$ is a maximal family of cut-free sets and there are sets $A, B \in \mathcal{A}$ with $|A| < |B|$ such that, for some $X \subseteq B$ the set $A \cup X$ is 1-cut free. Then, by maximality, all cut-free components of $B \setminus X$ are of size at most $|A|$. We will show that in this situation there exists a small set X_0 such that $X_0 \cap X \neq \emptyset$, for all sets X with the above properties. The main argument in the proof below consists of the following technical lemma.

Lemma 4.3.22. *Let $\mathfrak{M}$ be a 1-cut-free transition system and $\mathcal{A}$ a family of disjoint nonempty subsets $A \subseteq M$. If there exists a set $X \subseteq M$ such that X is a cut-free component of $M \setminus A$ for every $A \in \mathcal{A}$, then $|\mathcal{A}| \leq 3$.*

Proof. By Lemma 4.3.13, we can construct an increasing sequence of 1-cut-free sets $X = U_0 \subset U_1 \subset \cdots \subset M$ with $|U_1 \setminus U_0| \leq 3$. If $|\mathcal{A}| > 3$ then there is some $A \in \mathcal{A}$ with $A \cap (U_1 \setminus U_0) = \emptyset$. Therefore, $U_1 \subseteq M \setminus A$ and $X \subset U_1$ is no maximal 1-cut-free set of $M \setminus A$. Contradiction. □

Lemma 4.3.23. *Let $\mathfrak{M}$ be a 1-cut-free transition system and k, m numbers with $6km < |M|$. If $\mathcal{A}$ is a nonempty family of nonempty subsets $A \subseteq M$ of size $|A| \leq m$ such that, for all $A \in \mathcal{A}$, every cut-free component of $M \setminus A$ is of size at most k, then there exists a set $X \subseteq M$ of size $|X| \leq 3m$ such that $A \cap X \neq \emptyset$ for all $A \in \mathcal{A}$.*

Proof. Suppose that such a set X does not exist. Let $\mathcal{B} \subseteq \mathcal{A}$ be a maximal family of pairwise disjoint sets. Since $X := \bigcup \mathcal{B}$ is a set with $A \cap X \neq \emptyset$ for all $A \in \mathcal{A}$ it follows that $|\mathcal{B}| > 3$. Fix four distinct sets A, B_0, B_1, $B_2 \in \mathcal{B}$, and let Y_i, $i \in I$, be the family of cut-free components of $M \setminus A$.

Suppose that Z is a cut-free component of $M \setminus B$ for $B \in \mathcal{A} \setminus \{A\}$. If $Z \cap Y_i \neq \emptyset$ for some component with $B \cap Y_i = \emptyset$ then $Y_i \subseteq Z$ since Y_i is 1-cut free in $M \setminus B$. On the other hand, if $A \cap Z = \emptyset$ then $Z \subseteq Y_i$, for some $i \in I$, as Z is 1-cut free in $M \setminus A$. Since $|Z| \leq k$ it further follows that, if $A \cap Z \neq \emptyset$, then $Z \setminus A$ is the union of at most $k - 1$ components Y_i.

For every $B \in \mathcal{A} \setminus \{A\}$ let

- $C_0(B)$ be the union of those components Z of $M \setminus B$ with $Z \cap A \neq \emptyset$;
- $C_1(B)$ contain the components Z of $M \setminus B$ contained in some set Y_i with $B \cap Y_i \neq \emptyset$;
- $C_2(B)$ be the union of those Y_i which are cut-free components of both $M \setminus A$ and $M \setminus B$.

Then $C_0(B) \cup C_1(B) \cup C_2(B) = M \setminus B$ and

$$\begin{aligned} |C_0(B)| &\leq k|A| \leq km\,, \\ |C_1(B)| &\leq (k-1)|B| \leq (k-1)m\,, \\ |M \setminus C_2(B)| &\leq |B| + |C_0(B)| + |C_1(B)| \leq 2km\,, \end{aligned}$$

which implies that

$$|C_2(B_0) \cap C_2(B_1) \cap C_2(B_2)| \geq |M| - 3 \cdot 2km > 0\,.$$

Consequently, there exists a component

$$Y_i \subseteq C_2(B_0) \cap C_2(B_1) \cap C_2(B_2)\,.$$

But, according to the preceding lemma, Y_i cannot be a cut-free component of $M \setminus C$ for every $C \in \{A, B_0, B_1, B_2\}$. Contradiction. □

With these preparations we are able to prove the main theorem of this sections. Either a transition system contains a family of large disjoint cut-free sets, or its partition width is bounded.

Theorem 4.3.24. *Let $\mathfrak{M}$ be a transition system and $k < \omega$. If there is no family A_i, $i < k$, of disjoint 1-cut-free sets of size $|A_i| \geq m$ then there exists a partial partition refinement of $\mathfrak{M}$ of width $\alpha^{k(k+9)/2}$ and granularity $\frac{1}{5}(6^k - 1)(m + \frac{1}{5}) - \frac{k}{5}$.*

Proof. By Lemma 4.3.9, it is sufficient to prove the result for each cut-free component of $\mathfrak{M}$. Thus, w.l.o.g. we can assume that $\mathfrak{M}$ is 1-cut free.

We construct the partial partition refinement by induction on k. If $k = 1$ then $\mathfrak{M}$ does not contain a cut-free component of size at least m. Consequently, there exists a partial partition refinement of $\mathfrak{M}$ of width 1 and granularity m.

Now suppose that $k > 1$. Let A_i, $i < k$, be a maximal family of disjoint 1-cut-free sets with $|A_0| \leq \cdots \leq |A_{k-1}|$. By assumption, $|A_0| < m$.

(A) If there is no index $n > 0$ such that $6|A_{n-1}| + 1 < |A_n|$ then, by induction, it follows that

$$\begin{aligned} |A_{i+1}| \leq 6|A_i| + 1 &\leq 6 \cdot \left(6^i(m + \tfrac{1}{5}) - \tfrac{1}{5}\right) + 1 \\ &= 6^{i+1}(m + \tfrac{1}{5}) - \tfrac{1}{5}\,, \end{aligned}$$

which implies that

$$\begin{aligned} |A_0| + \cdots + |A_{k-1}| &\leq \sum_{i<k} \left(6^i(m + \tfrac{1}{5}) - \tfrac{1}{5}\right) \\ &= \tfrac{1}{5}(6^k - 1) \cdot (m + \tfrac{1}{5}) - \tfrac{k}{5}\,. \end{aligned}$$

Define $B_0 := A_0 \cup \cdots \cup A_{k-1}$ and $B_1 := M \setminus B_0$. Since every cut-free component of B_1 is of size less than m there exists a partial partition refinement of B_1 of width 1 and granularity m. By Lemma 4.3.21, we have $\mathrm{eti}^1_0(B_0/B_1) \leq k$. Thus, attaching the partial partition refinement of B_1 to the cut (B_0, B_1) we obtain a refinement of width at most α^k and the desired granularity.

(B) Now consider the case that there is such an index n. Let $B_0 := \bigcup_{i<n} A_i$, $B_1 := \bigcup_{i\geq n} A_i$, and, for $l \geq n$, define

$$C_l := \{\, v \in A_l \mid \mathrm{eti}^1_0(A_i/v) > 1 \text{ for some } i < n \,\}\,.$$

The maximality of A_i, $i < k$, implies that, for every $v \in C_l$, the set $A_l \setminus \{v\}$ does not contain a 1-cut-free set of size greater than $|A_{n-1}|$. Since $|A_l| > 6|A_{n-1}|$ we can apply Lemma 4.3.23 and it follows that $|C_l| \leq 3$. Consequently, the set $C := \bigcup_l C_l$ is of size $|C| \leq 3(k-n) \leq 3k$.

Since $\mathrm{eti}^1_o(A_i/A_l \setminus C_l) = 1$ it follows that

$$\begin{aligned}
&\mathrm{eti}^1_o(A_i/\textstyle\bigcup_{l\geq n}(A_l \setminus C_l)) = 1\\
\Rightarrow\ &\mathrm{eti}^1_o(B_o/B_1 \setminus C) \leq n\\
\Rightarrow\ &\mathrm{eti}^1_o(B_1 \setminus C/B_o) \leq \alpha^n\\
\Rightarrow\ &\mathrm{eti}^1_o(B_1/B_o) \leq \alpha^n + |C| \leq \alpha^k + 3k\,.
\end{aligned}$$

Furthermore, we have

$$\mathrm{eti}^1_o(B_o/B_1) \leq \mathrm{eti}^1_o(B_o/B_1 \setminus C) \cdot \mathrm{eti}^1_o(B_o/C) \leq n \cdot \alpha^{|C|} \leq k\alpha^{3k}.$$

If B_1 contains $k - n + 1$ disjoint 1-cut-free sets of size at least m then the family consisting of these sets together with $A_1, \ldots, A_{n-1}$ contradicts the maximality of A_i, $i < k$. If $n > 1$ then $k - n + 1 < k$ and we can apply the induction hypothesis to obtain a partial partition refinement $(U_v)_{v\in T}$ of B_1 of width at most $\alpha^{(k-1)(k+8)/2}$ and granularity

$$\frac{1}{5}(6^{k-1} - 1)(m + 1/5) - \frac{k-1}{5} < \frac{1}{5}(6^k - 1)(m + 1/5) - k/5\,.$$

Since
$$\begin{aligned}
&\mathrm{eti}^1_o(U_v/M \setminus U_v)\\
&\leq \mathrm{eti}^1_o(U_v/B_1 \setminus U_v) \cdot \mathrm{eti}^1_o(U_v/B_o)\\
&\leq \alpha^{(k-1)(k+8)/2} \cdot (\alpha^k + 3k)\\
&\leq \alpha^{(k-1)(k+8)/2} \cdot (\alpha^k + \alpha^k)\\
&\leq \alpha^{(k-1)(k+8)/2+k+1}\\
&= \alpha^{k(k+9)/2-3}
\end{aligned}$$

it follows that the partial partition refinement of $\mathfrak{M}$ obtained by attaching $(U_v)_v$ to the cut (B_o, B_1) is of width at most

$$\max\{k\alpha^{3k},\ \alpha^{k(k+9)/2-3}\} \leq \alpha^{k(k+9)/2}.$$

(c) It remains to consider the case that $n = 1$. Since $\mathfrak{M}$ is 1-cut free there exists a set $D \subseteq B_1$ of size $|D| \leq 3$ such that $A_o \cup D$ is 1-cut free. If $B_1 \setminus D$ contains $k-1$ disjoint 1-cut-free sets of size at least m then the family of these together with $A_o \cup D$ would contradict the maximality of A_i, $i < k$. Hence, by induction hypothesis, there exists a partial partition refinement $(U_v)_v$ of $B_1 \setminus D$ of width at most $\alpha^{(k-1)(k+8)/2}$ and granularity

$$\frac{1}{5}(6^{k-1} - 1)(m + 1/5) - \frac{k-1}{5}\,.$$

Since $\operatorname{eti}^1_0(U_v/M \smallsetminus U_v)$
$\leq \operatorname{eti}^1_0(U_v/(B_1 \smallsetminus D) \smallsetminus U_v) \cdot \operatorname{eti}^1_0(U_v/A_0) \cdot \operatorname{eti}^1_0(U_v/D)$
$\leq \alpha^{(k-1)(k+8)/2} \cdot (\alpha + 3k) \cdot \alpha^3$
$\leq \alpha^{(k-1)(k+8)/2} \cdot \alpha^{1+k} \cdot \alpha^3$
$= \alpha^{k(k+9)/2}$

it follows that the partial partition refinement of $\mathfrak{M}$ obtain by attaching $(U_v)_v$ to the cut (B_0, B_1) is of width at most

$$\max\{k\alpha^{3k},\ \alpha^{k(k+9)/2}\} \leq \alpha^{k(k+9)/2}. \qquad \square$$

Corollary 4.3.25. *Let $\mathfrak{M}$ be a finite transition system and $k < \omega$. If there is no family A_i, $i < k$, of disjoint 1-cut-free sets of size $|A_i| \geq m$ then*

$$\operatorname{pwd}_1 \mathfrak{M} \leq \max\Big\{\alpha^{k(k+9)/2},\ \tfrac{1}{5}(6^k - 1)(m + \tfrac{1}{5}) - \tfrac{k}{5}\Big\}$$

Corollary 4.3.26. *Let $\mathfrak{M}$ be a finite transition system and $k < \omega$. If there is no family A_i, $i < k$, of disjoint 1-cut-free sets of size*

$$|A_i| \geq \frac{\frac{10}{3}|M| + k}{6^k - 1} - \frac{1}{5}$$

then there is a cut (B_0, B_1) of $\mathfrak{M}$ of order $\alpha^{\alpha^{k(k+9)/2}}$ with $|B_0|, |B_1| \leq \frac{2}{3}|M|$.

Proof. By the preceding theorem there exists a partial partition refinement $(U_v)_{v \in T}$ of $\mathfrak{M}$ of width $\alpha^{k(k+9)/2}$ and granularity

$$\frac{1}{5}(6^k - 1)\left(\frac{\frac{10}{3}|M| + k}{6^k - 1} - \frac{1}{5} + \frac{1}{5}\right) - \frac{k}{5} = \frac{2}{3}|M|.$$

Fix some $v \in T$ such that $|U_v| > \frac{2}{3}|M|$ and $|U_{v0}|, |U_{v1}| \leq \frac{2}{3}|M|$. By symmetry we may assume that $|U_{v0}| \geq |U_{v1}|$. Define $B_0 := U_{v0}$ and $B_1 := U_{v1} \cup (M \smallsetminus U_v)$. The cut (B_0, B_1) is of the desired order and

$$|B_1| = |M| - |B_0| \leq |M| - \frac{1}{2}|U_v|$$
$$\leq |M| - \frac{1}{2} \cdot \frac{2}{3}|M| = \frac{2}{3}|M|. \qquad \square$$

4.4 Ribbons

A theorem of Menger states that, if $\mathfrak{G} = (V, E)$ is a graph and $X, Y \subseteq V$ sets such that no set of size less than k separates X from Y, then there

are k disjoint paths connecting X and Y. In the present section we try to derive an analogue to this result. In the proof of their Excluded Grid Theorem Robertson and Seymour [64] use such paths for the columns of the grid whose rows were obtained as described in in the previous section.

Definition 4.4.1. Let $X, Y, Z \subseteq M$ be disjoint. X is *k-separated* from Y over Z if there is a partition $Z = A \mathbin{\dot\cup} B$ with *k-separated*

$$\mathrm{eti}^1_0(Y \cup B / X \cup A) < k\,.$$

We will repeatedly make use of the following observation which immediately follows from the definition.

Lemma 4.4.2. *If X is not k-separated from Y over Z and $Z = A \mathbin{\dot\cup} B \mathbin{\dot\cup} C$ then $X \cup A$ is not k-separated from $Y \cup B$ over C.*

Alternatively, we can express k-separatedness also in terms of cuts or separations.

Lemma 4.4.3. *If X is k-separated from Y over Z then there exist*

(1) *a cut $(A, B) \in C^1_{\alpha^{k-1}}(Z)$ with $X \subseteq A$ and $Y \subseteq B$, and*

(2) *a separation $(A, B) \in S_{\alpha^k}(Z)$ with $X \subseteq A \setminus B$ and $Y \subseteq B \setminus A$.*

Proof. (1) Let (A, B) be a cut of Z such that $\mathrm{eti}^1_0(Y \cup B / X \cup A) < k$. Then $\mathrm{eti}^1_0(X \cup A / Y \cup B) \leq \alpha^{k-1}$ and, hence, $(X \cup A, Y \cup B) \in C^1_{\alpha^{k-1}}(Z)$.
(2) Follows from (1) and Lemma 4.1.2. □

We will show that, if X is not k-separated from Y over Z, then we can find something like a system of k disjoint paths from X to Y. The first step consists in defining an ordering of Z that induces a notion of distance of an element from X.

In the remainder of this section we will make frequent use of the following notation. If $(A_i)_i$ is a seqeuence of sets then we denote the union $\bigcup_{i>n} A_i$ by $A_{>n}$. The sets $A_{\geq n}$, $A_{<n}$, and $A_{\leq n}$ are defined analogously.

Definition 4.4.4. Let $X, Y, Z \subseteq M$.

(a) A *stratification* of Z from X to Y is a finite sequence $(A_i)_{i<l}$ of disjoint nonempty sets such that *stratification*

(1) $Z = \bigcup_{i<l} A_i$,

(2) for all $i < l$ and every $b \in Y \cup A_{>i}$ there exists some $c \in Y$ such that $b \simeq^{\mathrm{o}}_{X \cup A_{<i}} c$.

(b) A stratification $(A_i)_{i<l}$ *refines* $(B_i)_{i<m}$ iff there is a non-decreasing surjective function $\mu : [l] \to [m]$ such that $A_i \subseteq B_{\mu(i)}$ for all $i < l$.

We will construct a stratification $(A_i)_i$ by induction on i. If A_i is already defined then we consider all external types over $X \cup A_{\leq i}$ realised in $Y \cup (Z \setminus A_{\leq i})$. We say that $A_{\leq i}$ *distinguishes* those elements whose type is not realised in Y. These elements form the next stage A_{i+1}.

distinguish

Definition 4.4.5. Let $\mathfrak{M}$ be a transition system. For $X, Y, Z \subseteq M$ the set of elements of Z distinguished by X is

$D_Z^Y(X)$

$$D_Z^Y(X) := X \cup \{\, a \in Z \mid a \not\simeq^{\mathrm{o}}_X c \text{ for all } c \in Y \,\}.$$

Note that the operator $D_Z^Y(X)$ is monotone in X and Z. We use the stages of the fixed-point induction of D_Z^Y to define a stratification of Z.

Lemma 4.4.6. *If X is not k-separated from Y over Z then there is some $W \subseteq Z$ such that*

(1) *X is not k-separated form Y over W;*

(2) *W admits a stratification $(A_i)_{i<l}$ with $\mathrm{eti}^1_{\mathrm{o}}(Y \cup A_i / X \cup A_{<i}) \geq k$ for all $i < l$;*

(3) *$W = \bigcup_{n<\omega}(D_W^Y)^n(X)$.*

Proof. Let $W := \bigcup_{n<\omega}(D_Z^Y)^n(X)$. Then the third condition is satisfied since $D_W^Y(V) = D_Z^Y(V) \cap W$ for $V \subseteq M$.

For the first one, suppose that there is a partition $W = A \cup B$ such that $\mathrm{eti}^1_{\mathrm{o}}(Y \cup B / X \cup A) < k$. Since $\mathrm{eti}^1_{\mathrm{o}}(Y \cup B \cup (Z \setminus W) / X \cup A) \geq k$ there is some $b \in Z \setminus W$ such that $b \not\simeq^{\mathrm{o}}_{X \cup A} c$ for all $c \in Y \cup B$. Hence, $b \not\simeq^{\mathrm{o}}_{X \cup W} c$ for all $c \in Y$ which implies that $b \in D_Z^Y(W) = W$. Contradiction.

Finally, note that the sequence $(A_i)_{i<l}$ with

$$A_i := (D_W^Y)^{i+1}(X) \setminus (D_W^Y)^i(X)$$

forms a stratification of W. □

Our analogue of a system of disjoint paths consists of a stratification $(A_i)_i$ together with a bijection between consecutive stages of the stratification. These bijections map elements $a \in A_i$ to some $b \in A_{i+1}$ such that a distinguishes b from Y. This behaviour is formalised in the following definition.

Definition 4.4.7. Let X, Y, $Z \subseteq M$ be disjoint. Fix a set $Y_{\mathrm{o}} \subseteq Y$ of representatives of $Y/\simeq^{\mathrm{o}}_X$, i.e., a set such that

$$|Y_{\mathrm{o}}| = \mathrm{eti}^1_{\mathrm{o}}(Y_{\mathrm{o}}/X) = \mathrm{eti}^1_{\mathrm{o}}(Y/X),$$

$\iota : Y \to Y_{\mathrm{o}}$

and let $\iota : Y \to Y_{\mathrm{o}}$ be the function mapping elements $b \in Y$ to the unique $b_{\mathrm{o}} \in Y_{\mathrm{o}}$ with $b_{\mathrm{o}} \simeq^{\mathrm{o}}_X b$.

$S(a)$

(a) The set of elements distinguished by $a \in Z$ is

$$S(a) := \{\, b \in Y \setminus Y_0 \mid b \not\simeq^0_{X\cup\{a\}} \imath b \,\},$$

and, for $A \subseteq Z$, we define $\sigma(A)$

$$\sigma(A) := \Big|\bigcup_{a\in A} S(a)\Big|.$$

(b) A set $A \subseteq Z$ is called *independent* to Y over X if there exists an injective function $\mu : A \to Y \setminus Y_0$ such that *independent* $\mu : A \to Y \setminus Y_0$

$$\mu(a) \in S(a) \qquad \text{for all } a \in A.$$

(c) A set $A \subseteq Z$ is *free* to Y over X if $\sigma(A_0) \geq |A_0|$ for all $A_0 \subseteq A$. *free*

Remark. Note that σ can be equivalently defined by

$$\sigma(A) := \operatorname{eti}^1_0(Y/X \cup A) - \operatorname{eti}^1_0(Y/X).$$

Lemma 4.4.8. *Let X, Y, $Z \subseteq M$ be disjoint finite sets. Every set $A \subseteq Z$ that is free to Y over X is also independent to Y over X.*

Proof. If $A \subseteq Z$ is free then, by Hall's theorem, there exists a system of distinct representatives $b_a \in S(a)$, $a \in A$, for the family $(S(a))_{a\in A}$. We obtain the desired function $\mu : A \to Y \setminus Y_0$ by setting $\mu(a) := b_a$. □

Lemma 4.4.9. *Let X, Y, $Z \subseteq M$ be disjoint. The family of subsets $A \subseteq Z$ that are independent to Y over X forms a matroid.*

Proof. This statement is just a reformulation of Theorem 7.3.1 of Welsh [80]. The original proofs are due to Edmonds and Fulkerson [35] and Mirsky and Perfect [54]. □

To obtain an injection $\mu : A \to Y \setminus Y_0$ it is therefore sufficient to prove the existence of large free sets.

Lemma 4.4.10. *Let X, Y, $Z \subseteq M$ be disjoint finite sets and let $A \subseteq Z$ be a maximal set free to Y over X. Then* $\operatorname{eti}^1_0(Y/X \cup A) = \operatorname{eti}^1_0(Y/X \cup Z)$.

Proof. Suppose otherwise. Then there is some $b \in Z \setminus A$ with

$$\operatorname{eti}^1_0(Y/X \cup A \cup \{b\}) > \operatorname{eti}^1_0(Y/X \cup A).$$

Fix $c_0, c_1 \in Y$ such that $c_0 \simeq^0_{X\cup A} c_1$ and $c_0 \not\simeq^0_{X\cup A\cup\{b\}} c_1$. Obviously, this implies $c_0 \simeq^0_{X\cup A_0} c_1$ and $c_0 \not\simeq^0_{X\cup A_0\cup\{b\}} c_1$ for all $A_0 \subseteq A$. It follows that

$$\begin{aligned}
\sigma\big(A_0 \cup \{b\}\big) &= \operatorname{eti}^1_0(Y/X \cup A_0 \cup \{b\}) - \operatorname{eti}^1_0(Y/X) \\
&\geq \operatorname{eti}^1_0(Y/X \cup A_0) + 1 - \operatorname{eti}^1_0(Y/X) \\
&= \sigma(A_0) + 1 \\
&\geq |A_0| + 1 = |A_0 \cup \{b\}|,
\end{aligned}$$

for all $A_0 \subseteq A$, which implies that the set $A \cup \{b\}$ is also free. Contradiction. □

After all these preparations we are finally able to define what exactly we mean by a system of disjoint paths.

k-ribbon

Definition 4.4.11. Let $X, Y, Z \subseteq M$ be disjoint finite sets and $k < \omega$. A *k-ribbon* from X to Y in Z is a sequence $\rho = (\bar{a}^n)_{n<l}$ of k-tuples $\bar{a}^n \subseteq Y \cup Z$ satisfying the following conditions:

(1) $(A_n)_{n<l}$ is a stratification of $A_{<l}$ where $A_n := \bar{a}^n \setminus Y$.

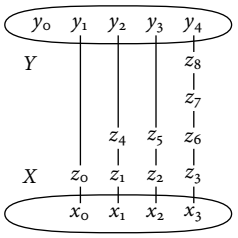
(2) $\bar{a}^{l-1} \subseteq Y$.

(3) If $a_i^n \in Y$ then $a_i^m = a_i^n$ for all $m \geq n$. Conversely, if $a_i^m = a_j^n$ then $i = j$ and, by (1), necessarily $a_i^m \in Y$.

(4) A_n is free to $Y \cup A_{n+1}$ over $X \cup A_{<n}$. In particular, for all $n < l-1$ and all $i < k$ with $a_i^n \in Z$, there is some $j < k$ with $a_j^n = a_j^{n+1} \in Y$ such that $a_i^{n+1} \simeq^{\circ}_{X \cup A_{<n}} a_j^{n+1}$ and $a_i^{n+1} \not\simeq^{\circ}_{X \cup A_{<n} \cup \{a_i^n\}} a_j^{n+1}$. We denote the mapping taking a_i^{n+1} to a_j^{n+1} by ι.

thread

The sequences $(a_i^n)_{n<l}$, for $i < k$, are called the *threads* of ρ.



Example. In the graph on the left there exists a 5-ribbon $(\bar{a}^n)_{n<5}$ from X to Y where

$$\bar{a}^0 := y_0\, z_0\, z_1\, z_2\, z_3$$
$$\bar{a}^1 := y_0\, y_1\, z_4\, z_5\, z_6$$
$$\bar{a}^2 := y_0\, y_1\, y_2\, y_3\, z_7$$
$$\bar{a}^3 := y_0\, y_1\, y_2\, y_3\, z_8$$
$$\bar{a}^4 := y_0\, y_1\, y_2\, y_3\, y_4$$

The desired analogue of Menger's theorem can now be stated in the following way.

Theorem 4.4.12. *If X is not k-separated from Y over Z then there exists a k-ribbon from X to Y_0 in Z, for every subset $Y_0 \subseteq Y$ with* $\mathrm{eti}^1_0(Y_0/X \cup Z) \geq k$.

Proof. We prove the claim by induction on $|Z|$. W.l.o.g. we may assume that $|Y_0| = k$.

By Lemma 4.4.6, we may assume that $Z = \bigcup_{i<\omega}(D_Z^Y)^i(X)$ and that $D_n := (D_Z^Y)^{n+1}(X) \setminus (D_Z^Y)^n(X)$ is a stratification of Z. As usual, we denote the union $\bigcup_{i<n} D_i$ by $D_{<n}$. Let l be the minimal index such that $\mathrm{eti}^1_0(Y_0/X \cup D_{<l}) = k$.

By reverse induction, we define an increasing sequence of sets

$$A_0 \subseteq \cdots \subseteq A_l = Y_0$$

and a sequence of sets $B_n \subseteq D_n$, $n < l$, such that

- $\operatorname{eti}^1_{\mathrm{o}}(A_n \cup B_n / X \cup D_{<n}) = |A_n \cup B_n| = k$;
- $\operatorname{eti}^1_{\mathrm{o}}(A_n / X \cup D_{<n}) = |A_n| = \operatorname{eti}^1_{\mathrm{o}}(Y_{\mathrm{o}} / X \cup D_{<n})$.

We start with $A_l := Y_{\mathrm{o}}$ and $B_l := \varnothing$. Suppose that A_{n+1} and B_{n+1} are already defined. Let

$$m_n := \operatorname{eti}^1_{\mathrm{o}}(A_{n+1} \cup B_{n+1} / X \cup D_{<n}) = \operatorname{eti}^1_{\mathrm{o}}(A_{n+1} / X \cup D_{<n})$$

where the second equality holds since $(D_n)_n$ is a stratification of Z.

Fix some $A_n \subseteq A_{n+1}$ with $\operatorname{eti}^1_{\mathrm{o}}(A_n / X \cup D_{<n}) = m_n$, and let $B'_n \subseteq D_n$ be a set of maximal size free to $A_{n+1} \cup B_{n+1}$ over $X \cup D_{<n}$.

First we show that $\operatorname{eti}^1_{\mathrm{o}}(B'_n / X \cup D_{<n}) \geq k - m_n$. By Lemma 4.4.10, we know that

$$\operatorname{eti}^1_{\mathrm{o}}(A_{n+1} \cup B_{n+1} / X \cup D_{<n} \cup B'_n) = k\,.$$

Suppose that $\operatorname{eti}^1_{\mathrm{o}}(B'_n / X \cup D_{<n}) < k - m_n$. Then $B'_n \subset D_n$ is a proper subset of D_n and, hence, $|B'_n| < |Z|$. Since the set $X \cup D_{<n}$ is not k-separated from $(Y \cup Z) \setminus (X \cup D_{<n} \cup B'_n)$ over B'_n we can apply the induction hypothesis and there exists a k-ribbon $\rho = (\bar{a}^i)_i$ from $X \cup D_{<n}$ to $A_{n+1} \cup B_{n+1}$ in B'_n. But $\bar{a}^{\mathrm{o}} \subseteq A_{n+1} \cup B_{n+1} \cup B'_i$ implies that

$$\begin{aligned} k &= \operatorname{eti}^1_{\mathrm{o}}(\bar{a}^{\mathrm{o}} / X \cup D_{<n}) \\ &\leq \operatorname{eti}^1_{\mathrm{o}}(A_{n+1} \cup B_{n+1} \cup B'_n / X \cup D_{<n}) \\ &\leq \operatorname{eti}^1_{\mathrm{o}}(A_{n+1} \cup B_{n+1} / X \cup D_{<n}) + \operatorname{eti}^1_{\mathrm{o}}(B'_n / X \cup D_{<n}) \\ &< m_n + (k - m_n) \end{aligned}$$

which is a contradiction.

Hence, $\operatorname{eti}^1_{\mathrm{o}}(B'_n / X \cup D_{<n}) \geq k - m_n$ and we can fix a free subset $B_n \subseteq B'_n$ of size $k - m_n$. By Lemma 4.4.8, there exists an injective function $\mu_n : B_n \to (A_{n+1} \cup B_{n+1}) \setminus A_n$ such that

$$\mu_n(b) \not\asymp^{\mathrm{o}}_{X \cup D_{<n} \cup \{b\}} \mu\mu_n(b) \qquad \text{for all } b \in B_n\,.$$

Since $|A_{n+1} \cup B_{n+1}| = |A_n| + |B_n|$ any such function is bijective.

Having defined A_n and B_n, for $n < l$, it remains to construct the desired k-ribbon $\rho = (\bar{a}^n)_{n<l}$. Again we proceed by inverse induction on n. Let $\bar{a}^l$ be an enumeration of Y_{o}. Suppose that $\bar{a}^{n+1} \subseteq A_{n+1} \cup B_{n+1}$ is already defined. Consider the function $\mu_n : B_n \to (A_{n+1} \cup B_{n+1}) \setminus A_n$ from above and set

$$a^n_i := \begin{cases} \mu_n^{-1}(a^{n+1}_i) & \text{if } a^{n+1}_i \in \operatorname{rng} \mu_n\,, \\ a^{n+1}_i & \text{otherwise.} \end{cases}$$ □

The above theorem can slightly be improved by considering a family of sets Y_i, $i \in I$, instead of a single one. To shorten our notation we set $Y_J := \bigcup_{i \in J} Y_i$ for $J \subseteq I$.

Proposition 4.4.13. *Let $\mathfrak{M}$ be a finite transition system, $X \subseteq M$, and $r < \omega$. Let Y_i, $i \in I$, be a family of disjoint sets and set $Z := M \setminus (X \cup Y_I)$.*

If there exists no set $J \subseteq I$ of size $|J| < r$ such that $Y_{I \setminus J}$ is $(r+1)$-separated from X over Z, then we can find

- *a set $J \subseteq I$ of size $|J| = r$,*
- *an $(r+1)$-ribbon $\rho = (\bar{a}^i)_i$ to X over Y_J in Z,*
- *an element $c \in X$, and*
- *an injective function $\mu : J \to \bar{a}^{\mathrm{o}}$*

such that $c \not\simeq^{\mathrm{o}}_{Y_i} \mu(i)$ for every $i \in J$.

Proof. In order to apply the previous theorem we fix a binary relation E_λ of $\mathfrak{M}$. For every $i \in I$, we add a new element a_i to $\mathfrak{M}$ that is connected by an E_λ-edge (a_i, b) to each $b \in Y_i$. Then we have

$$b_{\mathrm{o}} \simeq^{\mathrm{o}}_{a_i} b_1 \quad \text{iff} \quad b_{\mathrm{o}}, b_1 \in Y_i \text{ or } b_{\mathrm{o}}, b_1 \notin Y_i \,.$$

Let $A := \{ a_i \mid i \in I \}$. We claim that A is not $(r+1)$-separated from X over $Z \cup Y_I$.

Let (B_{o}, B_1) be a cut of $A \cup Z \cup Y_I \cup X$ with $A \subseteq B_{\mathrm{o}}$ and $X \subseteq B_1$. Set $K := \{ i \in I \mid Y_i \cap B_1 \neq \varnothing \}$. If $|K| \geq r$ then

$$\mathrm{eti}^1_{\mathrm{o}}(B_1/B_{\mathrm{o}}) \geq \mathrm{eti}^1_{\mathrm{o}}(X \cup (B_1 \cap Y_I)/A) \geq 1 + |K| \geq r+1 \,.$$

Otherwise, since $Y_{I \setminus K} \subseteq B_{\mathrm{o}}$ is not $(r+1)$-separated from X over Z it also follows that

$$\mathrm{eti}^1_{\mathrm{o}}(B_1/B_{\mathrm{o}}) \geq \mathrm{eti}^1_{\mathrm{o}}(B_1 \setminus (Y_K \cup A)/B_{\mathrm{o}} \setminus (Y_K \cup A)) \geq r+1 \,.$$

Hence, there exists an $(r+1)$-ribbon $(\bar{a}^i)_{i<l}$ to X over A in $Z \cup Y_I$. Since

$$(D^X_{Z \cup Y_I})^n(A) = (D^X_Z)^{n-1}(Y_I) \cup A$$

it follows that

$$\bar{a}^{\mathrm{o}} \subseteq X \cup D^X_{Z \cup Y_I}(A) \setminus A \subseteq X \cup Y_I \,.$$

Consequently, $\mathrm{eti}^1_{\mathrm{o}}(\bar{a}^{\mathrm{o}}/A) = r+1$ implies that $\bar{a}^{\mathrm{o}} \cap X = \{c\}$ for some $c \in X$, and there exists an injective function $\mu : \bar{a}^{\mathrm{o}} \setminus X \to I$ such that

$$a^{\mathrm{o}}_i \in Y_{\mu(a^{\mathrm{o}}_i)} \qquad \text{for all } a^{\mathrm{o}}_i \notin X \,.$$

Since $a^1_i \not\simeq^{\mathrm{o}}_{a^{\mathrm{o}}_i} c$ for all $a^{\mathrm{o}}_i \notin X$ we obtain the desired ribbon by setting $\rho := (\bar{a}^i)_{\mathrm{o}<i<l}$ and $J := \operatorname{rng} \mu$. □

Since we are interested in proving that some structure admits MSO-coding it would be handy if the threads of a ribbon were (uniformly) definable. Unfortunately, this does not seem to be the case. The most we can do so far is to define an ordering of each thread.

Lemma 4.4.14. *There exists an* MSO*-formula* $\varphi(x, y; X, Y, Z)$ *such that, if* $\rho = (\bar{a}^n)_{n<l}$ *is a* k*-ribbon from* X *to* Y*, then*

$$\mathfrak{M} \models \varphi(a_i^m, a_i^n; X, Y, A_i) \quad \text{iff} \quad m < n\,,$$

where $A_i := \{\, a_i^n \mid n < l \,\}$.

Proof. Note that

$$\bigcup_{s<\omega} (D^Y_{A_i \smallsetminus \{a_i^n\}})^s(X) = X \cup \{\, a_i^m \mid m < n \,\}\,.$$

is the least fixed point of an MSO-definable monotonous operator. Hence, there is an MSO-formula $\varphi(x, y; X, Y, Z)$ stating that

$$x \in \bigcup_{s<\omega} (D^Y_{Z \smallsetminus \{y\}})^s(X)\,.$$

□

Above we proposed ribbons as suitable analogues for systems of disjoint paths. The next lemma shows that each thread of a ribbon can indeed be viewed as a path.

Lemma 4.4.15. *Let* $\rho = (\bar{a}^n)_{n<l}$ *be a* k*-ribbon from* X *to* Y*, and let* $A_i := \{\, a_i^n \mid n < l \,\}$ *for* $i < k$.

(a) *If* Y *is* 1*-cut free then so is* $Y \cup A_i$ *for every* $i < k$.

(b) *If* X *and* Y *are* 1*-cut free then so is* $X \cup Y \cup A_i$ *for every* $i < k$ *with* $A_i \smallsetminus Y \neq \varnothing$.

Proof. (a) By reverse induction on n, it follows from Lemma 4.3.2 that the sets $Y \cup \{a_i^n, \dots, a_i^{l-1}\}$ are 1-cut free for every $i < k$.

(b) $a_i^0 \not\approx^0_X b$ for all $b \in Y$ implies that

$$\mathrm{eti}^1_0(Y \cup A_i/X) > \mathrm{eti}^1_0(Y \cup \{a_i^0\}/X) > 1\,.$$

Hence, the claim follows from (a). □

In the remainder of this section we try to strengthen this connection between threads and cut freeness. In particular, we construct minimal subsets of a thread still exhibiting the above behaviour.

Definition 4.4.16. Let $\mathfrak{M}$ be a transition system and $X, Y \subseteq M$.

(a) A *pseudopod* of X is a nonempty sequence $(a_i)_{i<n}$ of elements such that $\mathrm{eti}^1_0(X \cup A_{>k}/a_k) > 1$, for every $k < n$, where as usual $A_{>k} := \{\, a_i \mid i > k \,\}$. The element a_0 is called the *end* of $(a_i)_i$. — *pseudopod*, *end*

(b) A *pseudopod connection* of X to Y is a pseudopod $(a_i)_i$ of X such that either $a_0 \in Y$ or $\mathrm{eti}^1_0(Y/a_0) > 1$. — *pseudopod connection*

(c) A pseudopod $(a_i)_i$ of X is *minimal* if there is no proper subsequence with the same end that forms a pseudopod of X. — *minimal*

(d) The *union* of two pseudopods $\bar{a}$ and $\bar{b}$ of X is the sequence $(\bar{b} \smallsetminus \bar{a})\bar{a}$. — *union*

The following properties immediately follow from the respective definitions.

Lemma 4.4.17. *Let X be a 1-cut-free set.*

(a) *If ρ is a k-ribbon to X then every thread of ρ is a pseudopod.*

(b) *If X is 1-cut free and $(a_i)_{i<n}$ a pseudopod of X, then $X \cup A_{\geq k}$ is 1-cut free for all k.*

(c) *If X and Y are 1-cut free and $(a_i)_i$ is a pseudopod connection of X to Y, then $X \cup \bar{a} \cup Y$ is 1-cut free.*

(d) *The union of two pseudopods is again a pseudopod.*

The structure of a minimal pseudopod is especially simple.

Lemma 4.4.18. *Let $\mathfrak{M}$ be a transition system and $(a_i)_{i<n}$ a minimal pseudopod of $X \subseteq M$.*

(a) $\mathrm{eti}^1_0(A_{>k+1}/a_k) = 1$ *for all* $k < n-1$.

(b) $\mathrm{eti}^1_0(a_{k+2}a_{k+1}/a_k) > 1$ *for every* $k < n-2$.

Proof. (a) Suppose there is some index k such that $\mathrm{eti}^1_0(A_{>k+1}/a_k) > 1$ and assume that k is minimal with this property. Then $a_{k+1} \simeq^0_{A_{<k}} a_{k+2}$ which implies that

$$\mathrm{eti}^1_0\big(X \cup (A_{>i} \setminus \{a_{k+1}\}) \,/\, a_i\big) > 1 \qquad \text{for } i < k\,.$$

Consequently, the subsequence obtained by omitting a_{k+1} is still a pseudopod in contradiction to the minimality of $\bar{a}$.

(b) If $\mathrm{eti}^1_0(X/a_k) > 1$ then, by (a), the subsequence $a_0, \dots, a_k, a_{n-1}$ is a pseudopod and $\bar{a}$ is not minimal. If $\mathrm{eti}^1_0(a_{k+2}a_{k+1}/a_k) = 1$ then, by (a), $\mathrm{eti}^1_0(A_{>k}/a_k) = 1$, which implies that $\mathrm{eti}^1_0(X \cup \{a_{n-1}\}/a_k) > 1$. Hence, the subsequence $a_0, \dots, a_k, a_{n-1}$ is a pseudopod in contradiction to the minimality of $\bar{a}$. □

4.5 Meshes and weaves

With the technical results of the previous sections it is possible to translate the core of the original proof of the Excluded Grid Theorem (see [64]). We still assume that $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \bar{P})$ is a finite transition system and $\alpha := 4^{|\Lambda|}$.

We start by specifying what we mean by a 'grid'. We will consider two grid-like configurations. The first one, called a *mesh*, is a translation of the concept of two families of disjoint connected sets such that each set of one family intersects every set of the other one. When defining the other configuration, called a *preweave*, we have a system of disjoint paths and a family of connected sets in mind where every set intersects each path.

Definition 4.5.1. Let X and Y_i, $i < n$, be disjoint sets. X *connects* the family $(Y_i)_i$ if, for every family $(Y_i')_i$ of disjoint 1-cut-free sets with $Y_i' \supseteq Y_i$, the set $X \cup \bigcup_i Y_i'$ is 1-cut free.

X connects $(Y_i)_i$

We call a set X *connecting* if there exists a family $(Y_i)_i$ of disjoint sets, disjoint from X, such that X connects $(Y_i)_i$.

connecting set

Definition 4.5.2. An (m, n)*-mesh* consists of a family A_i, $i < m$, of disjoint 1-cut-free sets and a family B_i, $i < n$, of disjoint nonempty sets such that every B_k connects the A_i, $i < m$.

(m, n)*-mesh*

Definition 4.5.3. Let $X, Y, Z \subseteq M$ be disjoint.

(a) A set $C \subseteq Z$ *separates* X from Y over Z if there is a separation (A, B) of $X \cup Z \cup Y$ with $X \subseteq A$, $Y \subseteq B$, and $A \cap B \subseteq C$.

C separates X from Y

(b) C *separates* X from Y *modulo* D over Z if $C \cup D$ separates X from Y over Z.

C separates modulo D

Lemma 4.5.4. *If there is no set of size n separating X from Y modulo D over Z then there exists a* $\lfloor \log_\alpha n \rfloor$*-ribbon from X to Y in* $Z \setminus D$.

Proof. Let $r := \lfloor \log_\alpha n \rfloor$. There exists no separation (A, B) of $X \cup Z \cup Y$ with $X \subseteq A$, $Y \cup D \subseteq B$, and $|A \cap B| \leq n$. By Lemma 4.4.3, it follows that X is not r-separated from $Y \cup D$ over $Z \setminus D$. Hence, there exists an r-ribbon $\rho = (\bar{a}^i)_{i<l}$ from X to $Y \cup D$ in $Z \setminus D$.

In the same way as above it follows that $X \cup D$ is not r-separated from Y over $Z \setminus D$. This implies that $\mathrm{eti}^1_0(Y/X \cup Z) \geq r$. Consequently, we can choose ρ such that $\bar{a}^{l-1} \subseteq Y$. □

Definition 4.5.5. Let $\mathfrak{M}$ be a transition system and X, Y, $Z \subseteq M$ disjoint finite sets. An $(m, n; r)$*-preweave* consists of an m-ribbon ρ from X to Y in $M \setminus Z$ and a family of disjoint sets B_i, $i < n$, each of which is the union of at most r connecting sets, such that each B_i separates X from Y modulo Z.

$(m, n; r)$*-preweave*

Below we will show that a transition system which does not admit (k, n)-separations contains a large mesh or a large preweave. But what we are actually looking for is an MSO-definable pairing function. The notions of a mesh and a preweave are but a first approximation that does not seem to suffice for defining such functions in monadic second-order logic.

In particular the notion of a $(m, n; r)$-preweave is unsatisfactory. It would be much better if we could demand $r = 1$. The problem we are facing is that there does not seem to be a notion of *where* a thread intersects one of the sets B_i. If such a concept were available then we could take a subset of the threads and a subfamily of the B_i such that all the threads intersect each B_i in the same connected component.

The first step in the proof consists in deriving a condition for the existence of a preweave. We need one technical lemma that applies Proposition 4.4.13 to obtain a family of paths.

Lemma 4.5.6. *Let X and Y_i, $i < k$, be disjoint sets. If there is no set $J \subseteq [k]$ of size $|J| < r$ such that there exists a set of size α^{r+1} separating X from $\bigcup_{i\notin J} Y_i$ over $M \setminus \bigcup_{i\in J} Y_i$, then there exist a set $I \subseteq [k]$ of size $|I| = r$ and a family of disjoint sets P_i, $i \in I$, disjoint from $X \cup \bigcup_{i\in I} Y_i$ such that P_i connects X and Y_i.*

Proof. By Lemma 4.4.3 and Proposition 4.4.13, there exists a set $I \subseteq [k]$ of size $|I| = r$ and an $(r+1)$-ribbon ρ to X over $\bigcup_{i\in I} Y_i$ in $M \setminus \bigcup_{i<k} Y_i$. The threads of ρ are the desired sets P_i connecting Y_i with X. □

Lemma 4.5.7. *Let $\mathfrak{M}$ be a finite transition system. Let X and Y_i, $i < n$, be disjoint connecting sets and $r < \omega$ a number such that*

(1) *there is no set $I \subseteq [n]$ of size $|I| = r$ and no family of disjoint sets P_i, $i \in I$, disjoint from $X \cup \bigcup_{i\in I} Y_i$ such that P_i connects X and Y_i, for all $i \in I$, and*

(2) *there is no set $I \subseteq [n]$ of size $|I| \leq m$ such that there exists a set of size $m - |I|$ separating X from $\bigcup_{i\notin I} Y_i$ over M.*

Then $\mathfrak{M}$ contains a

$$\big(\lfloor \log_\alpha(m - l(\alpha^{r+1}/r + 1))\rfloor,\ \lfloor l/r\rfloor;\ r-1\big)\text{-preweave}$$

for every $l \leq n$.

Proof. Mirroring the proof of (2.9) in [62] we construct sets $C_k \subseteq M$ and disjoint sets $I_k \subseteq [n]$ by induction on k. Suppose that C_i and J_i are already defined for $i < k$. Let $J := [n] \setminus \bigcup_{i<k} I_i$. By (1) and Lemma 4.5.6, there exists a set $I_k \subseteq J$ of size $|I_k| < r$ and a separation (A, B) of $M \setminus \bigcup_{i\in I_k} Y_k$ with $X \subseteq A$, $Y_i \subseteq B$ for $i \in J \setminus I_k$, and $A \cap B = C_k$ for some set C_k of size $|C_k| \leq \alpha^{r+1}$ disjoint from Y_i, $i \in I_k$.

Fix $l \leq n$. We can perform the above construction for at least $\lfloor n/r\rfloor \geq \lfloor l/r\rfloor$ steps. Let

$$C := \bigcup_{k<\lfloor l/r\rfloor} C_k, \qquad Z_k := \bigcup_{i\in I_k} Y_i\,,$$
$$I := \bigcup_{k<\lfloor l/r\rfloor} I_k, \qquad U := \bigcup_{i\notin I} Y_i\,.$$

If there were a separation (A, B) of M with $X \subseteq A$, $U \subseteq B$, and $A \cap B = C \cup D$ for some set D of size $|D| \leq \beta := m - l(\alpha^{r+1}/r + 1)$, then

$$\begin{aligned}|I| + |C| + |D| &\leq \sum_{k<\lfloor l/r\rfloor} (|I_k| + |C_k|) + |D| \\ &< l + l\alpha^{r+1}/r + m - l(\alpha^{r+1}/r + 1) = m\end{aligned}$$

would contradict (2). Therefore, there is no set of size β separating X from U modulo C over M. By Lemma 4.5.4, there exists a $\lfloor \log_\alpha \beta\rfloor$-ribbon ρ from X to U in $M \setminus C$. Since $Z_k \cup C$ separates X and U for every k it follows that $\mathfrak{M}$ contains an

$$\big(\lfloor \log_\alpha \beta\rfloor,\ \lfloor l/r\rfloor;\ r-1\big)\text{-preweave.}$$ □

Corollary 4.5.8. *Let $\mathfrak{M}$ be a finite transition system that does not contain a $(\theta_1, \theta_2; r-1)$-preweave. Let X and Y_i, $i < n$, be disjoint connecting sets and set*

$$\beta := \alpha^{r+1}/r + 1\,, \quad m := r\beta\theta_2 + \alpha^{\theta_1}\,, \quad \text{and} \quad l := r\theta_2\,.$$

At least one of the following two conditions is satisfied:

(1) *There is a set $I \subseteq [n]$ of size $|I| = r$ and a family of disjoint sets P_i, $i \in I$, disjoint from $X \cup \bigcup_{i\in I} Y_i$ such that P_i connects X and Y_i for all $i \in I$.*

(2) *There are sets $C \subseteq M$ and $I \subseteq [n]$ of size $|C| + |I| \leq m$ such that C separates X from $\bigcup_{i\notin I} Y_i$.*

Proof. If $n < l$ then (2) holds for $I := [n]$. Otherwise, the result follows from the preceding lemma since

$$\lfloor l/r \rfloor = \lfloor r\theta_2/r \rfloor = \theta_2\,,$$
$$\text{and} \quad \lfloor \log_\alpha(m - \beta l) \rfloor = \lfloor \log_\alpha\bigl(r\beta\theta_2 + \alpha^{\theta_1} - r\beta\theta_2\bigr) \rfloor = \theta_1\,. \qquad \square$$

It remains to show that, if a transition system $\mathfrak{M}$ does not contain a preweave, then it contains a mesh or there exists a sunflower splitting $\mathfrak{M}$ into several parts of bounded size. The following function is used as bound of the size of the core of this sunflower.

Definition 4.5.9. Let $\zeta_{\theta_1\theta_2}(2, n) := \alpha^n$ and $\zeta_{\theta_1\theta_2}(k, n)$

$$\zeta_{\theta_1\theta_2}(k+1, n) := f(n)\bigl[1 + k\zeta_{\theta_1\theta_2}(k, f(n)+1)\bigr]$$

where $$f(n) := \theta_2(\alpha^{n+1} + n) + \alpha^{\theta_1} + n\,.$$

Lemma 4.5.10. *Let $\mathfrak{M}$ be a finite transition system that does not contain a $(\theta_1, \theta_2; n-1)$-preweave, and let $k \geq 2$. For every family of disjoint 1-cut-free sets $Y_i \subseteq M$, $i < k$, and all numbers $n < \omega$ at least one of the following statements is true:*

(1) *There are disjoint nonempty sets Z_i, $i < n$, such that $(Y_i)_{i<k}$ and $(Z_i)_{i<n}$ form a (k, n)-mesh.*

(2) *For some $l \leq k$, there exists a sunflower $(A_i)_{i<l}$ with domain M and core X of size $|X| \leq \zeta_{\theta_1\theta_2}(k, n)$ such that no petal $A_i \setminus X$ contains elements from every Y_j.*

Proof. ($k = 2$) If (2) does not hold then there is no separation (A_0, A_1) of order $|A_0 \cap A_1| \leq \alpha^n$ with $Y_0 \subseteq A_0$ and $Y_1 \subseteq A_1$. Hence, Y_0 is not n-separated from Y_1 and, by Theorem 4.4.12, there exists an n-ribbon ρ to Y_1 over Y_0. The threads of ρ satisfy (1).

($k > 2$) Let C_i, $i < N$, be a maximal family of disjoint sets such that each C_i connects all the Y_j except possibly Y_{k-1}. If $N < n$ then

(1) fails for $Y_0, \dots, Y_{k-2}$, and by induction hypothesis there exists a sunflower as in (2). This sunflower shows that (2) also holds for $Y_0, \dots, Y_{k-1}$. Thus, we may assume that $N \geq n$ and that $C_0, \dots, C_{N-n}$ do not connect $Y_0, \dots, Y_{k-1}$. By Corollary 4.5.8, one of the following cases occurs:

(a) There is a set $I \subseteq [N - n + 1]$ of size $|I| = n$ and a family of disjoint sets P_i, $i \in I$, disjoint from $Y_{k-1} \cup \bigcup_{j \in I} C_j$ such that P_i connects Y_{k-1} and C_i for all $i \in I$. In this case (1) holds with $Z_i := C_i \cup P_i$ since, if $Y'_i \supseteq Y_i$ is 1-cut free, for $i < k$, then so is $C_i \cup Y'_0 \cup \cdots \cup Y'_{k-2}$ and, thus, also $P_i \cup C_i \cup Y'_0 \cup \cdots \cup Y'_{k-1}$.

(b) There are sets $D \subseteq M$ and $I \subseteq [N - n + 1]$ of size

$$|D| + |I| \leq m := \theta_2(\alpha^{n+1} + n) + \alpha^{\theta_1}$$

such that D separates Y_{k-1} from every C_i, $i \in [N-n+1] \setminus I$. Let (A, B) be a separation with $A \cap B = D$, $Y_{k-1} \subseteq A$, and $C_i \subseteq B$ for $i \in [N-n+1] \setminus I$. Set $Y'_i := Y_i \cap A$. If there were more than $|D| + |I| + n \leq m + n$ disjoint 1-cut-free subsets of A that connect $Y'_0, \dots, Y'_{k-2}$, then more than $|I| + n$ of these would be disjoint from D, and those together with C_i, $i \in [N - n] \setminus I$, would form a family of more than N disjoint 1-cut-free components in contradiction to the maximality of N. Hence, there are no sets $Z_i \subseteq A$, $i < m + n + 1$, that form an $(k - 1, m + n + 1)$-mesh together with $Y'_0, \dots, Y'_{k-2}$.

Applying the induction hypothesis to the structure $\mathfrak{M}|_A$ we obtain a sunflower $(A'_i)_{i<l}$ with core $X' \subseteq A$ of size $|X'| \leq \zeta_{\theta_1\theta_2}(k-1, m+n+1)$ and $l \leq k-1$ such that no petal $A'_i \setminus X'$ contains elements from all sets $Y_0, \dots, Y_{k-2}$. By Lemma 4.1.6, there exists a set $X \supseteq X' \cup D$ of size

$$|X| \leq \big((k-1)\zeta_{\theta_1\theta_2}(k-1, m+n+1) + 1\big)(m+n) = \zeta_{\theta_1\theta_2}(k, n)$$

such that the sequence $(A_i)_{i<l+1}$ with

$$A_i := \begin{cases} A'_i \cup X & \text{for } i < l, \\ B \cup X & \text{for } i = l, \end{cases}$$

forms a sunflower with core X.

It remains to prove that no petal $A_i \setminus X$ contains elements from every Y_j, $j < k$. For $i = l$, we have $(B \setminus D) \cap Y_{k-1} = \varnothing$. Suppose that, for some $i < l$, we have $((A'_i \cup X) \setminus X) \cap Y_j \neq \varnothing$ for every $j < k$. Then

$$(A'_i \setminus X) \cap Y_j = (A'_i \setminus X) \cap Y'_j \neq \varnothing$$

for all $j < k$ in contradiction to the choice of $(A'_i)_i$. □

Combining the preceding lemmas we obtain the following result which can be regarded as a very weak form of an Excluded Grid Theorem.

Theorem 4.5.11. *Let $\theta_1, \theta_2, \theta_3, \theta_4 > 1$ be numbers such that*

$$\beta := 2\alpha^{\alpha^{\theta_3(\theta_3+9)/2}} \leq \zeta_{\theta_1\theta_2}(\theta_3, \theta_4)\,,$$

and define

$$\theta_5 := \zeta_{\theta_1\theta_2}(\theta_3, \theta_4) \quad \textit{and} \quad \theta_6 := \frac{3}{10}(\theta_3+1)(6^{\theta_3}-1)\,.$$

Every finite transition systems $\mathfrak{M}$ that does not contain a (θ_3, θ_4)-mesh or a $(\theta_1, \theta_2; \theta_4 - 1)$-preweave admits (θ_5, θ_6)-separations.

Proof. If there is a separation $(A, B) \in S_\beta(M)$ with $|A \setminus B|, |B \setminus A| \leq \frac{2}{3}|M|$ then we are done, since $\beta \leq \theta_5$ and $\frac{2}{3} \leq 1 - \theta_6^{-1}$. Otherwise, by Corollary 4.3.26, there exists a family of θ_3 disjoint 1-cut-free sets C_i, $i < \theta_3$, of size

$$|C_i| \geq \frac{\frac{10}{3}|M| + \theta_3}{6^{\theta_3}-1} - \frac{1}{5} > (\theta_3+1)\theta_6^{-1}|M| - \frac{1}{5} \geq \theta_3\theta_6^{-1}|M|\,.$$

By Lemma 4.5.10, there is some sunflower $(A_i)_i$ with core X of size $|X| \leq \zeta_{\theta_1\theta_2}(\theta_3, \theta_4) = \theta_5$ such that no petal $A_i \setminus X$ contains elements from every C_k.

For each $v \in C_0$ fix indices $j(v)$ and $0 < i(v) < \theta_3$ such that $v \in A_{j(v)}$ and $(A_{j(v)} \setminus X) \cap C_{i(v)} = \varnothing$. There is a subset $D \subseteq C_0$ of size

$$|D| \geq \theta_3^{-1}|C_0| \geq \theta_6^{-1}|M|$$

such that $i(u) = i(v)$ for all $u, v \in D$. W.l.o.g. assume that $i(v) = 1$ for $v \in D$. Define

$$B_0 := \bigcup\{\, A_i \mid D \cap (A_i \setminus X) \neq \varnothing \,\}$$
$$\text{and} \quad B_1 := \bigcup\{\, A_i \mid D \cap (A_i \setminus X) = \varnothing \,\}\,.$$

Then (B_0, B_1) is a separation of order $|B_0 \cap B_1| = |X| \leq \theta_5$. Furthermore, it follows that

$$|B_0 \setminus B_1| \leq |M| - |C_1| \leq (1 - \theta_6^{-1})|M|\,,$$
$$\text{and} \quad |B_1 \setminus B_0| \leq |M| - |D| \leq (1 - \theta_6^{-1})|M|\,,$$

since $C_1 \cap B_0 = \varnothing$ and $D \subseteq B_0$. □

Remark. Makowsky and Rotics introduce in [52] the 2-colour width of a transition system $\mathfrak{M}$ as the least number k such that there exists some set $X \subseteq M$ of size $\frac{1}{3}|M| \leq |X| \leq \frac{2}{3}|M|$ with $\mathrm{eti}^1_0(X/\overline{X}) \leq k$. Using this notion we obtain the following corollary to the above theorem:

Given θ_1, θ_2, θ_3, and θ_4 we can compute a number k such that, if $\mathfrak{M}$ is a finite transition systems that does not contain a (θ_3, θ_4)-mesh or a $(\theta_1, \theta_2; \theta_4 - 1)$-preweave, then the 2-colour width of every substructure of $\mathfrak{M}$ is at most k.

If we could prove that, in addition, $\mathfrak{M}$ strongly admits (θ_5, θ_6)-separations then we could use Proposition 4.2.6 to bound the partition width of $\mathfrak{M}$. The next theorem shows that it is possible to improve the above result by constructing separations that split large sets $Z \subseteq M$. Unfortunately, in the proof of Proposition 4.2.6 we need such separations for small Z.

Theorem 4.5.12. *Let $\mathfrak{M}$ be a finite transition system without (θ_3, θ_4)-meshes and $(\theta_1, \theta_2; \theta_4 - 1)$-preweaves, and let $Z \subseteq M$ be a set which has no partial partition refinement of width $\alpha^{\theta_3(\theta_3+9)/2}$ and granularity $\frac{1}{5}\big(6^{\theta_3}-1\big)\big(\theta_3\theta_6^{-1}|Z| + \frac{1}{5}\big) - \frac{k}{5}$. There exists a separation $(A, B) \in S_{\theta_5}(M)$ such that*

$$\big|(A \setminus B) \cap Z\big|, \ \big|(B \setminus A) \cap Z\big| \leq (1 - \theta_6^{-1})|Z|\,,$$

where

$$\theta_5 := \zeta_{\theta_1\theta_2}(\theta_3, \theta_4) \quad \textit{and} \quad \theta_6 := \frac{3}{10}(\theta_3 + 1)(6^{\theta_3} - 1)\,.$$

Proof. By Theorem 4.3.24, there exists a family of θ_3 disjoint 1-cut-free sets $C_i \subseteq Z$, $i < \theta_3$, of size $|C_i| \geq \theta_3\theta_6^{-1}|Z|$.

Hence, Lemma 4.5.10 implies that there is some sunflower $(A_i)_i$ with domain M and core X of size $|X| \leq \zeta_{\theta_1,\theta_2}(\theta_3, \theta_4) = \theta_5$ such that no petal $A_i \setminus X$ contains elements from every C_k.

For each $v \in C_0$ fix indices $j(v)$ and $0 < i(v) < \theta_3$ such that $v \in A_{j(v)}$ and $(A_{j(v)} \setminus X) \cap C_{i(v)} = \varnothing$. As above we can find a subset $D \subseteq C_0$ of size

$$|D| \geq \theta_3^{-1}|C_0| \geq \theta_6^{-1}|Z|$$

such that $i(u) = i(v)$ for all $u, v \in D$. W.l.o.g. assume that $i(v) = 1$ for all $v \in D$. Define

$$B_0 := \bigcup\big\{ A_i \;\big|\; D \cap (A_i \setminus X) \neq \varnothing \big\}$$
$$\text{and} \quad B_1 := \bigcup\big\{ A_i \;\big|\; D \cap (A_i \setminus X) = \varnothing \big\}\,.$$

Then (B_0, B_1) is a separation of order $|B_0 \cap B_1| = |X| \leq \theta_5$. Furthermore, $C_1 \cap B_0 = \varnothing$, $D \subseteq B_0$, and $D, C_1 \subseteq Z$ implies that

$$\big|(B_0 \setminus B_1) \cap Z\big| \leq |Z| - |C_1| \leq (1 - \theta_6^{-1})|Z|\,,$$
$$\text{and} \quad \big|(B_1 \setminus B_0) \cap Z\big| \leq |Z| - |D| \leq (1 - \theta_6^{-1})|Z|\,. \qquad \square$$

5 Tree-Interpretable Structures

Having defined a class of structures with a simple monadic theory we now try to find suitable subclasses where the monadic theory of each structure is decidable. We cannot hope to obtain a precise characterisation of when the MSO-theory of a structure is decidable. For instance, by coding a suitable nonrecursive set, we can easily construct even trees whose first-order theory has an arbitrary high Turing degree. Therefore, we aim at finding a subclass as large as possible such that we can still give a meaningful characterisation.

Throughout this and the following two chapters all structures are assumed to be of finite signature.

5.1 The Caucal hierarchy

A general method to obtain classes of structures with certain desirable properties consists in fixing one or several such structures and considering the closure of this set under operations preserving said properties. If, furthermore, the class $\mathcal{K}$ is obtained from finitely many base structures by operations each of which can be encoded by a finite object, then every structure in $\mathfrak{M} \in \mathcal{K}$ has a finite representation, namely, the sequence of the operations one has to apply to the base structures to obtain $\mathfrak{M}$.

If one is interested in monadic second-order logic the canonical structure to start with is the binary tree $\mathfrak{T}_2 := (2^{<\omega}, \preceq, \mathrm{suc}_0, \mathrm{suc}_1)$. As operations we can use MSO-functors. Since $\mathfrak{A} \times \mathfrak{T}_2 \leq_{\mathrm{MSO}} \mathfrak{T}_2$ for any finite structure $\mathfrak{A}$ we can restrict ourselves to interpretations and iterations. $\mathfrak{T}_2$

Definition 5.1.1. Let $\mathcal{C} := \bigcup_{n<\omega} \mathcal{C}_n$ where $\mathcal{C}_0$ is the class of all finite structures and $\mathcal{C}_n$, for $n > 0$, is the class of all structures $\mathfrak{M}$ such that there exists an injective interpretation $\mathcal{I} : \mathfrak{M} \leq_{\mathrm{MSO}} \mathfrak{T}_2^{(n-1)}$ where $\mathfrak{T}_2^{(n)}$ is the n-th iteration of $\mathfrak{T}_2$. The sequence $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots$ is called the *Caucal hierarchy*. $\mathcal{C}, \mathcal{C}_n$ *Caucal hierarchy*

Note that, by Lemma 1.3.15, we can equivalently define $\mathcal{C}_{n+1}$ to be the class of all structures obtained from $\mathfrak{T}_2$ by a finite number of

injective MSO-interpretations and at most n iterations.

We list some basic properties which immediately follow from the definition.

Proposition 5.1.2. *Each level $\mathcal{C}_n$ of the Caucal hierarchy is closed under injective* MSO*-interpretations. In particular, it is closed under*

(1) *isomorphisms,*

(2) *finite unions,*

(3) *definable expansions,*

(4) *expansion by finitely many constants, and*

(5) *substructures with definable universe.*

Lemma 5.1.3. *Every structure $\mathfrak{M} \in \mathcal{C}$ is of finite partition width.*

We have chosen the definition of $\mathcal{C}$ with the decidability of MSO in mind. Actually, a slightly stronger result holds.

Theorem 5.1.4. *There exists an algorithm which, given a formula $\varphi(\bar{x}) \in$ MSO + C, an injective interpretation $\mathcal{I} : \mathfrak{M} \leq_{\text{MSO}} \mathfrak{T}_2^{(n)}$, the number n, and a tuple $\bar{w} \subseteq T_2^{(n)}$, decides whether $\mathfrak{M} \models \varphi(\mathcal{I}(\bar{w}))$.*

In particular, the (MSO + C)*-theory of every structure $\mathfrak{M} \in \mathcal{C}$ is decidable.*

Proof. By the Interpretation Lemma, we can decide $\mathfrak{M} \models \varphi(\mathcal{I}(\bar{w}))$ by checking whether $\mathfrak{T}_2^{(n)} \models \varphi^{\mathcal{I}}(\bar{w})$. Note that every word $v \in 2^\omega$ is definable in $\mathfrak{T}_2$ and, hence, so is every element $u \in T_2^{(n)}$. Consequently, we can replace $\varphi^{\mathcal{I}}(\bar{w})$ by a sentence ψ. By Muchnik's theorem, it follows that we can construct another sentence $\hat{\psi}$ such that

$$\begin{aligned} \mathfrak{T}_2 \models \hat{\psi} \quad &\text{iff} \quad \mathfrak{T}_2^{(n)} \models \psi \\ &\text{iff} \quad \mathfrak{T}_2^{(n)} \models \varphi^{\mathcal{I}}(\bar{w}) \quad \text{iff} \quad \mathfrak{M} \models \varphi(\mathcal{I}(\bar{w})). \end{aligned}$$ □

Theorem 5.1.5. *Let $\mathfrak{M} \in \mathcal{C}$. The* GSO*-theory of $\mathfrak{M}$ is decidable if and only if $\mathfrak{M}$ is of finite tree width. The same holds for* GSO + C*.*

Proof. According to Lemma 1.2.12 and Theorem 1.2.13, if $\mathfrak{M}$ is of finite tree width then it is uniformly sparse and GSO + C collapses to MSO + C which is decidable. Conversely, if $\mathfrak{M}$ is of infinite tree width then its GSO-theory is undecidable by Theorem 1.2.9. □

Originally, Caucal [16] defined his hierarchy for transition systems only. At the lowest level he started with the class of all finite transition systems and, to obtain the next level, he constructed the unfoldings of the systems in the current one and then applied an inverse rational substitution.

Carayol and Wöhrle [13] (see also Carayol and Colcombet [12]) have shown that the hierarchy constructed in this way is strict and that each level is closed under injective MSO-interpretations. Moreover, the n-th level contains the $(n-1)$-th iteration of the binary tree and that every transition systems in level n can be obtained from this iteration by an injective MSO-interpretation. It follows that the hierarchy obtained in this way equals the one we get when we restrict $\mathcal{C}_n$, $n < \omega$, to transition systems.

Recent results of Caucal et. al. indicate that, analogous to the class of prefix-recognisable graphs, one can characterise the levels of the Caucal hierarchy by suitable models of pushdown automata, term rewriting systems, or systems of equations. For example, one can encode a system S of VR-equations $x_0 = t_0, \dots, x_n = t_n$ as a graph by taking the disjoint union of the terms t_i and replacing every leaf labelled by an unknown x_i by an edge to the root of the term t_i. That way, the least solution of S is the term T obtained by unravelling S. In particular, if $S \in \mathcal{C}_n$ then $T \in \mathcal{C}_{n+1}$ and, by the result of Carayol and Wöhrle, $\mathrm{val}(T) \in \mathcal{C}_{n+1}$. In fact, also the converse is true. Carayol and Colcombet [12] have shown that each transition system in $\mathcal{C}_{n+1}$ can be described by a VR-term which is the least solution of a system of equations in $\mathcal{C}_n$.

Finally, let us mention that the Caucal hierarchy does not contain all structures with decidable MSO-theory.

Example (Carayol and Wöhlre [13]). Let $\mathfrak{T} = (T, \preceq)$ be the tree with universe

$$T := \{\, 0^n 1^k \mid k < \beth_n(1) \,\}.$$

The decidability of the MSO-theory of $\mathfrak{T}$ can be obtained by a simple application of the composition method or by automata-theoretic arguments. On the other hand, one can show that $\mathfrak{T} \notin \mathcal{C}$.

5.2 Tree-interpretable structures

We turn to an investigation of $\mathcal{C}_1$, the lowest level of the Caucal hierarchy. This is a very natural class which can be defined in several different ways.

tree interpretable

Definition 5.2.1. A structure $\mathfrak{M}$ of finite signature is called *tree interpretable* if $\mathfrak{M} \leq_{\mathrm{MSO}} \mathfrak{T}_2$.

We will show in Proposition 5.2.6 that the class of tree-interpretable structures coincides with $\mathcal{C}_1$.

All tree-interpretable graphs are of finite clique width. On the other hand, their tree width can be unbounded as the example of the infinite clique $K_{\aleph_0}$ shows.

A result of Courcelle [25] which was extended to tree-interpretable graphs by Barthelmann [4] shows that being of finite tree width imposes a strong restriction on the structure of a tree-interpretable graph. It directly follows from the results of Section 2.4.

Proposition 5.2.2. *Let $\mathfrak{M}$ be a tree-interpretable structure. The following statements are equivalent:*

(1) *$\mathfrak{M}$ is HR-equational.*

(2) *$\mathfrak{M}$ has finite tree width.*

(3) *The Gaifman graph $\mathcal{G}(\mathfrak{M})$ is uniformly sparse.*

(4) *$\mathcal{G}(\mathfrak{M})$ does not contain the subgraph $K_{n,n}$ for some $n < \aleph_0$.*

Recently, this result has been extended to all levels of the Caucal hierarchy by Colcombet [17].

Although the definition of tree-interpretable structures by interpretations is quite elegant, in actual proofs it is most of the time easier to work with a more concrete characterisation in terms of languages.

Employing the correspondence between MSO-formulae and tree automata we can generalise the characterisation of the class of prefix-recognisable graphs by relations of the form $W(U \times V)$ to arbitrary relational structures.

branching structure

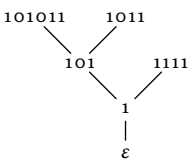
Definition 5.2.3. The *branching structure* of words $x_0, \dots, x_{n-1} \in 2^{<\omega}$ is the partial order $(X, \preceq, x_0, \dots, x_{n-1})$ with universe

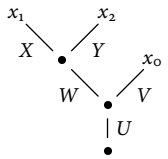
$$X := \{\varepsilon\} \cup \{\, x_i \sqcap x_j \mid i, j < n \,\}.$$

branching points

The elements of X are called *branching points*.

Example. The branching structure of 1111, 1011, 101011 is depicted to the left.





Note that for a fixed number of words there are only finitely many non-isomorphic branching structures.

Proposition 5.2.4. *An n-ary relation $R \subseteq (2^{<\omega})^n$ is* MSO*-definable in $\mathfrak{T}_2$ if and only if R is a finite union of relations R_i of the following form:*

(1) *All tuples $\bar{x} \in R_i$ have the same branching structure (up to isomorphism).*

(2) *For all pairs of adjacent branching points u, v, there exist regular languages $W_{u,v}$ such that $\bar{x} \in R_i$ if and only if, for each such pair u, v, the word $u^{-1}v$ belongs to $W_{u,v}$.*

Proof. ($\Leftarrow$) Clearly, every relation of this form is MSO-definable.

($\Rightarrow$) We show that, if R is MSO-definable, then the labels of paths between branching points are regular. For simplicity we assume that the relation $R \subseteq 2^{<\omega} \times 2^{<\omega}$ is binary. Let $\mathcal{A} = (Q, \wp[2], \Delta, q_0, \Omega)$ be the tree automaton associated to the MSO-definition of R in $\mathfrak{T}_2$.

For $q \in Q$, let $S_q \subseteq 2^{<\omega} \times 2^{<\omega} \times 2^{<\omega}$ be the set of all triples (w, u, v) with $u \sqcap v = \varepsilon$ such that there exists an accepting run ρ of $\mathcal{A}$ for the tree $T_{\{wu\},\{wu\}}$ with $\rho(w) = q$ (recall Definition 1.4.2). Let

$$\begin{aligned}
S_q^{\varnothing} &:= S_q \cap \{(w, \varepsilon, \varepsilon)\}, \\
S_q^{0} &:= \{\, (w, u, \varepsilon) \in S_q \mid u \neq \varepsilon \,\}, \\
S_q^{1} &:= \{\, (w, \varepsilon, v) \in S_q \mid v \neq \varepsilon \,\}, \\
S_q^{0,1} &:= \{\, (w, u, v) \in S_q \mid u, v \neq \varepsilon \,\}.
\end{aligned}$$

Let W_q^c, U_q^c, and V_q^c be the projections of S_q^c onto the respective coordinates. Then

$$R = \bigcup_{\substack{q \in Q \\ c \subseteq [2]}} S_q^c = \bigcup_{\substack{q \in Q \\ c \subseteq [2]}} W_q^c (U_q^c \times V_q^c).$$

It remains to prove that W_q^c, U_q^c, and V_q^c are regular. Let $\mathrm{occ}(t)$ denote the set of labels which occur at some vertex of the tree t. We classify the states of $\mathcal{A}$ according to the set of labels which can appear in trees that are accepted from this state.

$$\begin{aligned}
Q_{\varnothing} &:= \big\{ q \in Q \mid \mathrm{occ}(t) = \{\varnothing\} \text{ for all trees } t \text{ accepted from } q \big\} \\
Q_0 &:= \big\{ q \in Q \mid \mathrm{occ}(t) = \{\varnothing, \{0\}\} \text{ for all } t \text{ accepted from } q \big\} \\
Q_1 &:= \big\{ q \in Q \mid \mathrm{occ}(t) = \{\varnothing, \{1\}\} \text{ for all } t \text{ accepted from } q \big\} \\
Q_{0,1} &:= \big\{ q \in Q \mid \mathrm{occ}(t) = \{\varnothing, \{0,1\}\} \text{ or } \{\varnothing, \{0\}, \{1\}\} \\
&\qquad\qquad \text{for all trees } t \text{ accepted from } q \big\}
\end{aligned}$$

If $W_q^c = \varnothing$ then we are done. Otherwise, W_q^c is recognised by the automaton $\big(Q,\ [2],\ \Delta(W_q^c),\ q_0,\ \{q\}\big)$ with transition relation

$$\begin{aligned}
&\{\, (p, 0, p') \mid (p, \varnothing, p', p_0) \in \Delta,\ p, p' \in Q_{0,1},\ p_0 \in Q_{\varnothing} \,\} \\
\cup\ &\{\, (p, 1, p') \mid (p, \varnothing, p_0, p') \in \Delta,\ p, p' \in Q_{0,1},\ p_0 \in Q_{\varnothing} \,\}.
\end{aligned}$$

We may assume that $U_q^c \neq \varnothing$ and $U_q^c \neq \{\varepsilon\}$. Hence, $0 \in c$. Let q_s be a new state. The automaton $\big(Q \cup \{q_s\},\ [2],\ \Delta(U_q^c),\ q_s,\ F\big)$ recognises U_q^c where

$$F := \{\, p \in Q \mid (p, \{0\}, p_0, p_0') \in \Delta,\ p_0, p_0' \in Q_{\varnothing} \,\}$$

and the transition relation is

$$\begin{aligned}
&\{\, (q_s, 0, p) \mid (q, c \setminus \{0\}, p, p') \in \Delta,\ p \in Q_0,\ p' \in Q_{\overline{c}} \,\} \\
\cup\, &\{\, (q_s, 1, p) \mid (q, c \setminus \{0\}, p', p) \in \Delta,\ p \in Q_0,\ p' \in Q_{\overline{c}} \,\} \\
\cup\, &\{\, (p, 0, p') \mid (p, \varnothing, p', p_0) \in \Delta,\ p, p' \in Q_0,\ p_0 \in Q_\varnothing \,\} \\
\cup\, &\{\, (p, 1, p') \mid (p, \varnothing, p_0, p') \in \Delta,\ p, p' \in Q_0,\ p_0 \in Q_\varnothing \,\},
\end{aligned}$$

where $\overline{c} := \wp[2] \setminus c$. The automaton for V_q^c can be defined analogously. □

Example. For the branching structure in the previous example, a relation would be defined by five regular languages U, V, W, X, and Y with $R = U(V \times W(X \times Y))$.

Definition 5.2.5. Let $\mathfrak{M}$ be a tree-interpretable structure. Fixing an interpretation we can assume that the universe $M \subseteq \Sigma^{<\omega}$ is regular and each relation R is specified by regular languages as in the preceding proposition. The *syntactic congruence* $\sim$ of $\mathfrak{M}$ (w.r.t. this interpretation) is the intersection of the syntactic congruences of all these languages. We denote the index of $\sim$ by I.

syntactic congruence
$\sim$
I

If some elements of a tree-interpretable structure are encoded by several words it becomes difficult to apply pumping arguments since the words obtained by pumping may encode the same element. Fortunately, for each tree-interpretable structure $\mathfrak{M}$, we can choose an interpretation where this does not happen.

Proposition 5.2.6. *If $\mathfrak{M} \leq_{\mathrm{MSO}} \mathfrak{T}_2$ then there is an injective interpretation $\mathcal{I} : \mathfrak{M} \leq_{\mathrm{MSO}} \mathfrak{T}_2$.*

Proof. We prove that, for all regular languages $M \subseteq 2^{<\omega}$ and every MSO-definable equivalence relation $E \subseteq M \times M$, there is a regular language $M_0 \subseteq M$ which contains exactly one element of each E-class. Then the desired interpretation is obtained by replacing the formula defining the universe M of $\mathfrak{M}$ by the one defining M_0.

Denote the E-class of x by $[x]$, define $p_{[x]} := \bigsqcap [x]$, and set $s_x := (p_{[x]})^{-1}x$. Finally, let I be the number of states of the automaton associated with E. We claim that each class $[x]$ contains an element of length less than $|p_{[x]}| + I$. Thus, one can define

$$M_0 := \{\, x \in M \mid s_x \leq_{\mathrm{ll}} s_y \text{ for all } y \in [x] \,\}$$

where the length lexicographic ordering $\leq_{\mathrm{ll}}$ is definable since the length of the words is bounded so that we only need to consider finitely many cases.

To prove the claim choose $x_0, x_1 \in [x]$ such that $x_0 \sqcap x_1 = p_{[x]}$. Since $(x_0, x_1) \in E$ there are regular languages U, V, and W such that

$W(U \times V) \subseteq E$ and $x_0 = wu$, $x_1 = wv$ for words $u \in U$, $v \in V$, and $w \in W$ with $w \preceq p_{[x]}$. If $|wu| \geq |p_{[x]}| + I$ then, by a pumping argument, there exists some $u' \in U$ such that $|p_{[x]}| \leq |wu'| < |p_{[x]}| + I$. Hence, $(wu', x_1) \in E$ is an element of the desired length. □

This result allows us to identify the elements a of a tree-interpretable structure $\mathcal{I} : \mathfrak{M} \leq_{\mathrm{MSO}} \mathfrak{T}_2$ with the unique word $\mathcal{I}^{-1}(a)$ encoding them. We will do so tacitly in the remainder of the thesis.

We conclude this section by comparing the class of tree-interpretable structures to the class of automatic structures which was introduced by Khoussainov and Nerode in [49]. In the following proofs we will use the characterisation of automatic and unary-automatic structures in terms of FO-interpretations given in Blumensath [6].

Proposition 5.2.7. *The class of tree-interpretable structures is strictly contained in the class of automatic structures.*

Proof. Strictness follows from the fact that model checking for MSO is decidable for tree-interpretable structures but not for all automatic ones.

We have to show that $\mathfrak{M} \leq_{\mathrm{MSO}} \mathfrak{T}_2$ implies $\mathfrak{M} \leq_{\mathrm{FO}} (\mathfrak{T}_2, \preceq, \mathrm{el})$ where el is the equal-length predicate. Using the characterisation from Proposition 5.2.4 it is sufficient to construct an FO-definition of a relation R that is defined by a certain branching structure and regular languages W_i as described above. By a simple modification of the usual translation of automata to FO-formulae on $(\mathfrak{T}_2, \preceq, \mathrm{el})$ (see e.g. [10, 6]) one obtains, for each W_i, a formula $\varphi_{W_i}(x, y)$ which states that $x \preceq y$ and the path from x to y is labelled by a word in W_i. Obviously, there also is a formula $\beta(\bar{x}, \bar{y})$ which holds iff $\bar{x}$ has a given branching structure with universe $\bar{x} \cup \bar{y}$. Thus, one can define R by

$$\psi(\bar{x}) := \exists \bar{y} \Big(\beta(\bar{x}, \bar{y}) \wedge \bigwedge_i \varphi_{W_i}(z_i, z_i') \Big)$$

where $z_i,\ z_i' \in \{x_0, \dots, y_0, \dots\}$ are the branching points corresponding to W_i. □

Proposition 5.2.8. *The class of unary-automatic structures is strictly contained in the class of tree-interpretable structures.*

Proof. Since $(\omega, s, \leq) \leq_{\mathrm{MSO}} \mathfrak{T}_2$, by Corollary 7.5 of [6], it is sufficient to construct an interpretation $(\mathbb{N}, \leq, (n \mid x)_n) \leq_{\mathrm{MSO}} (\omega, s, \leq)$. To do so we only need to define the divisibility predicates.

$$\varphi_{n|x}(x) := \forall X (X\varepsilon \wedge \forall y (Xy \to Xs^n y) \to Xx) .$$

For strictness, note that $\mathfrak{T}_2$ is tree interpretable but not unary automatic. □

5.3 MSO-FUNCTORS

In this section we will investigate under which MSO-functors the class of tree-interpretable structures is closed. Obviously, it is closed under MSO-interpretations and, by Lemma 1.3.6, also under products by finite structures. Further, one can show by pumping arguments that it is not closed under iterations, i.e., the Caucal hierarchy does not collapse to its first level. In the following we will show that the class of tree-interpretable structures is also closed under a special case of generalised sums which we call a *substitution*.

substitution
$\mathfrak{I}[\lambda : \mathfrak{M}_0, \dots, \mathfrak{M}_{n-1}]$

Definition 5.3.1. Let $\mathfrak{M}_0, \dots, \mathfrak{M}_{n-1}$ be τ-structures, and $\mathfrak{I}$ a structure of signature σ. Let $\lambda : I \to [n]$ be a function partitioning $\mathfrak{I}$ into sets $P_k := \lambda^{-1}(k)$ for $k < n$. The *substitution* $\mathfrak{I}[\lambda : \mathfrak{M}_0, \dots, \mathfrak{M}_{n-1}]$ is the structure $\mathfrak{N}$ with universe

$$N := \bigcup_{i \in I} M_{\lambda(i)} \times \{i\} .$$

and the following relations:

$$\begin{aligned} \mathrm{eq} &:= \left\{ \big((a,i),(b,j)\big) \in N \times N \;\middle|\; i = j \right\}, \\ P_k &:= \{\, (a,i) \in N \mid \lambda(i) = k \,\} \qquad\qquad \text{for } k < n . \end{aligned}$$

For each relation $R \in \tau$ of arity r, $\mathfrak{N}$ has a relation

$$R^1 := \left\{ \big((a_0,i),\dots,(a_{r-1},i)\big) \in N^r \;\middle|\; \bar{a} \in R^{\mathfrak{M}_{\lambda(i)}} \right\},$$

and for each relation $R \in \sigma$ of arity r, there is a relation

$$R^2 := \left\{ \big((a_0,i_0),\dots,(a_{r-1},i_{r-1})\big) \in N^r \;\middle|\; \bar{\imath} \in R^{\mathfrak{I}} \right\}.$$

Theorem 5.3.2. *Let $\mathfrak{M}_0, \dots, \mathfrak{M}_{n-1}$, and $(\mathfrak{I}, \bar{S}, P_0, \dots, P_{n-1})$ be tree-interpretable structures. Then so is $\mathfrak{I}[\lambda : \mathfrak{M}_0, \dots, \mathfrak{M}_{n-1}]$ where λ is the function which induces the partition $P_0, \dots, P_{n-1}$.*

Proof. Let # be a new symbol. We encode the element (x, i) by the word $i\#x$. Then the universe of the substitution becomes

$$M := \bigcup_{i<n} P_i \# M_i$$

and the relations are

$$\begin{aligned} \mathrm{eq} &:= \{\, (w\#u, w\#v) \mid w \in I,\ u, v \in M_i \text{ for some } i < n \,\}, \\ P_i &:= P_i \# M_i , \\ R^1 &:= \{\, (w\#u_0, \dots, w\#u_s) \mid w \in P_i,\ \bar{u} \in R^{\mathfrak{M}_i},\ i < n \,\}, \\ R^2 &:= \{\, (w_0\#u_0, \dots, w_s\#u_s) \mid \bar{w} \in R^{\mathfrak{I}},\ u_0, \dots, u_s \in \textstyle\bigcup_i M_i \,\} . \end{aligned}$$

□

5.4 Combinatorial lemmas

In the remainder of this chapter we turn to the investigation of algebraic properties of tree-interpretable structures. Throughout we will use the following notation. $\sim$ is the syntactic congruence of the given structure (w.r.t. a fixed interpretation), and I is its index. Recall that x/k is the prefix of x of length $|x| - k$.

I

We start with two combinatorial lemmas. The first one allows us to obtain information about the words encoding an element.

Lemma 5.4.1. *Let $\mathfrak{M}$ be a tree-interpretable structure and $\varphi(x, y)$ an $\mathrm{MSO}(\exists^\kappa)$-formula such that, for every $a \in M$, there are only finitely many elements $b \in M$ with $\mathfrak{M} \models \varphi(a, b)$. There exists a constant $k < \omega$ such that $\varphi(a, b)$ implies $b/k \prec a$. In particular, $|\varphi(a, M)| \in \mathcal{O}(|a|)$.*

Proof. Let $\sim$ be the syntactic congruence of the expansion $(\mathfrak{M}, \varphi^{\mathfrak{M}})$ and let $k := I$ be its index. Suppose, for a contradiction, that there are element $a, b \in M$ such that $\mathfrak{M} \models \varphi(a, b)$ and $|b| \geq |a \sqcap b| + I$. Then we can find words $b/k \preceq x \prec y \preceq b$ such that $x \sim y$. Let $u := x^{-1}y$ and $z := y^{-1}b$. Then $(a, b) \in \varphi^{\mathfrak{M}}$ implies that $(a, xu^i z) \in \varphi^{\mathfrak{M}}$ for all $i < \omega$. Contradiction. □

The other lemma states that the class of tree-interpretable structures is closed under expansion by Skolem functions.

Lemma 5.4.2. *Let $\mathfrak{M}$ be tree interpretable and $\varphi(x, \bar{y}) \in \mathrm{MSO}$. There exists a function $f : M^n \to M$ satisfying*

$$\mathfrak{M} \models \exists x \varphi(x, \bar{a}) \to \varphi(f(\bar{a}), \bar{a}) \qquad \text{for all } \bar{a} \in M^n$$

such that $(\mathfrak{M}, f)$ is tree interpretable.

Furthermore, we can choose f such that, for all sequences $(\bar{a}^i)_{i<\omega}$ satisfying

$$\mathfrak{M} \models \forall x \big(\varphi(x, \bar{a}^i) \to \varphi(x, \bar{a}^{i+1}) \big),$$

there is some $k < \omega$ such that $f(\bar{a}^i) = f(\bar{a}^k)$ for all $i \geq k$.

Proof. W.l.o.g. assume that $\varphi(x, \bar{y}) := Rx\bar{y}$ for some relation R. Let $\bar{a} \in M^n$. If there is some element $b \in M$ with $(b, \bar{a}) \in R$, then we can find such an element of length

$$|b| < |b \sqcap a_i| + I$$

for some $i < n$ since, otherwise, fixing some b of minimal length there would be words $b/k \preceq x \prec y \preceq b$ with $x \sim y$ and we could remove the factor between x and y to obtain a shorter word b' with $(b', \bar{a}) \in R$.

Let $\bar{a} \in M^n$ and $b \in M$. We define a linear order $\sqsubseteq_b$ on $\bar{a}$ by

$$a_i \sqsubseteq_b a_k \;:\text{iff}\; b \sqcap a_i \prec b \sqcap a_k \text{ or both are equal and } a_i \leq_{\text{lex}} a_k\,.$$

Let $h(b, \bar{a})$ be the $\sqsubseteq_b$-maximal element of $\{a_0, \dots, a_{n-1}\}$.

For $\bar{a} \in M^n$, let $f(\bar{a})$ be the element b such that, in the following order,

(1) $h(b, \bar{a})$ is $\leq_{\text{lex}}$-maximal,

(2) $b \sqcap h(b, \bar{a})$ is $\preceq$-minimal,

(3) $(b \sqcap h(b, \bar{a}))^{-1} b$ is $\leq_{\text{ll}}$-minimal.

Since this function is MSO-definable in $\mathfrak{T}_2$, it follows that $(\mathfrak{M}, f)$ is tree interpretable.

Suppose that there exists a sequence $(\bar{a}^i)_{i<\omega}$ of parameters such that $(b, \bar{a}^i) \in R$ implies $(b, \bar{a}^{i+1}) \in R$ and the sequence $b^i := f(\bar{a}^i)$ is not eventually constant. Note that $b^i = b^k$ implies $b^l = b^k$ for all $i \leq l \leq k$. Hence, by considering an appropriate subsequence of $(\bar{a}^i)_{i<\omega}$, we may assume that all the b^i are different and, since for each b there are only finitely many b' with $b'/I \preceq b/I$, that

$$b^i/I \not\preceq b^j/I \qquad \text{for all } j < i < \omega\,.$$

Since $b^i/I \preceq h(b^i, \bar{a}^i)$ it follows that $h(b^j, \bar{a}^i) \neq h(b^i, \bar{a}^i)$ and, hence,

$$h(b^j, \bar{a}^i) \sqcap h(b^i, \bar{a}^i) \preceq b^j \qquad \text{for } j < i < \omega\,.$$

By induction on l we construct infinite sets $J_l \subseteq \omega$, indices $j_l \in J_l$, words $w_l \in \Sigma^{<\omega}$, and symbols $c_l \neq d_l$ satisfying

$$J_l \supseteq J_{l+1}, \qquad j_l < j_{l+1}, \qquad w_l \prec w_{l+1}\,,$$

such that

$$w_l c_l \preceq h(b^i, \bar{a}^i) \quad \text{and} \quad w_l d_l \preceq h(b^{j_l}, \bar{a}^i) \quad \text{for all } i \in J_l,\ i > j_l\,.$$

These conditions imply that $h(b^{j_k}, \bar{a}^i) \neq h(b^{j_{k'}}, \bar{a}^i)$ for all $k, k' \leq n$, $k \neq k'$ and every $i \in J_n$ such that $i > j_n$. As h can take only n different values this yields the desired contradiction. We start the construction by setting $J_0 := \omega$, $j_0 := 0$, and c_0 is the first letter of b^0 while $d_0 \neq c_0$ is arbitrary.

Given J_l, j_l, c_l, and d_l, we construct the next stage as follows. Since

$$h(b^{j_l}, \bar{a}^i) \sqcap h(b^i, \bar{a}^i) \preceq b^{j_l}$$

for all $i \in J_l$, $i > j_l$, there is some word w_{l+1} with

$$h(b^{j_l}, \bar{a}^i) \sqcap h(b^i, \bar{a}^i) = w_{l+1}$$

for infinitely many $i \in J_l$, $i > j_l$. Thus, we can find symbols c_{l+1}, d_{l+1} such that

$$w_{l+1}c_{l+1} \preceq h(b^i, \bar{a}^i) \qquad \text{and} \qquad w_{l+1}d_{l+1} \preceq h(b^{j_l}, \bar{a}^i)$$

for infinitely many $i \in J_l$, $i > j_l$. Finally, let J_{l+1} be the set of these indices and j_{l+1} the first element of J_{l+1}. □

5.5 Paths in tree-interpretable transition systems

In this section we investigate the set of paths in a fixed tree-interpretable transition system $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, \bar{P})$. By replacing each edge relation $E_\lambda = \bigcup_i W_i(U_i \times V_i)$ by several relations $E^i_\lambda := W_i(U_i \times V_i)$ we may assume that $E_\lambda = W_\lambda(U_\lambda \times V_\lambda)$ for regular languages U_λ, V_λ, $W_\lambda \subseteq \Sigma^{<\omega}$. We also add the relation $E_{\lambda^-} := (E_\lambda)^{-1}$ for each edge relation E_λ. Note that these operations do not affect the syntactic congruence $\sim$.

Remark. By Proposition 5.2.4, we could choose U_λ, V_λ, and W_λ such that

$$\begin{aligned} & (x, y) \in W_\lambda(U_\lambda \times V_\lambda) \\ \text{iff} \quad & x \sqcap y \in W_\lambda,\ (x \sqcap y)^{-1}x \in U_\lambda,\ \text{and } (x \sqcap y)^{-1}y \in V_\lambda\,. \end{aligned}$$

Definition 5.5.1. The *base point* of an edge $(a, b) \in W(U \times V)$ is the longest word w contained in W such that $w^{-1}a \in U$ and $w^{-1}b \in V$. The *spine* of a path is the sequence of the base points of its edges.

base point

spine

Definition 5.5.2.

(a) A *path above c* is a path $a_0, \ldots, a_n$ such that $c \preceq a_i$ for all i. *path above c*

(b) A path $a_0, \ldots, a_n$ is *bounded by l* if $|a_i| \leq l$ for all i. *bounded path*

(c) A sequence $a_0, \ldots, a_n$ is *k-increasing* if $|a_j| \geq |a_i| - k$ for all $i < j$. *k-increasing path*

(d) A path $a_0, \ldots, a_n$ with spine $w_0, \ldots, w_{n-1}$ is called *k-normal* if the path and its spine are k-increasing and $a_i/k \preceq a_j$ for all $i \leq j$. *k-normal path*

The aim of this section is to show that every vertex can be reached by a k-normal path. The importance of such paths stems from the fact that, by following a k-normal path to a vertex x, one can compute certain information about x like its $\sim$-class. We start with some immediate observations.

Lemma 5.5.3. *Let $a_0, \ldots, a_n$ be a path with spine $w_0, \ldots, w_{n-1}$.*

(a) *For all $i < n-1$, either $w_i \preceq w_{i+1}$ or $w_i \succeq w_{i+1}$.*

(b) *If $w_0, \dots, w_{n-1}$ is k-increasing then $w_i/k \preceq w_j$ for all $i < j$.*

The next two lemmas can be used to find a k-normal path once we have shown how to obtain a path with k-increasing spine.

Lemma 5.5.4. *Let a_0, a_1, a_2 be a path with spine w_0, w_1. There exists a vertex a_1' of length*

$$|a_1'| < \max\{|w_0|, |w_1|\} + I$$

such that a_0, a_1', a_2 is a path with spine w_0, w_1.

Proof. W.l.o.g. we may assume that $w_1 \preceq w_0$. Suppose that $|a_1| \geq |w_0| + I$. Since $a_1 \succeq w_0$ there are prefixes $w_0 \preceq x \prec y \preceq a_1$ such that $(w_0)^{-1}x \sim (w_0)^{-1}y$. Setting $a_1' := x(y^{-1}a_1)$ we obtain a path a_0, a_1', a_2 with $|a_1'| < |a_1|$. Iterating this step sufficiently many times we obtain a vertex of the desired length. □

Lemma 5.5.5. *Let $w_0, \dots, w_{n-1}$ be a k-increasing spine of some path from x to y. There exists a path $a_0, \dots, a_n$ with the same spine from x to y such that*

$$a_i/(k + I - 1) \preceq w_j \qquad \text{for all } 0 < i \leq j < n\,.$$

Proof. By the preceding lemma we can replace each a_i for $0 < i < n$ by some a_i' with $|a_i'| < \max\{|w_{i-1}|, |w_i|\} + I$. Since $w_{i-1}/k \preceq w_i$ it follows that $a_i'/(k + I - 1) \preceq w_i \preceq w_j$ for all $j \geq i$. □

In the proofs below we frequently need to remove parts of a path and glue the remaining pieces together. The following construction is the main tool in this process.

Definition 5.5.6. Let $a_0, \dots, a_n$ be a path with spine $w_0, \dots, w_{n-1}$. Let x and y be words such that $x \preceq w_i$ for all $i < n$, that is, there are words $u_0, \dots, u_n, v_0, \dots, v_{n-1}$ such that

$$a_i = xu_i \quad \text{and} \quad w_i = xv_i\,.$$

shifting a path

Shifting the path from x to y yields the sequences $a_0', \dots, a_n'$ and $w_0', \dots, w_{n-1}'$ where

$$a_i' := yu_i \quad \text{and} \quad w_i' := yv_i\,.$$

Lemma 5.5.7. *Using the same notation as in the preceding definition, $x \sim y$ implies that $a_0', \dots, a_n'$ is a path with spine $w_0', \dots, w_{n-1}'$.*

Proof. Since

$$w_i' \sim w_i, \quad (w_i')^{-1}a_i' = w_i^{-1}a_i, \quad \text{and} \quad (w_i')^{-1}a_{i+1}' = w_i^{-1}a_{i+1}$$

it follows that $(a_i', a_{i+1}') \in E_\lambda$ iff $(a_i, a_{i+1}) \in E_\lambda$. □

Now we are ready to prove the main result needed to obtain k-normal paths.

Proposition 5.5.8. *Let $\mathfrak{M}$ be a tree-interpretable transition system with r binary relations. There is a constant K_0 such that, for all paths $a_0, \dots, a_n$ with spine $w_0, \dots, w_{n-1}$, there exists a path of length $m \leq n$ from a_0 to a_n with spine $w'_0, \dots, w'_m$ where*

$$|w'_i| < \max\{|w_0|, |w_{n-1}|\} + K_0 \qquad \text{for all } i \leq m.$$

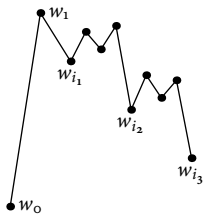
Proof. We proceed in several steps.

Claim 1. *If $|w_i| \geq \max\{|w_0|, |w_{n-1}|\} + I$ for every $0 < i < n-1$, then there exists a path $a'_0, \dots, a'_n$ from a_0 to a_n such that $|a'_i| < |a_i|$ for all $0 < i < n$.*

Since $w_{i-1} \prec w_i \preceq a_i$ the prerequisites imply that $w_0 \preceq w_{n-1} \prec w_i$ for all $0 < i < n-1$. Hence, either $w_0 \preceq w_{n-1}$ or $w_{n-1} \preceq w_0$. W.l.o.g. assume the latter. There exists a word x of length I such that $w_0 x \preceq w_i$ for all $0 < i < n-1$. Since $|x| = I$ there are prefixes $y \prec z \preceq x$ with $y \sim z$. The desired path is obtained by shifting the subpath $a_1, \dots, a_{n-1}$ from $w_0 z$ to $w_0 y$.

By Claim 1 we may assume that for each subpath $a_k, \dots, a_l$ there exists some index $k < i < l$ with $|w_i| < \max\{|w_k|, |w_{l-1}|\} + I$.

Claim 2. *If $|w_1| \geq |w_0| + rI^2$, then there exists a path $a'_0, \dots, a'_m$ from a_0 to a_n with $m < n$.*



Let $w_{i_0}, \dots, w_{i_t}$ be the subsequence of base points w_i such that

$$|w_k| > |w_i| \geq |w_0| \qquad \text{for all } 0 < k < i.$$

By assumption

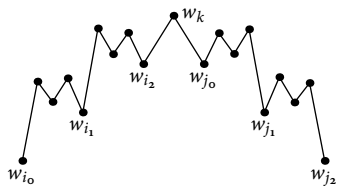
$$|w_{i_{k+1}}| < |w_{i_k}| < |w_{i_{k+1}}| + I$$

for all $k < t-1$. Hence, $t \geq rI$ and there exist indices $k < l$ in $\{i_0, \dots, i_t\}$ such that $w_k \sim w_l$ and $(a_k, a_{k+1}), (a_l, a_{l+1}) \in E_\lambda$ for some λ. Since $|w_k| > |w_l|$ there is some word x with $w_k = w_l x$ and $w_l x \preceq w_i$ for all $0 < i < k$. Let $(a'_i)_i$ be the path obtained from $(a_i)_i$ by shifting the subpath $a_1, \dots, a_k$ from $w_l x$ to w_l and removing the subpath $a_{k+1}, \dots, a_l$.

By Claim 2 we may further assume that $|w_{i+1}| - |w_i| < rI^2$ for all $i < n-1$. Define $K_0 := r^3 I^4 |\Sigma|^{rI^2}$. The third claim concludes the proof.

Claim 3. *There exists a path $a'_0, \dots, a'_m$ from a_0 to a_n with spine $w'_0, \dots, w'_{m-1}$ such that*

$$|w'_i| < \max\{|w_0|, |w_n|\} + r^3 I^4 |\Sigma|^{rI^2}.$$



Fix some base point w_k such that $|w_k|$ is maximal, and consider the subsequences $w_{i_0}, \dots, w_{i_s}$ and $w_{j_0}, \dots, w_{j_t}$ of base points w_i, for $i < k$, and w_j, for $j > k$, such that

$$|w_i| < |w_l| \qquad \text{for all } i < l \leq k,$$
$$|w_j| < |w_l| \qquad \text{for all } k \leq l < j.$$

Assume that $|w_k| \geq \max\{|w_0|, |w_n|\} + r^3 I^4 |\Sigma|^{rI^2}$. By assumption, this implies that

$$s \geq r^3 I^4 |\Sigma|^{rI^2} / (rI^2) = r^2 I^2 |\Sigma|^{rI^2}.$$

For $i \in \{i_0, \dots, i_s\}$ define $f(i) \in \{j_0, \dots, j_t\}$ such that $|w_{f(i)}| \geq |w_i|$ is minimal. We colour each $i \in \{i_0, \dots, i_s\}$ with the tuple

$$\chi(i) := \big([w_i]_\sim,\ [w_{f(i)}]_\sim,\ w_i^{-1} w_{f(i)},\ \lambda,\ \lambda'\big)$$

where λ and λ' are indices with $(a_i, a_{i+1}) \in E_\lambda$ and $(a_{f(i)}, a_{f(i)+1}) \in E_{\lambda'}$. (Note that $w_i \preceq w_l$ for all $i \leq l \leq f(i)$.) Since

$$|w_{f(i)}| < |w_i| + rI^2$$

there are less than $I^2 |\Sigma|^{rI^2} r^2$ different colours. Therefore, there are two indices $i, i' \in \{i_0, \dots, i_s\}$, $i < i'$, with $\chi(i) = \chi(i')$. Let $w_{i'} = w_i x$. Then $w_i x \preceq w_l$ for $i' \leq l < f(i')$ and the desired path is obtained from $a_0, \dots, a_n$ by removing the subpaths $a_{i+1}, \dots, a_{i'}$ and $a_{f(i')+1}, \dots, a_{f(i)}$ and by shifting the subpath $a_{i'+1}, \dots, a_{f(i')}$ from $w_i x$ to w_i. □

Corollary 5.5.9. *Let $\mathfrak{M}$ be a tree-interpretable transition system. All elements a, b in the same component of M are connected by a path bounded by* $\max\{|a|, |b|\} + K_0 + I$.

Proof. Let $a_0, \dots, a_n$ be a path from a to b whose spine $w_0, \dots, w_{n-1}$ satisfies

$$|w'_i| < \max\{|w_0|, |w_{n-1}|\} + K_0 \qquad \text{for all } i < n.$$

Applying Lemma 5.5.4 we obtain a path $a'_0, \dots, a'_n$ from a to b with

$$|a'_i| \leq \max\{|w_{i-1}|, |w_i|\} + I < \max\{|w_0|, |w_{n-1}|\} + K_0 + I \qquad \square$$

With these preparations we are able to prove the existence of k-normal paths.

Proposition 5.5.10. *Let $\mathfrak{M}$ be a tree-interpretable graph. There is a constant K such that each connected component of $\mathfrak{M}$ contains a vertex v, which we call its* root*, such that there are K-normal paths from v to all other vertices of the component.*

K

root

Proof. Let $K := K_0 + I - 1$. Choose v such that $|v|$ is minimal. Let $a_0, \dots, a_n$ be a path from v to some other vertex, and let $w_0, \dots, w_{n-1}$ be its spine. We transform it into a path with K_0-increasing spine as follows. Suppose there are indices $i < j$ such that $|w_j| < |w_i| - K_0$. Let $k < i$ be the greatest index such that $|w_k| < |w_j|$. By Proposition 5.5.8 there is a path $b_0, \dots, b_m$ from a_k to a_j whose spine is bounded by $|w_j| + K_0$. By iterating this operation we obtain a path with K_0-increasing spine. Applying Lemma 5.5.5 we obtain a path $a'_0, \dots, a'_n$ from v to a_n with $a'_i/K \preceq w_j \preceq a'_j, a'_{j+1}$ for all $0 < i \le j < n$. It remains to prove that $a'_0/K = v/K \preceq a'_1/K$. Since $|a'_1| \ge |v|$ it is sufficient to show $v/K \preceq a'_1$. Assume that $|v \sqcap a'_1| < |v| - K$. Then $|a'_1| \ge |v| \ge |v \sqcap a'_1| + K$. Thus, there exists some b with $v \sqcap a'_1 \preceq b \preceq a'_1$ and $|b| < |v \sqcap a'_1| + I < |v|$ such that $(v \sqcap a'_1)^{-1} b \sim (v \sqcap a'_1)^{-1} a'_1$. Therefore, $(v, a'_1) \in E_\lambda$ implies $(v, b) \in E_\lambda$. This is a contradiction since the connected component of v does not contain vertices of length less than $|v|$. □

We conclude this section with some results bounding the length of paths.

Lemma 5.5.11. *Let $k > 1$. If $a_0, \dots, a_n$ is a path with k-increasing spine $w_0, \dots, w_{n-1}$ then its length is bounded by*

$$n < \big(|w_{n-1}| - |w_0| + k + 1\big)|\Sigma|^{k+1}.$$

Proof. By assumption we have $w_i/k \preceq w_j$ for all $i < j$, i.e., there are words $x_{ij} \in \Sigma^{<\omega}$ such that $w_j = (w_i/k)x_{ij}$. Since there are

$$\Delta := \frac{|\Sigma|^{k+1} - 1}{|\Sigma| - 1}$$

words of length at most k, all sets of the form $\{x_{ii}, \dots, x_{im}\}$ with $m \ge i + \Delta$ contain some word x_{ij} of a greater length. It follows that $|w_j| > |w_i|$ and, by induction, there is some $\delta \le \Delta$ such that

$$|w_{n-1-\delta}| \ge |w_0| + \big\lfloor (n-1)/\Delta \big\rfloor.$$

Thus, $$|w_{n-1}| \ge |w_0| + \big\lfloor (n-1)/\Delta \big\rfloor - k$$

and
$$\begin{aligned} n &\le \big(|w_{n-1}| - |w_0| + k + 1\big)\Delta \\ &= \big(|w_{n-1}| - |w_0| + k + 1\big)\frac{|\Sigma|^{k+1} - 1}{|\Sigma| - 1} \\ &< \big(|w_{n-1}| - |w_0| + k + 1\big)|\Sigma|^{k+1}. \end{aligned}$$ □

Proposition 5.5.12. *Let $\mathfrak{M}$ be a tree-interpretable transition system. Every pair of vertices a, b in the same connected component of $\mathfrak{M}$ is connected by a path of length less than*

$$\big(|a| + |b| + 2K + 2\big)|\Sigma|^{K+1}.$$

Proof. By Proposition 5.5.10, there are K-normal paths from some vertex v to a and to b. Their concatenation yields a path from a to b whose length is bounded by

$$(|a| + K + 1)|\Sigma|^{K+1} + (|b| + K + 1)|\Sigma|^{K+1}$$

according to Lemma 5.5.11. □

Lemma 5.5.13. *Let $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, P)$ be a tree-interpretable transition system and $x \in M$. If there exists a k-normal path from x to some element in P, then we can find such a path of length less than $(I + 1)|\Sigma|^{k+1}$.*

Proof. Fix some k-normal path $a_0, \dots, a_n$ with spine $w_0, \dots, w_{n-1}$ starting in $a_0 = x$, ending in $a_n \in P$, and such that n is minimal. Let $w_{i_0}, \dots, w_{i_s}$ be the subsequence of those base points w_i such that $|w_j| > |w_i|$ for all $j > i$. By (the proof of) Lemma 5.5.11, it follows that $i_{l+1} - i_l < |\Sigma|^{k+1}$.

We claim that $s < I$. Otherwise, there would be indices $j < j'$ in $\{i_0, \dots, i_s\}$ such that $w_j \sim w_{j'}$, and we could obtain a shorter path by deleting the subsequence $a_{j+1}, \dots, a_{j'}$ and shifting the path $a_{j'+1}, \dots, a_n$ from $w_{j'}$ to w_j. This new path ends in some $a'_n \sim a_n$ that is also in P. It follows that $n < (s + 2)|\Sigma|^{k+1} \leq (I + 1)|\Sigma|^{k+1}$. □

Proposition 5.5.14. *Let $\mathfrak{M} = (M, (E_\lambda)_{\lambda\in\Lambda}, P)$ be a tree-interpretable transition system. There is a constant L such that, for every predicate P and all vertices $x, y \in P$ in the same connected component of $\mathfrak{M}$, there exists a path from x to y every subpath of which of length L contains some element of P.*

Proof. Fix some connected component of $\mathfrak{M}$. By Proposition 5.5.10, in contains some vertex v such that there are K-normal paths from v to every other vertex of the component. It is sufficient to construct paths with the desired property from v to x and from v to y.

Let $a_0, \dots, a_n$ be a K-normal path from v to x or y with spine $w_0, \dots, w_{n-1}$. By the preceding lemma, there exist K-normal paths of length less than $(I + 1)|\Sigma|^{K+1}$ from a_i to some element $z_i \in P$, for every $i < n$. If we insert these paths from a_i to z_i and back again into the original path, we obtain a path where every subpath of length $L := 2(I + 1)|\Sigma|^{K+1}$ contains a vertex in P. □

5.6 Substructures and back-and-forth equivalence

As in the previous section let $\sim$ be the intersection of the syntactic congruences of all languages appearing in the presentation of a tree-

interpretable structure.

Definition 5.6.1. Let $\mathfrak{M}$ be a structure and $n < \omega$. For sequences $\bar{a}, \bar{b} \in M^\alpha$, not necessarily finite, we define the *back-and-forth equivalence* $\bar{a} \cong_n \bar{b}$ by induction on n. $\bar{a} \cong_0 \bar{b}$ holds if the map $a_i \mapsto b_i$ is a partial isomorphism, and $\bar{a} \cong_{n+1} \bar{b}$ holds if

- for every c there exists some d such that $\bar{a}c \cong_n \bar{b}d$, and
- for every d there exists some c such that $\bar{a}c \cong_n \bar{b}d$.

back-and-forth equivalence $\bar{a} \cong_n \bar{b}$

For sets $A, B \subseteq M$, we write $A \cong_n B$ if there are enumerations $\bar{a}$ and $\bar{b}$ of A and B, respectively, such that $\bar{a} \cong_n \bar{b}$.

We start by deriving a sufficient condition for two tuples to be $\cong_n$-equivalent.

Definition 5.6.2. Let $n < \omega$. The *n-type* $\tau_n(w)$ of a word $w \in \Sigma^{<\omega}$ is defined by

$$\begin{aligned} \tau_0(w) &:= [w]_\sim, \\ \tau_{n+1}(w) &:= \{ (\tau_n(u), \tau_n(v)) \mid w = uv \}. \end{aligned}$$

n-type $\tau_n(w)$

Let $\sim_n \subseteq \Sigma^{<\omega} \times \Sigma^{<\omega}$ be the kernel of τ_n, i.e., $u \sim_n v$ iff $\tau_n(u) = \tau_n(v)$.

$u \sim_n v$

Lemma 5.6.3. *Let $\mathfrak{M}$ be a tree-interpretable structure and let $\bar{a}, \bar{b} \in M^n$ be tuples of the form $a_i = ux_i$ and $b_i = vx_i$ for $i < n$. If $u \sim v$ then the map $\bar{a} \mapsto \bar{b}$ is a partial isomorphism.*

Proof. Suppose that $\bar{a} \in R$ for some relation R. Note that the branching structures of $\bar{a}$ and $\bar{b}$ are isomorphic. To each edge e of this branching structure we can associate a regular language W_e as in Proposition 5.2.4. W.l.o.g. we may assume that, if e and e' are edges with a common first vertex and $w \in W_e$, $w' \in W_{e'}$ words, then $w \sqcap w' = \varepsilon$. Let $e = (\varepsilon, \bigsqcap \bar{a})$ be the edge at the root. There exists a word $w \in W_e$ such that $u \preceq w$, i.e., $w = uy$ and $b_i = vyz_i$ for some $y, z_i \in \Sigma^{<\omega}$. Since $vy \sim uy = w$ this implies that $vy \in W_e$ and, hence, $\bar{b} \in R$. □

Corollary 5.6.4. *Let $\mathfrak{M}$ be a tree-interpretable structure and $u, v \in \Sigma^{<\omega}$. If $u \sim v$ then the substructures induced by $u\Sigma^{<\omega}$ and $v\Sigma^{<\omega}$ are isomorphic.*

Proposition 5.6.5. *Let $\mathfrak{M}$ be a tree-interpretable structure. Let $\bar{a}$ be an enumeration of $M \cap u\Sigma^{<\omega}$ and $\bar{b}$ the corresponding one of $M \cap v\Sigma^{<\omega}$, that is, $v^{-1}b_i = u^{-1}a_i$. If $u \sim_n v$ then $\bar{a} \cong_n \bar{b}$.*

Proof. The case $n = 0$ is the preceding corollary. For $n > 0$ we verify the forth condition by induction on n. The back condition follows by symmetry. Suppose that $\bar{a} \sim_n \bar{b}$, and fix an arbitrary element $c \in M$.

We have to find some d such that $\bar{a}c \sim_{n-1} \bar{b}d$. Then, by induction hypothesis, it follows that $\bar{a}c \cong_{n-1} \bar{b}d$.

If $u \preceq c$ then $c = a_i$ for some i and we can set $d := b_i$. Otherwise, let $u_0 := u \sqcap c$ and $u_1 := (u_0)^{-1}u$. Since $u \sim_n v$, there exists a factorisation $v = v_0v_1$ with $v_0 \sim_{n-1} u_0$ and $v_1 \sim_{n-1} u_1$. Thus, we can choose $d := v_0(u_0^{-1}c)$. □

Given a tree-interpretable structure $\mathfrak{M}$ we would like to infer information about the encoding of elements by words. To this end, we choose the following approach: fixing a word $w \in \Sigma^{<\omega}$ we consider the class of substructures A of $\mathfrak{M}$ with $\bigsqcap A = w$. The next result constitutes the main tool to derive structural properties of these substructures.

Proposition 5.6.6. *Let $\mathfrak{M}$ be a tree-interpretable structure and $u \in \Sigma^{<\omega}$. Let $\mathcal{K}$ be a class of substructures of $\mathfrak{M}$ which is $\cong_n$-closed for some n, i.e.,*

$$A \cong_n B \quad \textit{implies} \quad A \in \mathcal{K} \Leftrightarrow B \in \mathcal{K}\,.$$

Up to isomorphism, $\mathfrak{M}|_{u\Sigma^{<\omega}}$ has only finitely many different substructures in $\mathcal{K}$ if and only if there is a constant k such that, for every word $w \succeq u$, there are at most k nonisomorphic substructures $A \in \mathcal{K}$ with $w = \bigsqcap B$.

Proof. ($\Rightarrow$) is trivial. For ($\Leftarrow$) suppose there are infinitely many nonisomorphic substructures $A_i \in \mathcal{K}$, $i < \omega$, with $A_i \subseteq u\Sigma^{<\omega}$. Set $w_i := \bigsqcap A_i$. There exists an infinite set $J \subseteq \omega$ of indices such that $w_i \sim_{n+1} w_j$ for all $i, j \in J$. Choose $k+1$ different indices $i_0, \dots, i_k \in J$. For simplicity, we may assume that these are $0, \dots, k$. Thus, $A_0, \dots, A_k$ induce $k+1$ nonisomorphic substructures of $\mathfrak{M}$ with $w_0 \sim_{n+1} \cdots \sim_{n+1} w_k$.

For $i \leq k$, define the map $p_i : w_i\Sigma^{<\omega} \to w_0\Sigma^{<\omega}$ by

$$p_i(x) := w_0(w_i^{-1}x)\,,$$

and let $B_i := p_i(A_i)$ be the image of A_i under p_i. Since all the p_i are partial isomorphisms, we have $B_i \cong A_i$ and, thus, $B_0, \dots, B_k$ induce $k+1$ nonisomorphic substructures of $\mathfrak{M}$ with the same infimum $w_0 = \bigsqcap B_i \succeq u$. This yields the desired contradiction, since $B_i \cong_n A_i$ implies that all the B_i are contained in $\mathcal{K}$. □

5.7 Tree-interpretable graphs

We apply the results of the previous section to study graphs. First, we count the number of nonisomorphic connected components.

Lemma 5.7.1. *Let $\mathfrak{G}$ be a tree-interpretable undirected graph. For all words $w \in \Sigma^{<\omega}$, there are*

(a) *at most $I|\Sigma| + 1$ connected components C with $w = \prod C$ and*

(b) *at most $I^2|\Sigma| + 1$ strongly connected components C with $w = \prod C$.*

Proof. (a) At most one component contains w. Let $m := I|\Sigma|$ and suppose there are $m+1$ components $C_0, \dots, C_m$ not containing w with $w = \prod C_i$ for $i \leq m$. Fix elements $a_i, b_i \in C_i$ with $wc_i \preceq a_i$ and $wd_i \preceq b_i$ for some $c_i \neq d_i$. Since each C_i is connected we can choose some path from a_i to b_i. Each such path must contain some edge (a_i', b_i') such that $a_i' = wc_ix_i$ and $b_i' = we_iy_i$ for some $e_i \neq c_i$ and words $x_i, y_i \in \Sigma^{<\omega}$. Since $m+1 > I|\Sigma|$ there are indices $i \neq j$ with $c_i = c_j$ and $x_i \sim x_j$. Thus, $(a_i', b_i') \in E$ implies $(a_j', b_i') \in E$. Therefore, the components C_i and C_j are connected and, hence, identical. Contradiction.

(b) Let $m := I^2|\Sigma|$ and suppose there are $m + 1$ components $C_0, \dots, C_m$ not containing w with $w = \prod C_i$. In the same way as above we can find edges (a_i, b_i) and (b_i', a_i'), for $i \leq I^2$, with a_i, a_i', b_i, $b_i' \in C_i$ such that

$$a_i = wc_ix_i\,, \quad a_i' = wc_ix_i'\,, \quad b_i = wd_iy_i\,, \quad b_i' = wd_i'y_i'\,.$$

Since $m + 1 > I^2|\Sigma|$, there are indices $i \neq j$ with $c_i = c_j$, $x_i \sim x_j$, and $x_i' \sim x_j'$. Consequently, we have edges (a_i, b_j) and (b_j', a_i'), and C_i and C_j are connected. Contradiction. □

Proposition 5.7.2. *A tree-interpretable graph $\mathfrak{G}$ has only finitely many nonisomorphic* (a) *connected components and* (b) *strongly connected components.*

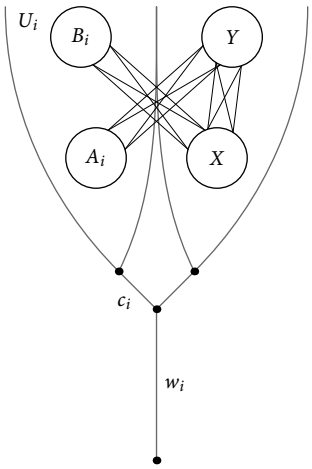
Proof. (a) Let $\mathcal{K}$ be the class of connected components. By the preceding lemma and Proposition 5.6.6, it is sufficient to show that $\mathcal{K}$ is $\cong_1$-closed. Let $B, C \subseteq A$ with $B \cong_1 C$. Then B is connected if and only if C is. Furthermore, a connected set X is maximal iff there is no element $a \in M \setminus X$ that is connected to some vertex $b \in X$. Consequently, $B \cong_1 C$ implies that B is a maximal connected component if and only if C is one. (b) follows in the same way. □

In Section 2.4 we have seen that the parameter $\beta(\mathfrak{G})$ which bounds the size of complete bipartite subgraphs plays an important role when studying graphs of bounded clique width. In the case of tree-interpretable graphs Barthelmann [4] has shown that, if $\beta(\mathfrak{G})$ is infinite, then we can find subgraphs of the form $K_{n,\aleph_0}$ or $K_{\aleph_0,n}$ for arbitrarily large $n < \aleph_0$. But note that, nevertheless, there still might be no subgraph $K_{\aleph_0,\aleph_0}$ as the counterexample $(\omega, \leq)$ shows.

Proposition 5.7.3. *Let $\mathfrak{G} = (V, E)$ be a tree-interpretable graph. If there are subgraphs of the form $K_{m,n}$ for arbitrary large finite m, n then there are subgraphs of the form $K_{\aleph_0,n}$ or $K_{n,\aleph_0}$ for all $n < \aleph_0$.*

Proof. W.l.o.g. we may assume that the alphabet $\Sigma = [2]$ is binary. A complete bipartite subgraph of $\mathfrak{G}$ is given by two sets $X, Y \subseteq V$ with $X \times Y \subseteq E$. We will represent such a subgraph by the pair (X, Y) which we will write as $X \times Y$. (Do not confuse the subgraph $X \times Y$ with the cartesian product of X and Y. In particular, we have $X \times \varnothing \neq X' \times \varnothing$ for $X \neq X'$.) Given a class $\mathcal{C}$ of bipartite subgraphs and a set $U \subseteq 2^{<\omega}$ we define the restriction of $\mathcal{C}$ to U by

$$\mathcal{C}\,|_U := \big\{\, (X \cap U) \times (Y \cap U) \;\big|\; X \times Y \in \mathcal{C} \,\big\}.$$



Let $\mathcal{K}$ be the class of finite bipartite subgraphs $X \times Y$ such that there is no subgraph $X' \times Y'$ with $X \times Y \subset X' \times Y' \subseteq E$.

We have to show that, if there are infinitely many nonisomorphic graphs in $\mathcal{K}$, then $\mathfrak{G}$ does contain the subgraphs $K_{n,\aleph_0}$ or $K_{\aleph_0,n}$ for arbitrarily large n. We will construct the following sequences:

$(w_i)_{i<\omega}$	of words $w_i \in 2^{<\omega}$,
$(c_i)_{i<\omega}$	of symbols $c_i \in [2]$,
$(U_i)_{i<\omega}$	of subsets $U_i \subseteq 2^{<\omega}$,
$(A_i, B_i)_{i<\omega}$	of subsets $A_i, B_i \subseteq U_i$,
$(\mathcal{K}_i)_{i<\omega}$	of classes $\mathcal{K}_i \subseteq \mathcal{K}\,|_{U_i}$,
$(\mathcal{Z}_i)_{i<\omega}$	of classes of subgraphs $X \times Y \subseteq U_i \times U_i$.

We will ensure that, for all $i < \omega$,

(1) $w_i c_i \preceq w_{i+1}$ and $U_i \subseteq w_i c_i 2^{<\omega}$,

(2) $\mathcal{K}_{i+1} \subseteq \mathcal{K}_i\,|_{U_{i+1}}$ contains graphs of unbounded size,

(3) $\varnothing \neq \mathcal{Z}_{i+1} \subseteq \mathcal{K}_i\,|_W$ where $W := U_i \setminus w_{i+1} c_{i+1} 2^{<\omega}$,

(4) $A_i \times Y \subseteq E$ and $X \times B_i \subseteq E$ for all $X \times Y \in \mathcal{Z}_i$,

(5) $X \times Y \subseteq A_i \times B_i$ for all $X \times Y \in \mathcal{K}_{i+1}$,

Since $X \neq \varnothing$ or $Y \neq \varnothing$ for all $X \times Y \in \mathcal{Z}_i$, $i < \omega$, there exists, for every $n < \omega$, an index m and subgraphs $X_i \times Y_i \in \mathcal{Z}_i$ for $i < m$ such that $X := \bigcup_{i<m} X_i$ or $Y := \bigcup_{i<m} Y_i$ is of size at least n. Note that (5) implies that the sets A_i and B_i are infinite. Hence, it follows that

$$K_{n,\aleph_0} \subseteq X \times A_m \subseteq E \qquad \text{or} \qquad K_{\aleph_0,n} \subseteq B_m \times Y \subseteq E.$$

It remains to describe the construction. For $X \times Y \in \mathcal{K}$ let

$$t(X, Y) := \big([w]_\sim,\ w^{-1}X^0/\!\sim,\ w^{-1}X^1/\!\sim,\ w^{-1}Y^0/\!\sim,\ w^{-1}Y^1/\!\sim\big)$$

where $w := \bigsqcap(X \cup Y)$ and $Z^c := Z \cap wc2^{<\omega}$ for $Z \in \{X, Y\}$. Note that the range of t is finite.

The construction proceeds in several steps. In order to avoid the special case of $i = 0$ in every definition below, we set $w_{-1} := \varepsilon$, $c_{-1} := \varepsilon$,

$U_{-1} := 2^{<\omega}$, and $\mathcal{K}_{-1} := \mathcal{K}$. Suppose that $\mathcal{K}_{i-1}$ and U_{i-1} have already been defined.

(a) By Proposition 5.6.6 and Corollary 5.6.4, we can choose $w_i \in U_{i-1}$ such that there are infinitely many nonisomorphic $X \times Y \in \mathcal{K}_{i-1}$ with $w_i = \bigsqcap X \cup Y$.

(b) There exists an infinite subclass $\mathcal{C} \subseteq \mathcal{K}_{i-1}$ containing subgraphs $X \times Y$ of unbounded size such that, for all $X \times Y, X' \times Y' \in \mathcal{C}$,

$$X, Y \subseteq w_i 2^{<\omega} \qquad \text{and} \qquad t(X, Y) = t(X', Y') \,.$$

Suppose that $t(X, Y) = ([w_i], A^0, A^1, B^0, B^1)$.

(c) There exists some $c_i \in [2]$ such that there are subgraphs $X \times Y \in \mathcal{C}$ of unbounded size with $X, Y \subseteq w_i c_i 2^{<\omega}$. Set $U_i := w_i c_i 2^{<\omega}$, $A_i := A^{c_i}$, and $B_i := B^{c_i}$.

(d) Let $\mathcal{K}_i := \mathcal{C}_{U_i}$ and $\mathcal{Z}_i := \mathcal{C}\restriction_W$ where $W := U_{i-1} \setminus w_i c_i 2^{<\omega}$. □

5.8 Tree-interpretable partial orders

We begin our investigation of tree-interpretable partial orders by looking at chains. The following two lemmas bound the number of, respectively, intervals of a chain and of pairwise incomparable chains that have the same infimum.

Lemma 5.8.1. *Let $\mathfrak{M} = (M, \leq, \bar{R})$ be a tree-interpretable structure partially ordered by $\leq$. For every chain $C \subseteq M$, every family $\mathcal{J}$ of pairwise disjoint nonempty intervals $J \subseteq C$, and every word $w \in \Sigma^{<\omega}$ there are at most $I + 1$ intervals $J \in \mathcal{J}$ with $w = \bigsqcap J$.*

Proof. There is at most one interval $J \in \mathcal{J}$ with $w \in J$. Suppose there are $I + 1$ intervals $J_0, \dots, J_I \in \mathcal{J}$ not containing w with $w = \bigsqcap J_i$ for $i \leq I$. Order the J_i such that $i < j$ implies $a < b$ for $a \in J_i$ and $b \in J_j$. By assumption, we can choose elements $a_i, b_i \in J_i$, for $i \leq I$, with $a_i < b_i$ such that $a_i = w c_i x_i$ and $b_i = w d_i y_i$ for some $c_i, d_i \in \Sigma$, $c_i \neq d_i$, and $x_i, y_i \in \Sigma^{<\omega}$. There are indices $i < j$ with $c_i x_i \sim c_j x_j$. Therefore, $a_i \leq b_i$ implies $a_j \leq b_i$. Contradiction. □

Lemma 5.8.2. *Let $\mathfrak{M} = (M, \leq, \bar{R})$ be a tree-interpretable partial order. For every family $\mathcal{J}$ of pairwise incomparable nonempty intervals and every word $w \in \Sigma^{<\omega}$ there are at most $I+1$ intervals $J \in \mathcal{J}$ with $w = \bigsqcap J$.*

Proof. There is at most one interval $J \in \mathcal{J}$ with $w \in J$. Suppose there are $I + 1$ intervals $J_0, \dots, J_I \in \mathcal{J}$ not containing w with $w = \bigsqcap J_i$ for $i \leq I$. We choose elements $a_i, b_i \in J_i$, for $i < I$, with $a_i < b_i$ such that

$a_i = wc_ix_i$ and $b_i = wd_iy_i$ for symbols $c_i \neq d_i$ and words $x_i, y_i \in \Sigma^{<\omega}$. Again, there are indices $i \neq j$ with $c_ix_i \sim c_jx_j$. Hence, $a_i \leq b_i$ implies $a_j \leq b_i$. But J_i and J_j are incomparable. Contradiction. □

One easy application of Proposition 5.6.6 is the following result.

Proposition 5.8.3. *Let $\mathfrak{M} = (M, \leq, \bar{P})$ be a tree-interpretable coloured linear order. Define $a \approx b$ iff the interval $[a, b]$ of M is dense. Up to isomorphism there are only finitely many $\approx$-classes of $\mathfrak{M}$.*

Proof. Let $\mathcal{K}$ be the set of $\approx$-classes. Each $B \in \mathcal{K}$ is either a singleton or a dense linear order. Since $\approx$ can be defined by

$$x \approx y \text{ : iff } \forall z \forall z' (x \leq z < z' \leq y \to \exists u (z < u < z')),$$

which has quantifier rank 3, it follows that $B \cong_4 C$ implies $B \in \mathcal{K}$ iff $C \in \mathcal{K}$. Therefore, we can apply Proposition 5.6.6 and, by Lemma 5.8.1, the claim follows. □

Next we try to develop a normal form for encodings of tree-interpretable partial orders.

Lemma 5.8.4. *Let $(M, \leq)$ be a tree-interpretable partial order. For $X \subseteq M$ and $w \in \Sigma^{<\omega}$ let $X_w := X \cap w\Sigma^{<\omega}$. For every chain $C \subseteq M$ and every word $w \in \Sigma^{<\omega}$ such that $C \cap w\Sigma^+ \neq \varnothing$ there is some $c \in \Sigma$ such that C_{wc} contains an upper bound for the set $\bigcup \{ C_{wd} \mid d \neq c \}$.*

Proof. Otherwise, there exists an increasing subsequence $(a_i)_{i<\omega}$ of C with $a_i \sqcap a_{i+1} = w$ since for each $a_i \in C_{wc}$ there is some $a_{i+1} \in C_{wd}$, for $d \neq c$, with $a_i < a_{i+1}$. Since Σ is finite there is some $c \in \Sigma$ such that there are infinitely many $a_i \in C_{wc}$. There exist indices $i < k$ such that $w^{-1}a_i \sim w^{-1}a_k$. By construction, $k > i + 1$. $a_i \sqcap a_{i+1} = a_k \sqcap a_{i+1}$ and $w^{-1}a_i \sim w^{-1}a_k$ implies that $a_i < a_{i+1}$ iff $a_k < a_{i+1}$. Contradiction. □

infix order $\leq_{\mathrm{i}}$

Definition 5.8.5. For $x, y \in 2^{<\omega}$, we define the *infix order* $\leq_{\mathrm{i}}$ by

$$x \leq_{\mathrm{i}} y \quad \text{: iff} \quad x1 \leq_{\mathrm{lex}} y1.$$

We obtain the following normal form for tree-interpretable partially ordered structures. This result will be crucial for the characterisation of tree-interpretable linear orders below.

Lemma 5.8.6. *Let $\mathfrak{M} = (M, \leq, \bar{R})$ be a tree-interpretable structure partially ordered by $\leq$. We can construct a tree-interpretable structure $\mathfrak{N} \cong \mathfrak{M}$ with universe $N \subseteq 2^{<\omega}$ such that its order is a subset of $\leq_{\mathrm{i}}$.*

Proof. Let $M \subseteq \Sigma^{<\omega}$. W.l.o.g. we may assume that $\Sigma = [2]$. We will encode each element $x = u_0 \cdots u_m \in M$ by some word of the form

$$\hat{x} := (b_0, i_0, u_0) \cdots (b_m, i_m, u_m)$$

over the alphabet $\Xi := [2] \times [k] \times \Sigma$ for some $k < \omega$. Once this is done the symbols of Ξ can be encoded in binary without changing their ordering. The additional components of $\hat{x}$ are defined by

$$b_l := \begin{cases} 0 & \text{if } u_0 \cdots u_{l-1} > x\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

and $$i_l := \max\{\, i+1 \mid \text{there is some word } z \text{ such that } (b_0, i_0, u_0)\cdots(b_{l-1}, i_{l-1}, u_{l-1})(b_l, i, 1-u_l)z \leq (b_0, i_0, u_0)\cdots(b_m, i_m, u_m)\,\}\,.$$

By induction on l and i_l, one can show that the above definition is sound. Lemma 5.8.1 implies that there exists a common upper bound k for the labels i_l.

The choice of $b_0, \ldots, b_m$ ensures that, for all words w, x, y,

$$w < w(b,i,c)x \;\Rightarrow\; b = 1\,,$$
$$w(b,i,c)x < w \;\Rightarrow\; b = 0\,,$$

and the second labelling implies that

$$w(b,i,c)x < w(b,j,d)y \;\Rightarrow\; i < j\,,$$

for $c \neq d$. Thus, using a suitable binary encoding of Ξ the order $\leq$ is contained in the infix ordering.

Clearly, all relations of $\mathfrak{M}$ are still tree interpretable in this encoding. It only remains to prove that the set of such encodings is regular. The formula

$$\vartheta(x,y) := \exists w \bigvee_{c\in\Sigma} (y = wc \wedge w < x)$$

states, for $x = u_0 \cdots u_m$ and $y = u_0 \cdots u_l$, that $b_l = 1$.

By induction on i, we construct an MSO-formula $\varphi_i(x)$ which states that the last symbol of the labelling $i_0 \cdots i_m$ should be i.

$$\varphi_i(x) := \exists w \bigvee_{c\in\Sigma} \Big(x = wc \wedge (\forall z < x)\Big(\bigvee_{d\neq c} wd \preceq z \to \bigvee_{j<i} \varphi_j(z)\Big) \wedge (\exists z < x)\Big(\bigvee_{d\neq c} wd \preceq z \wedge \varphi_{i-1}(z)\Big)\Big)$$

where we set $\varphi_{-1}(x) := \text{true}$. Let π be the projection from Γ to Σ. x is a correctly encoded word if and only if

$$(\forall y \preceq x)\big(\psi(y) \wedge \vartheta(\pi x, \pi y)\big)$$

holds where

$$\psi(y) := \bigvee_{i<k}\Big(\varphi_i(\pi y) \wedge \exists w \bigvee_{b,c<2} y = w(b,i,c)\Big)\,.$$

□

Corollary 5.8.7. *Let $\mathfrak{M}$ be a linearly ordered tree-interpretable structure. There exists a tree-interpretable structure $\mathfrak{N} \cong \mathfrak{M}$ with universe $N \subseteq 2^{<\omega}$ such that its order is $\leq_{\mathrm{i}}|_N$.*

Corollary 5.8.8. *Every tree-interpretable partial order can be completed to a tree-interpretable linear order.*

Proof. Let $(M, \leq)$ be a tree-interpretable partial order. By the preceding lemma we can assume that $\leq \subseteq \leq_{\mathrm{i}}$. Thus $(M, \leq_{\mathrm{i}}|_M)$ is the desired completion. □

Remark. The proofs of Lemmas 5.8.1 and 5.8.6 can easily be generalised to other levels of the Caucal hierarchy and to arbitrary partial orders of finite partition width.

For trees we can obtain a similar, but slightly weaker, normal form.

Lemma 5.8.9. *Let $(T, \leq)$ be a tree-interpretable tree with $T \subseteq \aleph_0^{<\omega}$. There exists an interpretation $\mathcal{I} : (T, \leq) \leq_{\mathrm{MSO}} \mathfrak{T}_2$ such that $\mathcal{I}(a) \not\leq \mathcal{I}(b)$ implies $a \not\leq b$ for all a, b in the domain of $\mathcal{I}$.*

Proof. Let $\pi : ([2] \times [2])^{<\omega} \to [2]^{<\omega}$ be the projection onto the first coordinate where we identify $[2] \times [2]$ with the set $\{00, 01, 10, 11\} \subseteq [2]^{<\omega}$. Fix an arbitrary injective interpretation $\mathcal{J} = \langle \delta, \varphi_{\leq} \rangle : (T, \leq) \leq_{\mathrm{MSO}} \mathfrak{T}_2$ and consider $T' := \pi^{-1}\mathcal{J}^{-1}(T)$. Let $\mathcal{J}' = \langle \delta', \varphi'_{\leq} \rangle$ be the interpretation with $\mathcal{J}'(x) = \mathcal{J}\pi(x)$ for $x \in T'$. We construct a formula $\psi(x)$ such that for every $a \in \operatorname{rng} \mathcal{J}'$ there exists exactly one $x \in \mathcal{J}'^{-1}(a)$ satisfying ψ. The second component of each word $x \in T'$ is used to satisfy the additional condition required above. If $x = (a_0, b_0) \cdots (a_l, b_l)$ then $\psi(x)$ states that

$$b_i = 1 \quad \text{iff} \quad \mathcal{J}(a_0 \cdots a_i) \not\leq \mathcal{J}(a_0 \cdots a_l)\,.$$

In particular, this implies that $b_l = 0$. The interpretation $\mathcal{I} := \langle \delta' \wedge \psi, \varphi'_{\leq} \rangle$ has the desired property. □

5.8.1 Linear orders

Applying Lemma 5.8.6 we can characterise tree-interpretable linear orders by systems of equations.

Proposition 5.8.10. *A coloured linear order $(M, \leq, \bar{P})$ is tree interpretable if and only if it is the canonical solution of a finite system of equations of the form*

$$x_i = x_j + x_k \qquad \text{or} \qquad x_i = c\,,$$

where $+$ denotes ordered sum and c is a chain of length 1 with colour c.

Proof. ($\Rightarrow$) Any such system is a special form of a system of equations of VR-terms. The solutions of those are tree interpretable.

($\Leftarrow$) For $w \in 2^{<\omega}$, let $A_w := M \cap w\Sigma^{<\omega}$. By Lemma 5.8.6 we can assume that $\leq = \leq_{\mathrm{i}} |_M$. It follows that

$$A_w = \begin{cases} A_{w0} + c + A_{w1} & \text{if } w \in M \text{ is coloured } c\,, \\ A_{w0} + A_{w1} & \text{if } w \notin M\,. \end{cases}$$

This infinite system of equations can be reduced to a finite one since $v \sim_1 w$ implies $(A_v, \leq) \cong (A_w, \leq)$. □

Another way to characterise the tree-interpretable linear orders is via closure under certain operations. It immediately follows from the preceding proposition that the class of tree-interpretable linear orders is closed under ordered sums and products.

Corollary 5.8.11. *Let $(\xi_0, \leq)$ and $(\xi_1, \leq)$ be tree-interpretable linear orders. Then $\xi_0 + \xi_1$ and $\xi_0 \cdot \xi_1$ are tree interpretable as well.*

Proof. Given systems of equations for ξ_0 and ξ_1 we can construct systems for $\xi_0 + \xi_1$ and $\xi_0 \cdot \xi_1$. □

Another operation tree-interpretable linear orders are closed under are dense shuffles.

Definition 5.8.12. Let $\mathfrak{M}_0, \ldots, \mathfrak{M}_n$ be linear orders. The *shuffle* of $\mathfrak{M}_0, \ldots, \mathfrak{M}_n$ is the linear ordering defined by the equations *shuffle*

$$\begin{aligned} x_0 &= x_1 + \mathfrak{M}_0 + x_1\,, \\ &\vdots \\ x_{n-1} &= x_n + \mathfrak{M}_{n-1} + x_n\,, \\ x_n &= x_0 + \mathfrak{M}_n + x_0\,. \end{aligned}$$

It turns out that these operations are sufficient to construct every tree-interpretable linear order.

Theorem 5.8.13. *A coloured linear order is tree interpretable if and only if it can be obtained from singletons by the operations of ordered sum, right-multiplication by ω and $-\omega$, and shuffle.*

Proof. Since tree-interpretable structures are closed under these operations all such orders are tree interpretable. For the other direction consider a system of equations of the form $x_i = x_k + x_l$ or $x_i = c$ defining a tree-interpretable order $\mathfrak{M}$.

We define the dependency preorder $\sqsupseteq$ of the variables as follows. Construct a graph with vertices $x_0, \ldots, x_{n-1}$ and, for each equation $x_i = x_k + x_l$, add edges $x_i \to x_k$ and $x_i \to x_l$. Then we set $x_i \sqsupseteq x_k$ if this graph contains a path from x_i to x_k.

We prove the claim by induction on the number of equivalence classes induced by $\sqsupseteq$. Suppose that $X := \{x_0, \dots, x_m\}$ is the maximal $\sqsupseteq$-class, and let $Y := \{y_0, \dots, y_n\}$ be the set of constants and other variables. By induction hypothesis, the orders that are the value of variables in Y can be obtained by finitely many applications of the above operations. To construct a term yielding $\mathfrak{M}$ we consider two cases.

(1) Suppose that all equations are of the form $x = y + x'$ or $x = x' + y$ for $x' \in X$ and $y \in Y$. By repeatedly replacing the variables $x' \in X$ by their definitions, we finally obtain equations of the form

$$x = z + x + z'$$

where z and z' are sums of variables in Y. Thus,

$$x = z\omega + z'(-\omega).$$

(2) Otherwise there are equations of the form $x_i = x_k + x_l$. Eliminate all equations with only one $x' \in X$ on the right-hand side by replacing x' by its definition. Then all equations are of the form

$$x_i = z + x_k + z' + x_l + z''$$

where z, z', and z'' are sums of variables in Y. By introducing a new variable y we can rewrite this equation as

$$\begin{aligned} x_i &= z + x_k + y\,, \\ y &= z' + x_l + z''\,, \end{aligned}$$

and by replacing x_l in the latter equation by its definition we obtain a system of equations of the form

$$x_i = z + x_k + x_l + z'\,.$$

For each such equation we define orders λ_i, μ_i, ρ_i by

$$\begin{aligned} \lambda_i &= z + \lambda_k\,, \\ \rho_i &= \rho_l + z'\,, \end{aligned}$$

and $\quad \lambda_i + \mu_i + \rho_i = x_i\,.$

The equations for λ_i and ρ_i are of the form above. Their solutions are

$$\lambda_i = (z_0 + \cdots + z_r)\omega, \qquad \text{and} \qquad \rho_i = (z'_0 + \cdots + z'_s)(-\omega)\,.$$

The μ_i can equivalently be defined by

$$\mu_i = \mu_k + \rho_k + \lambda_l + \mu_l\,.$$

Let ξ_i be the ordering obtained from μ_i by replacing the sum $\rho_k + \lambda_l$ in each such equation by a constant c_{kl}. Then μ_i can be obtained from ξ_i by substituting the c_{kl} by $\rho_k + \lambda_l$. Since the ξ_i are dense orders without end points each μ_i denotes the shuffle of the orders $\rho_k + \lambda_l$. □

The following theorem summarises the various characterisations we have obtained.

Theorem 5.8.14. *Let $\mathfrak{M}$ be a coloured linear order. The following statements are equivalent:*

(1) *$\mathfrak{M}$ is tree interpretable.*

(2) *$\mathfrak{M}$ is the solution of a system of equations of the form $x_i = x_k + x_l$ or $x_i = c$.*

(3) *$\mathfrak{M}$ can be obtained from singletons by ordered sum, right-multiplication by ω and $-\omega$, and shuffle.*

5.8.2 Well-orders

After having given a complete characterisation of all tree-interpretable linear orders we present some further results for the simpler case of well-orders.

Lemma 5.8.15. *Let $\mathfrak{M}$ be a well-ordered tree-interpretable structure. Then $|a_n| \in \Theta(n)$ for $n < \omega$ where a_n is the n-th element of $\mathfrak{M}$.*

Proof. Since the successor function is definable we have $|a + 1| < |a| + I$. Therefore, $|a_n| \in \mathcal{O}(n)$.

To show the other bound consider some element a_n. Lemma 5.4.1 implies that there exists a constant k such that $a/k \preceq a_N$ for every $a < a_n$. Therefore, there are at most $|\Sigma|^{k+1} \cdot |a_n|$ such elements and we have $|a_n| \geq n|\Sigma|^{-(k+1)} \in \Omega(n)$. □

We have seen that every unary-automatic structure is tree interpretable. The following result states the converse in the case of well-ordered structures.

Proposition 5.8.16. *Let $\mathfrak{M}$ be well-ordered of order type $\alpha < \omega^2$. $\mathfrak{M}$ is tree interpretable if and only if it is unary automatic.*

Proof. Since all unary-automatic structures are tree interpretable it remains to show the other direction. Let $\mathfrak{M}$ be well-ordered or order-type α where $\omega(n-1) \leq \alpha < \omega n$. For $i < n$, we denote the elements a of M with $\omega i \leq a < \omega(i+1)$ that are of length l by $a^i_{l,0}, \ldots, a^i_{l,m}$ where, according to Lemma 5.4.1, m is bounded by some constant k. Further, we require that $a^i_{l,0}, \ldots, a^i_{l,m}$ are sorted lexicographically. Applying the homomorphism $0 \mapsto 0^{nk}$ and $1 \mapsto 1^{nk}$ we can assume that M contains only elements whose length is a multiple of nk. To construct a unary presentation of $\mathfrak{M}$ we encode the element $a^i_{nkl,j}$ by the word $1^{nkl+ki+j}$. It remains to define, for each relation R, a formula φ_R such that

$$\left(\mathfrak{T}_2, \preceq, \mathrm{el}\right) \vDash \varphi_R\left(1^{nkl_0+ki_0+j_0}, \ldots, 1^{nkl_r+ki_r+j_r}\right)$$

iff $\quad \big(a^{i_0}_{nkl_0,j_0}, \dots, a^{i_r}_{nkl_r,j_r}\big) \in R.$

Since all tree-interpretable structures are automatic there is a formula ψ_R which defines R in $(\mathfrak{T}_2, \preceq, \mathrm{el})$. It can be used to define φ_R if we are able to decode $1^{nkl+ki+j}$ into $a^i_{nkl,j}$. Given a word $1^{nkl+ki+j}$ we search for all elements $a^i_{nkl,0}, \dots, a^i_{nkl,m}$ and pick the $(ki+j)$-th one. Let $\delta(x)$ be the formula defining the universe. We set

$$\vartheta_i(x) := \bigvee_{j<k} |x| \equiv (ki+j) \pmod{nk}$$

$$\eta_i(y) := (\exists^{=i+1} z \le y)(\forall u < z)\exists v(u < v < z)$$

and $\quad \chi(x,y) := \delta(y) \wedge |x| - nk < |y| \le |x| \wedge \bigwedge_{i<n} (\vartheta_i(x) \leftrightarrow \eta_i(y))$

which states that y is one of the $a^i_{nkl,j'}$, $j' < k$, if x is the encoding $1^{nkl+ki+j}$. For simplicity, consider the case of a unary relation R only. We can define R by

$$\varphi_R(x) := \bigvee_{\substack{i<n \\ m\le k}} \exists x_0 \cdots x_{m-1} \Big(\exists^{=m} y \chi(x,y) \wedge \bigwedge_{i<m} \chi(x,x_i) \wedge \bigwedge_{i<j} x_i <_{\mathrm{lex}} x_j \wedge \bigvee_{\substack{i<n \\ j<m}} \big(|x| = |x_0| + ki + j \wedge \psi_R(x_j) \big) \Big).$$

□

We conclude this section with a characterisation of all tree-interpretable well-orders. These results already follow from the characterisation of tree-interpretable linear orders above. But, since the present case is much simpler, we also give a direct proof.

Proposition 5.8.17. *$(\omega^\omega, \le)$ is not tree interpretable.*

Proof. Assume otherwise. By Proposition 5.8.10, there exists a finite system of equations of the form

$$x_i = x_k + x_l \qquad \text{or} \qquad x_i = 1.$$

Let $(\xi_i, \le)$ be the order that the canonical solution assigns to x_i. W.l.o.g. we can assume that all the ξ_i are nonempty. Since each $(\xi_i, \le)$ is an interval of $(\omega^\omega, \le)$ it does not contain an infinite descending chain and, hence, all the ξ_i are ordinals. Consider the equation $x_i = x_k + x_l$ for those x_i with $\xi_i = \omega^\omega$. By assumption, $\xi_l \neq 0$ and, thus, $\xi_k < \xi_i = \omega^\omega$. On the other hand, $\xi_k + \xi_l = \omega^\omega$ which implies $\xi_l = \omega^\omega$. Thus, we can assume that $l = i$. The equation $x_i = x_k + x_i$ has the solution $\xi_i = \xi_k \omega$. But $\xi_k \omega = \omega^\omega$ implies $\xi_k = \omega^\omega$. Contradiction. □

Theorem 5.8.18. *Let α be an ordinal. $(\alpha, \le)$ is tree interpretable if and only if $\alpha < \omega^\omega$.*

5.9 Tree-interpretable groups

The investigation of infinite structures with finite presentations has its origins in group theory. As this field remains an important area for the application of finitely presented structures it is natural to ask which groups are tree interpretable.

There are two different ways to represent finitely generated groups as structures. Either multiplication is treated as binary function or one just includes several unary functions denoting the multiplication by a generator. We have already seen in Corollary 3.5.8 that, if the first version is chosen, no infinite group is of finite partition width and, consequently, no such group is tree interpretable.

Lemma 5.9.1. *A group $(G, \cdot)$ is tree interpretable if and only if G is finite.*

Example. $(\mathbb{Z}, +)$ is not tree interpretable.

Cayley graph

The second type of presentation is called the *Cayley graph* of a group. Given a set $S \subseteq G$ of semigroup generators, the Cayley graph of $\mathfrak{G}$ is the structure

$$\Gamma(\mathfrak{G}, S) := \big(G, (f_e)_{e \in S}\big)$$

where $f_e(x) := xe$. Since $\Gamma(\mathfrak{G}, S) \leq_{\mathrm{MSO}} \mathfrak{G}$, the requirement that the Cayley graph is tree interpretable is weaker than the one that $\mathfrak{G} \leq \mathfrak{T}_2$. It turns out that we indeed obtain a larger class of groups using this representations. Thus we will say that a finitely generated group is tree interpretable iff its Cayley graph is so.

Example. Let $\mathfrak{G}$ be the free group of two generators a and b. Its Cayley graph is tree interpretable. Let $S := \{a, b, a^{-1}, b^{-1}\}$. The universe consists of all words over S which are reduced, that is, they do not contain any of the following factors:

$$aa^{-1}, \quad a^{-1}a, \quad bb^{-1}, \quad b^{-1}b\,.$$

The multiplication by a takes words w not ending in a^{-1} to wa and words of the form $w = ua^{-1}$ to u. Hence, we can write f_a in the form

$$\begin{aligned} f_a = (\varepsilon \times a) &\cup S^{<\omega}(a \times aa) \cup S^{<\omega}(a^{-1} \times \varepsilon) \\ &\cup S^{<\omega}(b \times ba) \cup S^{<\omega}(b^{-1} \times b^{-1}a)\,. \end{aligned}$$

The other generators can be defined similarly. It follows that $\Gamma(\mathfrak{G}, S)$ is tree interpretable.

We will show that the class of tree-interpretable groups coincides with the class of context-free groups introduced by Muller and Schupp [56, 57].

context-free group

Definition 5.9.2. A group $\mathfrak{G}$ is *context-free* if there exists a set $S \subseteq G$ of semigroup generators such that the language $\{ w \in S^{<\omega} \mid w = 1 \}$ is context-free.

Lemma 5.9.3. *A group is tree interpretable if and only if it is context-free.*

Proof. Muller and Schupp [56, 57] have shown that a group is context-free if and only if its Cayley graph is isomorphic to the configuration graph of a pushdown automaton. This is equivalent to being isomorphic to a structure of the form $(V, (E_a)_a)$ where V is a regular language and each relation E_a is of the form

$$E_a = \bigcup_{i<n} W_i(\{u_i\} \times \{v_i\}) .$$

Clearly, each such structure is tree interpretable. For the converse, note that, if

$$f_a = \bigcup_{i<n} W_i(U_i \times V_i)$$

is a function, then every set V_i must be a singleton as, otherwise, we would have $f_a(wu) = wv$ and $f_a(wu) = wv'$ for words $w \in W_i$, $u \in U_i$, and $v, v' \in V_i$ with $v \neq v'$. Furthermore, since f_a is injective, we have $|U_i| = 1$ by the same reasoning. □

The class of context-free groups is well investigated and has several characterisations.

Theorem 5.9.4. *Let $\mathfrak{G}$ be a finitely generated group. The following statements are equivalent:*

(1) *$\mathfrak{G}$ is context-free.*

(2) *$\mathfrak{G}$ is virtually free.*

(3) *$\Gamma(\mathfrak{G}, S)$ has only finitely many nonisomorphic ends.*

(4) *$\Gamma(\mathfrak{G}, S)$ is isomorphic to the configuration graph of a pushdown automaton.*

(5) *$\Gamma(\mathfrak{G}, S)$ is κ-triangulable for some finite κ.*

(6) *$\Gamma(\mathfrak{G}, S)$ is tree interpretable.*

(7) *$\Gamma(\mathfrak{G}, S) \in \mathcal{C}$.*

(8) *$\Gamma(\mathfrak{G}, S)$ has finite tree width.*

(9) *$\Gamma(\mathfrak{G}, S)$ has finite partition width.*

(10) *The* MSO*-theory of $\Gamma(\mathfrak{G}, S)$ is decidable.*

The equivalence of (1) – (4) was shown by Muller and Schupp in [56, 57], while characterisations (8) and (10) are from Kuske and Lohrey [50]. The equivalence of (8) and (9) follows from Theorem 2.4.6. Finally, (6) ⇒ (7) ⇒ (9). We will prove a special case of the equivalence of (6) and (8).

Theorem 5.9.5 (Ly [51]). *A finitely generated group $\mathfrak{G}$ is context-free if and only if $\Gamma(\mathfrak{G}, S)$ has a tree decomposition $(F_v)_v$ of finite width where every component F_v is connected.*

Proof. (⇒) By results of Muller and Schupp [56, 57], there exists a set $S \subseteq G$ of semigroup generators and some prefix closed set $V \subseteq S^{<\omega}$ such that $\Gamma(\mathfrak{G}, S) = (V, (f_e)_{e\in S})$ and each function f_e is of the form

$$f_e = \bigcup_{i<n} W_i(\{u_i\} \times \{v_i\}) .$$

The family $(F_w)_w$ with $F_w := wS^{\leq I} \cap V$ is a tree decomposition since

$$|u|,\ |f_e(u)| < |u \sqcap f_e(u)| + I ,$$

implies that $u, f_e(u) \in F_{u\sqcap f_e(u)}$ for all $u \in 2^{<\omega}$. Furthermore, each component F_w is connected since the universe V is prefix closed.

(⇐) Suppose there exists a tree decomposition $(F_v)_v$ with $|F_v| \leq k$. We will construct a k-triangulation of $\Gamma(\mathfrak{G}, S)$. Let $a_0, \ldots, a_{n-1}$ be a cycle where each edge is either an actual edge of $\Gamma(\mathfrak{G}, S)$ or represents a path of length at most k. Let T be the subtree of the decomposition which contains edges of the cycle. The triangulation is constructed by induction on the size of T. Consider a leaf F_v of T containing $a_i, \ldots, a_{i+m}$ but neither a_{i-1} nor a_{i+m+1}. Note that any two vertices in F_v are connected by a path of length at most k. If $m = 1$ the edge (a_i, a_{i+1}) is also contained in the predecessor of F_v which, thus, can be deleted from T. Otherwise, add a path of length at most k from a_i to a_{i+k} to the cycle. By induction hypothesis there is a k-triangulation of the cycle $a_0, \ldots, a_i, a_{i+k}, \ldots, a_n$. By adding paths, say, from a_i to each a_{i+j}, for $j < k$, it can be completed to a triangulation of the whole cycle. □

6 Axiomatisations

Each tree-interpretable structure can be encoded by a finite amount of information, namely, by an MSO-interpretation in the binary tree. Therefore, it should not be surprising that every such structure can be axiomatised in a suitable logic. In the present chapter we will show that each tree-interpretable structure $\mathfrak{M}$ is finitely $\mathrm{GSO}(\exists^\kappa)$-axiomatisable, i.e., there is a $\mathrm{GSO}(\exists^\kappa)$-sentence $\psi_{\mathfrak{M}}$ such that $\mathfrak{N} \models \psi_{\mathfrak{M}}$ if and only if $\mathfrak{N} \cong \mathfrak{M}$.

Actually, we will prove the slightly stronger statement that, for each tree-interpretable structure $\mathfrak{M}$, there is a colouring χ of the guarded tuples such that the coloured structure $(\mathfrak{M}, \chi)$ is $\mathrm{MSO}(\exists^\kappa)$-axiomatisable. That is, the axiom consists of a sequence of existential non-monadic second-order quantifiers followed by an $\mathrm{MSO}(\exists^\kappa)$-formula.

Roughly, the proof consists in defining a forest $(\mathfrak{F}, \chi)$ in $(\mathfrak{M}, \chi)$ in such a way that the original structure $(\mathfrak{M}, \chi)$ can be reconstructed from $(\mathfrak{F}, \chi)$. Then the theorem follows from the corresponding, but much simpler, result for forests.

6.1 The congruence colouring

If x is a word, we denote by $\mathrm{suf}_k\, x$ the suffix of x of length k. The axiomatisation uses colourings of elements and of pairs of elements that are of the following form:

$\mathrm{suf}_k\, x$

Definition 6.1.1. (a) Let $\approx\, \subseteq \Sigma^{<\omega} \times \Sigma^{<\omega}$ be a congruence of finite index and let $k \in \mathbb{N}$. The $(\approx, k)$-*congruence colouring* $\chi^k_\approx$ maps words $x \in \Sigma^{<\omega}$ to the pair

congruence colouring $\chi^k_\approx$

$$\chi^k_\approx(x) := \big([x/k]_\approx,\ \mathrm{suf}_k\, x\big)$$

and pairs $x, y \in \Sigma^{<\omega}$ to

$$\chi^k_\approx(x, y) := \big(\chi^k_\approx(w^{-1}x),\ \chi^k_\approx(w^{-1}y)\big)$$

where $w := x \sqcap y$.

(b) A $(\approx', k')$-colouring χ' *refines* the $(\approx, k)$-colouring χ if $\approx' \subseteq \approx$ and $k' \geq k$. We denote this fact by $\chi' \geq \chi$. The *common refinement* of the $(\approx_0, k_0)$-colouring χ_0 and the $(\approx_1, k_1)$-colouring χ_1 is the $(\approx_0 \cap \approx_1,\ \max\{k_0, k_1\})$-colouring denoted by $\chi_0 \sqcup \chi_1$.

refining a colouring $\chi' \geq \chi$
common refinement $\chi_0 \sqcup \chi_1$

χ-expansion

Definition 6.1.2. The *χ-expansion* $(\mathfrak{M}, \chi)$ of $\mathfrak{M}$ expands $\mathfrak{M}$ by unary and binary relations for each colour class where the binary colour classes consists only of pairs (x, y) which are guarded.

The restriction to guarded pairs is essential since GSO allows only quantification over relations of this form. Below we frequently will need to obtain the value $\chi(x, y)$ for pairs (x, y) which are not guarded. These values must be computed explicitly from available data. This is where the k-normal paths of Section 5.5 come into play.

Lemma 6.1.3. *Let* $\chi_1 \geq \chi_0$.

(a) *There exists a function* f *with* $\chi_0 = f \circ \chi_1$.

(b) $(\mathfrak{M}, \chi_0)$ *is* FO*-interpretable in* $(\mathfrak{M}, \chi_1)$.

Lemma 6.1.4. *Let* $\mathfrak{M}$ *be a tree-interpretable structure,* $\approx$ *a congruence of finite index, and* k *a constant. The* $\chi^k_\approx$*-expansion* $(\mathfrak{M}, \chi^k_\approx)$ *of* $\mathfrak{M}$ *is also tree interpretable.*

Proof. It is sufficient to note that, since $\approx$ is of finite index, each $\approx$-class forms a regular language. □

coding a path

We say that a set P of vertices *codes* a path between x and y if every element of P except for x and y is connected to exactly two other elements in P whereas x and y are connected to exactly one such element. Clearly, not every path can be coded in this way. Fortunately, for our purposes it is sufficient that, if there exists a k-normal path between two vertices, then we can obtain a codable k-normal path between them by removing some vertices.

Lemma 6.1.5. *For every number* k *and each colour* c *there exists an* MSO*-formula* $\varphi_c(P, x, y)$ *such that, for all graphs* $\mathfrak{G}$ *and all* $(\approx, k)$*-congruence colourings* χ *we have* $(\mathfrak{G}, \chi) \models \varphi_c(P, x, y)$ *if and only if* P *codes a* k*-normal path from* x *to* y *and* $\chi((x \sqcap y)^{-1}y) = c$.

Proof. We label the elements $z \in P$ by the $(\approx, k')$-colour of $(x/k)^{-1}z$ for some $k \leq k' \leq 2k$. Since $x/k \preceq y$ we can compute $\chi((x \sqcap y)^{-1}y)$ from $\chi(x)$ and the label of y. To decide whether a given labelling is correct note that, if (z, z') is an edge of the path and z is labelled $([u], w)$, then the label of z' consists of the suffix w' of z' of length $\min\{2k, |w| + |z'| - |z|\}$ and the $\approx$-class of $(x/k)^{-1}z'(w')^{-1}$ both of which can be calculated from the colour of z. Note that, since the path is k-normal, we can ensure that the length of the stored suffix is at least $2k - k = k$. □

6.2 Forests

We start slowly by showing that forests are finitely axiomatisable. We regard forests as partial orders such that the elements below any given one form a finite linear order. For any partial order $(A, <)$ let $\downarrow x := \{ z \in A \mid z < x \}$ and $\uparrow x := \{ z \in A \mid x < z \}$. $\downarrow x, \uparrow x$

Lemma 6.2.1. *Let $\mathfrak{T} := (T, \leq)$ be a tree-interpretable forest and $\chi^I_\sim$ the $(\sim, I)$-congruence colouring. $\chi^I_\sim(x) = \chi^I_\sim(y)$ implies that $\uparrow x \cong \uparrow y$.*

Proof. For each $b \in T$, there are only finitely many $a \leq b$. By Lemma 5.4.1, it follows that $a \leq b$ implies $a/I \preceq b$. Therefore, $\uparrow a \subseteq (a/I)\Sigma^{<\omega}$, and the function $f : (x/I)\Sigma^{<\omega} \to (y/I)\Sigma^{<\omega}$ mapping $(x/I)z$ to $(y/I)z$ is the desired isomorphism. □

Theorem 6.2.2. *If $\mathfrak{T} := (T, \leq)$ is a tree-interpretable forest and $\chi \geq \chi^I_\sim$ then the structure $(\mathfrak{T}, \chi)$ is finitely $\mathrm{FO}(\exists^\kappa)$-axiomatisable.*

Proof. Let $T_0 \subseteq T$ be the set of minimal elements, and we denote by $S(x)$ the set of immediate successors of $x \in T$. For $X \subseteq T$ let $\mu(X)$ be the function which maps each colour c to the number of elements $x \in X$ coloured c.

We claim that a structure $\mathfrak{X} := (X, \leq, \chi')$ is isomorphic to $(\mathfrak{T}, \chi)$ if and only if

(1) $\leq$ is a partial order such that, for all $x \in X$, the set $\downarrow x$ is either empty or it forms a finite linear order,

(2) $\mu(X_0) = \mu(T_0)$ where $X_0 \subseteq X$ is the set of minimal elements, and

(3) $\mu(S(x)) = \mu(S(u))$ for all $x \in X$ and $u \in T$ with $\chi'(x) = \chi(u)$.

Clearly, all these conditions can be expressed in $\mathrm{FO}(\exists^\kappa)$.

To prove the nontrivial direction we construct an isomorphism $h : X \to T$ given some order $\mathfrak{X}$ that satisfies the above conditions. Note that (1) implies that $\mathfrak{X}$ is a forest. Let $\mathrm{ht}(x) := |\downarrow x|$. We construct h as the limit of partial isomorphisms

$$h_i : \{ x \in X \mid \mathrm{ht}(x) \leq i \} \to \{ u \in T \mid \mathrm{ht}(u) \leq i \}, \qquad i < \omega,$$

as follows.

($i = 0$) Since $\mu(X_0) = \mu(T_0)$ there is a bijection $h_0 : X_0 \to T_0$ that preserves the colouring.

($i > 0$) For each $x \in X$ with $\mathrm{ht}(x) = i - 1$ we choose a colour preserving bijection $g_x : S(x) \to S(h_{i-1}x)$. Note that (3) ensures its existence. h_i is the extension of h_{i-1} by all the g_x.

Using the preceding lemma it is easy to show that h is well-defined and indeed an isomorphism. □

6.3 Partial orders

The next step consists in extending the result to tree-interpretable partial orders $\mathfrak{M} := (M, \leq)$ for which there is a constant $n \in \mathbb{N}$ such that $x \leq y$ implies $x/n \preceq y/n$ for all $x, y \in M$. To do so we have to define a forest in $\mathfrak{M}$. When speaking of paths we always consider undirected paths in this section, that is, we ignore the direction of the edges.

$x \sqsubseteq y$

$x \equiv y$

Definition 6.3.1. Let $x \sqsubseteq y$ iff $x/n \preceq y/n$ and there exists an undirected $\leq$-path $z_0, \dots, z_m$ from x to y with $x/n \preceq z_i/n$ for all $i \leq m$. Further, define $x \equiv y$ iff $x \sqsubseteq y$ and $y \sqsubseteq x$.

Lemma 6.3.2. *$(M, \sqsubseteq)/\equiv$ is a forest.*

Proof. It is sufficient to show that $\downarrow[x]_\equiv$ is a linear order for all $[x]_\equiv \in M/\equiv$. Suppose that $[y]_\equiv, [z]_\equiv \sqsubseteq [x]_\equiv$. Then $y/n, z/n \preceq x/n$ and, by symmetry, we may assume that $y/n \preceq z/n$. We claim that $[y]_\equiv \sqsubseteq [z]_\equiv$.

By definition, there are undirected $\leq$-paths $y_0, \dots, y_l$ from y to x and $z_0, \dots, z_m$ from z to x such that $y/n \preceq y_i/n$ and $z/n \preceq z_i/n$ for all i. $y/n \preceq z/n$ implies $y/n \preceq z_i/n$ and the path $y_0, \dots, y_l, z_{m-1}, \dots, z_0$ leading from y to z witnesses that $y \sqsubseteq z$. □

Using the result of the previous section we first prove that $(M, \sqsubseteq, \chi)$ is axiomatisable by defining a suitable copy of $(M, \sqsubseteq)/\equiv$ in it.

Lemma 6.3.3. *The subset $M_0 \subseteq M$ which consists of the lexicographically minimal elements of each $\equiv$-class is* MSO*-definable in $(M, \sqsubseteq, \chi^n_\sim)$.*

Proof. Since $x \equiv y$ implies $x/n = y/n$, one can determine whether $x \leq_{\mathrm{lex}} y$ by looking at $\mathrm{suf}_n x$ and $\mathrm{suf}_n y$. This information is contained in the colouring $\chi^n_\sim$. □

Proposition 6.3.4. *There is a congruence-colouring χ_0 such that the order $(M, \sqsubseteq, \chi)$ is finitely* MSO$(\exists^\kappa)$*-axiomatisable for every $\chi \geq \chi_0$.*

Proof. Let $\mathfrak{B} := (M, \sqsubseteq)$, and let $\delta(x)$ be the formula defining M_0 in $(\mathfrak{B}, \chi)$. We set $\chi_0 := \chi^n_\sim \sqcup \chi^{I_t}_{\sim_t}$ where $\sim_t$ is the syntactic congruence corresponding to $\mathfrak{B}/\equiv$ and I_t is its index.

A structure $(\mathfrak{X}, \chi') := (X, \sqsubseteq', \chi')$ is isomorphic to $(\mathfrak{B}, \chi)$ if and only if there is an isomorphism $f : (\mathfrak{X}/\equiv', \chi') \to (\mathfrak{B}/\equiv, \chi)$ such that $[x]_{\equiv'} \cong f[x]_\equiv$ for all $x \in X$ where $\equiv' := \sqsubseteq' \cap \sqsupseteq'$. This condition is equivalent to the following ones:

(1) $\delta^{\mathfrak{X}}$ contains exactly one element of each $\equiv'$-class of X.

(2) $(\delta^{\mathfrak{X}}, \sqsubseteq', \chi') \cong (\delta^{\mathfrak{B}}, \sqsubseteq, \chi)$.

(3) $[x]_{\equiv'} \cong [a]_\equiv$ for all $x \in X$ and $a \in M$ with $\chi'(x) = \chi(a)$.

(1) and (3) are easily expressed in MSO. (2) can be checked since $\chi \geq \chi^{I_t}_{\sim_t}$ and therefore the forest $(\delta^{\mathfrak{B}}, \sqsubseteq, \chi) \cong (\mathfrak{B}/\equiv, \chi)$ is $\mathrm{FO}(\exists^\kappa)$-axiomatisable. □

In order to transfer the axiomatisability result from $(M, \sqsubseteq)$ to $\mathfrak{M}$, we have to show that each of the structures is definable in the other one.

Lemma 6.3.5. *Let $k := \max\{K, n\}$.*

(a) *If $x \sqsubseteq y$ then there is a k-normal path $z_0, \dots, z_m$ from x to y with $x/n \preceq z_i/n$ for all i.*

(b) *If $x \equiv y$ then there exists an undirected $\leq$-path $z_0, \dots, z_m$ from x to y with $x/n \preceq z_i/n$ and $|z_i| \leq |x| + k$ for all $i \leq m$.*

(c) *The relation $\sqsubseteq$ is* MSO*-definable in $(\mathfrak{M}, \chi^k_\sim)$.*

(d) *$(M, \sqsubseteq, \chi)$ is* MSO*-definable in $(\mathfrak{M}, \chi)$ for all $\chi \geq \chi^k_\sim$.*

Proof. (a) is implied by Proposition 5.5.10, and (b) follows from Corollary 5.5.9 since x and y are connected by a path above $x/n = y/n$.

(c) We have $x \sqsubseteq y$ iff there is a k-normal undirected path $z_0, \dots, z_m$ from x to y with $x/n \preceq z_i/n$ for all i. Thus, $x \sqsubseteq y$ iff there is a k-normal path P such that each initial segment P' of P leads to some vertex z with $|z| \geq |x|$. It follows from Lemma 6.1.5 that the condition $|z| > |x|$ can be expressed by an MSO-formula.

(d) By (c) it remains to define the colouring $\chi(x, y)$ for $x \sqsubseteq y$. This can by done, by Lemma 6.1.5, since there exists a k-normal path from x to y. □

Lemma 6.3.6. *$(\mathfrak{M}, \chi)$ is* MSO*-definable in $(M, \sqsubseteq, \chi)$ for all $\chi \geq \chi^n_\sim$.*

Proof. Since $\leq$ is tree interpretable and $x \leq y$ implies $x/n \preceq y/n$, there are sets $U([w]) \subseteq \Sigma^{\leq I}$ and $V([w]) \subseteq \Sigma^{<\omega}/\sim$ for every class $[w] \in \Sigma^{<\omega}/\sim$ such that $x \leq y$ iff

$$x/n \preceq y/n, \quad w^{-1}x \in U([w]), \quad \text{and} \quad [w^{-1}y] \in V([w]).$$

where $w := x \sqcap y$. Since $x \leq y$ implies $x \sqsubseteq y$, all of the above conditions can be expressed in MSO using $\chi(x)$, $\chi(y)$, and $\chi(x, y)$. The colouring of $(\mathfrak{M}, \chi)$ is definable for the same reason. □

Theorem 6.3.7. *Let $\mathfrak{M} := (M, \leq)$ be a tree-interpretable partial order and let $n \in \mathbb{N}$ be a constant such that $x \leq y$ implies $x/n \preceq y/n$ for all $x, y \in M$. There is a congruence colouring χ_0 such that $(\mathfrak{M}, \chi)$ is finitely* $\mathrm{MSO}(\exists^\kappa)$*-axiomatisable for every $\chi \geq \chi_0$.*

Proof. Let χ'_0 be the colouring of Proposition 6.3.4. We set $\chi_0 := \chi'_0 \sqcup \chi^K_\sim \sqcup \chi^n_\sim$. Let $\mathcal{I}$ be the MSO-definition of $(M, \sqsubseteq, \chi)$ in $(\mathfrak{M}, \chi)$. By the preceding lemmas, a structure $(\mathfrak{X}, \chi')$ is isomorphic to $(\mathfrak{M}, \chi)$ if and only if $\mathcal{I}(\mathfrak{X}, \chi') \cong \mathcal{I}(\mathfrak{M}, \chi)$. The claim follows since $\mathcal{I}(\mathfrak{M}, \chi)$ is $\mathrm{MSO}(\exists^\kappa)$-axiomatisable by Proposition 6.3.4. □

6.4 The general case

Finally, we consider an arbitrary tree-interpretable structure $\mathfrak{M}$. For the reduction to the previous case we define, as above, a partial order $\leq$ and show that the structures $(M, \leq)$ and $\mathfrak{M}$ are definable within each other.

$x \vdash y$
$x \leq y$

Definition 6.4.1. Let $x \vdash y$ if $x/I \leq y/I$ and the pair (x, y) is guarded. Let $\leq$ be the reflexive and transitive closure of $\vdash$.

Lemma 6.4.2. *$(M, \leq, \chi)$ is* MSO*-definable in $(\mathfrak{M}, \chi)$ for all $\chi \geq \chi^I_\sim$.*

Proof. The relation $\vdash$ is FO-definable, since one can tell whether $x/I \leq y/I$ holds by looking at $\chi^I_\sim(x, y)$. Thus, $\leq$, its reflexive and transitive closure, is MSO-definable.

To show that the colouring is also definable we prove that, for each colour c of χ, there is a formula $\varphi_c(x, y)$ such that

$$(\mathfrak{M}, \chi) \vDash \varphi_c(x, y) \quad \text{iff} \quad x \leq y \text{ and } \chi(x, y) = c\,.$$

If $x \vdash y$ then there is a relation R and a tuple $\bar{a} \in R$ with $x, y \in \bar{a}$. Hence, $\chi(x, y)$ is available in $(\mathfrak{M}, \chi)$. Thus, there is a formula $\varphi^{\vdash}_c(x, y)$ which expresses that $x \vdash y$ and $\chi(x, y) = c$. We have $x \leq y$ iff there is a path $x = z_0 \vdash \cdots \vdash z_n = y$. Note that $z_i \vdash z_{i+1}$ implies $z_i/I \preceq z_{i+1}/I$. Therefore, we can compute $\chi(x, z_{i+1})$ from $\chi(x, z_i)$ and $\chi(z_i, z_{i+1})$. □

The proof of the converse is more involved and requires an investigation of the branching structure of a tuple.

reduct

Definition 6.4.3. Let $\bar{a}, \bar{b} \in M^n$. We say that $\bar{a}$ is a *reduct* of $\bar{b}$ iff

(1) the branching structures of $\bar{a}$ and $\bar{b}$ are isomorphic,

(2) $\bigsqcap \bar{a} \sim \bigsqcap \bar{b}$,

(3) $(a_i \sqcap a_j)^{-1}(a_k \sqcap a_l) \sim (b_i \sqcap b_j)^{-1}(b_k \sqcap b_l)$ for all indices such that $a_i \sqcap a_j \preceq a_k \sqcap a_l$,

(4) $|a_i| < |\bigsqcap \bar{a}| + nI$ for all $i < n$.

reduced

A tuple is called *reduced* if it is a reduct of itself.

Lemma 6.4.4. *If $\bar{a}$ is a reduct of $\bar{b}$ and $\bar{b} \in R$ then $\bar{a} \in R$.*

We can check whether a tuple $\bar{a}$ belongs to a relation R by using the characterisation of Proposition 5.2.4. To do so we need the $\sim$-class of $u^{-1}v$ for branching points u and v of $\bar{a}$.

coding a branching structure

Definition 6.4.5. Let $\bar{a} \in M^n$. The elements $b_{ik} \in M$, for $i, k < n$, *code the branching structure* of $\bar{a}$ if

(1) $b_{ii} = a_i$ for $i < n$,

(2) $b_{ik}/nI \prec a_i \sqcap a_k \preceq b_{ik}$ for all i, k, and

(3) if $a_i \sqcap a_k \prec a_i \sqcap a_l$ then $b_{ik} \vdash b_{il}$ for $i, k, l < n$.

Given b_{ik} and b_{il} we can compute the $\sim$-class of $(a_i \sqcap a_k)^{-1}(a_i \sqcap a_l)$. Hence, if we can show that such elements always exist and that they are definable, then we are almost done.

Lemma 6.4.6. *Let $\chi \geq \chi_{\sim}^{nI}$. For each branching structure $\mathfrak{X}$ there exists a formula $\beta_{\mathfrak{X}}(\bar{x}, \bar{y})$ such that $(M, \leq, \chi) \models \beta_{\mathfrak{X}}(\bar{a}, \bar{b})$ if and only if the branching structure of $\bar{a}$ is $\mathfrak{X}$ and it is coded by $\bar{b}$.*

Proof. For all $i, k < n$ we have to express that $b_{ik}/m = a_i \sqcap a_k$ for some $m < nI$. Since $b_{ik} \vdash b_{ii} = a_i$ and $b_{ik} \vdash b_{kk} = a_k$ this can be determined by looking at $\chi(b_{ik}, a_i)$ and $\chi(b_{ik}, a_k)$. The verification of the other conditions can be done easily. □

Lemma 6.4.7. *Let R be an n-ary relation of $\mathfrak{M}$ and $\bar{a} \in R$. There are elements $b_{ik} \in M$, $i, k < n$, coding the branching structure of $\bar{a}$.*

Proof. W.l.o.g. assume that $\Sigma = [2]$. Let

$$J_{ik} := \{\, j < n \mid a_i \sqcap a_k \prec a_i \sqcap a_j \,\}.$$

We define tuples $\bar{c}_{ik}$, for $i, k < n$, by induction on $|J_{ik}|$ such that $\bar{c}_{ik}|_{J_{ik}}$ is a reduct of $\bar{a}|_{J_{ik}}$. If $J_{ik} = \varnothing$ let $\bar{c}_{ik} := \bar{a}$. Otherwise, let j, l be indices such that the branching points $a_i \sqcap a_l$ and $a_j \sqcap a_k$ are the immediate successors of $a_i \sqcap a_k$. Let

$$\bar{d}_{ik} := \bar{c}_{il}|_{J_{ik}} \cup \bar{c}_{jk}|_{J_{ki}} \cup \bar{a}|_{\overline{J_{ik} \cup J_{ki}}}.$$

Choose $\bar{c}_{ik}$ such that

$$\bar{c}_{ik}|_{J_{ik} \cup J_{ki}} \text{ is a reduct of } \bar{d}_{ik}|_{J_{ik} \cup J_{ki}} \quad \text{and} \quad \bar{c}_{ik}|_{\overline{J_{ik} \cup J_{ki}}} = \bar{a}|_{\overline{J_{ik} \cup J_{ki}}}.$$

Finally, set $b_{ik} := (c_{ik})_k$. Since, by construction, $\bar{d}_{ik}|_{\overline{J_{ki}}} \cup \bar{c}_{ik}|_{J_{ki}} \in R$ we have

$$b_{ik} = (c_{ik})_k \vdash (d_{ik})_l = (c_{il})_l = b_{il}. \qquad \square$$

At last, we are able to prove the other direction.

Lemma 6.4.8. *The structure $(\mathfrak{M}, \chi)$ is* MSO*-definable in $(M, \leq, \chi)$ for every $\chi \geq \chi_{\sim}^{nI}$ where n is the maximal arity of relations of $\mathfrak{M}$.*

Proof. Let R be an n-ary relation of $(\mathfrak{M}, \chi)$. We prove the claim by induction on n.

($n = 1$) $R \subseteq \Sigma^{<\omega}$ is regular with a coarser congruence than $\sim$. Thus, we can determine whether $a \in R$ by looking at $\chi(a)$.

($n > 1$) W.l.o.g. assume that all tuples $\bar{a} \in R$ have the same branching structure. For all branching points $a_i \sqcap a_k$ with immediate successor $a_i \sqcap a_l$ let W_{ikl} be the regular language such that $\bar{a} \in R$ iff $(a_i \sqcap a_k)^{-1}(a_i \sqcap a_l) \in W_{ikl}$ for all such i, k, l. By the preceding lemma

it follows that $\bar{a} \in R$ if and only if there are elements b_{ik}, $i, k < n$, coding the branching structure of $\bar{a}$ and constants $m_{ik} < nI$ such that $b_{ik}/m_{ik} = a_i \sqcap a_k$ and $(b_{ik}/m_{ik})^{-1}(b_{il}/m_{il}) \in W_{ikl}$ for all admissible i, k, l. Since $b_{ik} \vdash b_{il}$ we can check the latter condition by looking at $\chi(b_{ik}, b_{il})$. □

Theorem 6.4.9. *Let $\mathfrak{M}$ be a tree-interpretable structure. There is a congruence colouring χ_0 such that $(\mathfrak{M}, \chi)$ is finitely* MSO($\exists^\kappa$)*-axiomatisable for all $\chi \geq \chi_0$.*

Proof. The proof is completely analogous to the one of Theorem 6.3.7. Let χ_0' be the colouring of Theorem 6.3.7 for the structure $(M, \leq)$, and set $\chi_0 := \chi_0' \sqcup \chi_\sim^{nI}$ where n is the maximal arity of relations of $\mathfrak{M}$. Let $\mathfrak{X}$ be a structure. By the preceding lemmas $(\mathfrak{M}, \chi)$ and $(M, \leq, \chi)$ are MSO-definable within each other. Let $\mathcal{I} : (M, \leq, \chi) \leq_{\text{MSO}} (\mathfrak{M}, \chi)$ be the corresponding interpretation. It follows that $\mathfrak{X} \cong (\mathfrak{M}, \chi)$ iff $\mathcal{I}(\mathfrak{X}) \cong \mathcal{I}(\mathfrak{M}, \chi)$. The later condition is MSO($\exists^\kappa$)-expressible by Theorem 6.3.7. □

Since GSO($\exists^\kappa$) allows quantification over colourings χ we obtain as an immediate corollary the following result.

Theorem 6.4.10. *Every tree-interpretable structure is finitely* GSO($\exists^\kappa$)*-axiomatisable.*

As GSO($\exists^\kappa$) collapses to MSO($\exists^\kappa$) on uniformly k-sparse structures we obtain the following result of Courcelle [22].

Corollary 6.4.11. *Every HR-equational structure is finitely* MSO($\exists^\kappa$)*-axiomatisable.*

6.5 Lower bounds

We have shown that every tree-interpretable structure is finitely GSO($\exists^\kappa$)-axiomatisable. Can we do better? In this section we show that at least the quantifiers $\exists^{\aleph_0}$ and $\exists^{\aleph_1}$ are needed. Since all tree-interpretable structures are countable we obviously can do without the ones for higher cardinalities.

For a logic $\mathfrak{L}$ let $\mathfrak{L}_m$ denote the set of $\mathfrak{L}$-formulae of quantifier rank at most m where we count both first- and second-order quantifiers. The following statements about the expressivity of MSO_m and $\text{MSO}_m(\exists^{\aleph_0})$ can easily be proved using the corresponding versions of the Ehrenfeucht-Fraïssé game.

Lemma 6.5.1. (a) *For every $m < \omega$ there exists a constant k such that two sets A and B are* MSO_m*-equivalent if and only if*

$$|A| = |B| \quad \text{or} \quad |A|, |B| \geq k\,.$$

(b) *For every $m < \omega$ there exists a constant k such that two sets A and B are* $\mathrm{MSO}_m(\exists^{\aleph_0})$*-equivalent if and only if*

$$|A| = |B|\,, \quad \text{or} \quad k \leq |A|, |B| < \aleph_0\,, \quad \text{or} \quad |A|, |B| \geq \aleph_0\,.$$

(c) *Any two infinite sets are* $\mathrm{MSO}(\exists^{\aleph_0})$*-equivalent.*

Lemma 6.5.2. *For all* $\mathrm{GSO}(\exists^\kappa)$*-sentences φ there exists an* $\mathrm{MSO}(\exists^\kappa)$*-sentence φ' such that*

$$\mathfrak{T} \vDash \varphi \quad \text{iff} \quad \mathfrak{T} \vDash \varphi' \quad \text{for every tree } \mathfrak{T}\,.$$

Proof. This is a special case of Theorem 1.2.13. Since each vertex has at most one predecessor one can code a set of edges by the set of their second components. This way each quantifier over sets of edges can be replaced by a monadic quantifier. □

Theorem 6.5.3. *There exists a tree-interpretable tree which is not* $\mathrm{GSO}(\exists^{\aleph_0})$*-axiomatisable.*

Proof. The tree $K_{1,\aleph_0}$ is tree interpretable and the preceding lemmas imply that $K_{1,\aleph_0} \equiv_{\mathrm{GSO}(\exists^{\aleph_0})} K_{1,\aleph_1}$. □

This shows that we cannot do without all cardinality quantifiers even if we allow infinitely many axioms. But do we really need non-monadic second-order quantifiers?

Open Problem. Are there tree-interpretable structures that are not (finitely) $\mathrm{MSO}(\exists^\kappa)$-axiomatisable?

6.6 Applications

Proposition 6.6.1. *Let $\mathfrak{M} = \mathrm{val}(T)$ for some $Y_{C,\tau}$-term $T \subseteq 2^{<\omega}$. If $\mathfrak{M}$ is finitely* $\mathrm{MSO}(\exists^\kappa)$*-axiomatisable then $\mathfrak{M}$ is tree interpretable. The same holds for $Y^<_{C,\tau}$-terms.*

Proof. Let $\varphi \in \mathrm{MSO}(\exists^\kappa)$ be the axiom for $\mathfrak{M}$. By Proposition 3.1.9, there exists an MSO-interpretation $\mathcal{V}$ such that $\mathcal{V}(T) = \mathrm{val}(T)$ for every $Y_{C,\tau}$-term T. The class of terms T with $\mathrm{val}(T) = \mathfrak{M}$ is finitely $\mathrm{MSO}(\exists^\kappa)$-axiomatisable since

$$T \vDash \varphi^{\mathcal{V}} \quad \text{iff} \quad \mathcal{V}(T) = \mathfrak{M} \vDash \varphi\,.$$

Thus, the set of such trees forms a regular tree language and contains a regular tree T_0. $T_0 \leq_{\mathrm{MSO}} \mathfrak{T}$ implies $\mathfrak{M} = \mathrm{val}(T_0) \leq_{\mathrm{MSO}} \mathfrak{T}$. □

Corollary 6.6.2 (Courcelle [20, 22]). *Let $\mathfrak{M}$ be a countable structure of finite tree width. $\mathfrak{M}$ is finitely* $\mathrm{MSO}(\exists^\kappa)$*-axiomatisable if and only if it is HR-equational.*

Lemma 6.6.3. *There is a $Y_{C,\tau}$-term $T \subseteq 2^{<\omega}$ such that $\mathfrak{M} := \mathrm{val}(T)$ is finitely* GSO*-axiomatisable but not tree interpretable.*

Proof. Let $\mathfrak{M} := (\mathbb{N}, \leq, E, P)$ where $E := \mathbb{N} \times \mathbb{N}$ and $P \subseteq \mathbb{N}$ is any arithmetical but non-recursive set. Then $\mathfrak{M}$ is not tree interpretable.

On the other hand, because of E the expressive power of GSO equals full second-order logic. Thus, addition and multiplication are definable in $\mathfrak{M}$ and we can construct an axiom using the FO-definition of P in $(\mathbb{N}, +, \cdot)$.

Finally, there exists a term denoting $\mathfrak{M}$ since $\mathcal{I} : (\mathbb{N}, \leq, E) \leq_{\mathrm{MSO}} \mathfrak{T}_2$ implies $(\mathbb{N}, \leq, E, P) \leq_{\mathrm{MSO}} (\mathfrak{T}_2, \mathcal{I}^{-1}(P))$. □

Lemma 6.6.4. *Let $\mathfrak{M}$ be a tree-interpretable structure and $a \in M$. The orbit O of a under automorphisms is* $\mathrm{GSO}(\exists^\kappa)$*-definable.*

Proof. If $\mathfrak{M}$ is tree interpretable then so is $(\mathfrak{M}, a)$. Let $\varphi(x)$ be the $\mathrm{GSO}(\exists^\kappa)$-formula obtained from the axiom of $(\mathfrak{M}, a)$ by replacing every occurrence of the constant a by the variable x. It follows that

$$b \in O \quad \text{iff} \quad (\mathfrak{M}, b) \cong (\mathfrak{M}, a) \quad \text{iff} \quad \mathfrak{M} \models \varphi(b)\,. \qquad \square$$

Lemma 6.6.5 (Pélecq [59]). *Let $\mathfrak{M}$ be a tree-interpretable structure of finite tree width and let O be the orbit of $a \in M$ under automorphisms. Then $(\mathfrak{M}, O)$ is tree interpretable.*

Proof. O is $\mathrm{GSO}(\exists^\kappa)$-definable by the preceding lemma. Since $\mathfrak{M}$ is of finite tree width it follows that O is even $\mathrm{MSO}(\exists^\kappa)$-definable and, therefore, $(\mathfrak{M}, O) \leq_{\mathrm{MSO}} \mathfrak{T}_2$. □

Theorem 6.6.6 (Courcelle [22]). *Given two tree-interpretable structures $\mathfrak{M}$ and $\mathfrak{N}$ of finite tree width one can decide whether $\mathfrak{M} \cong \mathfrak{N}$.*

Proof. Although not stated explicitly, the construction of the axiom in the previous section is effective. Thus, in order to determine whether $\mathfrak{M} \cong \mathfrak{N}$ one can construct the $\mathrm{GSO}(\exists^\kappa)$-formula $\varphi_{\mathfrak{M}}$ which axiomatises $\mathfrak{M}$ and check whether $\mathfrak{N} \models \varphi_{\mathfrak{M}}$. □

Open Problem. Is isomorphism decidable for all tree-interpretable structures?

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Symbol Index

Chapter 1

$[n]$ $\{0, \dots, n-1\}$, 1
$\beth_\alpha(\kappa)$ iterated exponentiation, 1
$\mathrm{Th}_{\mathfrak{L}}(\mathfrak{M})$ $\mathfrak{L}$-theory, 2
$\preceq_{\mathfrak{L}}$ $\mathfrak{L}$-elementary extension, 2
$\equiv_{\mathfrak{L}}$ $\mathfrak{L}$-equivalence, 2
φ^X relativisation, 2
$\varphi^{\mathfrak{M}}$ defined relation, 2
$\kappa^{<\alpha}$ tree, 2
ε empty sequence, 2
$x \sqcap y$ common prefix, 2
x/k prefix of x, 2
$\leq_{\mathrm{lex}}$ lexicographic order, 2
$\leq_{\mathrm{ll}}$ length-lexicographic order, 2
$\kappa \to (\mu)^\lambda_\chi$ homogeneous partition, 2
$R(m)^d_p$ Ramsey number, 3
$\mathcal{G}(\mathfrak{M})$ Gaifman graph, 4
$\mathfrak{M}^{\mathcal{I}}$ incidence structure, 4
$\mathrm{twd}\,\mathfrak{M}$ tree width, 5
$\mathfrak{N} \leq_{\mathfrak{L}} \mathfrak{M}$ interpretation, 8
$\mathfrak{M} \times \mathfrak{N}$ product, 9
$\mathfrak{M}^*$ iteration, 11
$\bigcup_{i \in \mathfrak{I}} \mathfrak{M}_i$ generalised sum, 14
$\langle\!\langle \varphi; \bar{S} \rangle\!\rangle_{\mathfrak{M}}$ function defined by φ, 24

Chapter 2

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ϑ_a merging of vertices, 37
$\oplus_S$ disjoint union, 37
$\mathrm{cwd}\,\mathfrak{M}$ clique width, 38
$\Delta(\mathfrak{M})$ degree, 46
$\beta(\mathfrak{M})$ size of $K_{\kappa,\kappa}$, 47

Chapter 3

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$\bigcup_i^\Theta$ symmetric sum, 53
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$\Upsilon_{C,\tau}$ signature with symmetric sum, 54
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tp_Δ Δ-type, 58
etp_Δ external Δ-type, 58
mtp_Δ monadic Δ-type, 58
emtp_Δ monadic external Δ-type, 58
$\approx^\Delta_U$ type equivalence, 59
$\simeq^\Delta_U$ type equivalence, 59
ti^n_Δ type index, 59
eti^n_Δ external type index, 59
mti^n_Δ monadix type index, 59
emti^n_Δ monadic external type index, 59
$\mathrm{pwd}_n\,\mathfrak{M}$ partition width, 59
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$\sim$	syntactic congruence, 152
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$\sim_n$	n-type equivalence, 163
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Chapter 6

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$\chi_0 \sqcup \chi_1$	common refinement, 179
$\downarrow x$	ideal generated by x, 181
$\uparrow x$	filter generated by x, 181

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