Axiomatising tree-interpretable structures

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Abstract Generalising the notion of a prefix-recognisable graph to arbitrary relational structures we introduce the class of tree-interpretable structures. We prove that every tree-interpretable structure is finitely axiomatisable in guarded second-order logic with cardinality quantifiers.

1 Introduction

In recent years the investigation of algorithmic properties of *infinite* structures has become an established part of computer science. Its applications range from algorithmic group theory to databases and automatic verification. Infinite databases, for example, were introduced to model geometric and, in particular, geographical data (see [16] for an overview). In the field of automatic verification several classes of infinite transition systems and corresponding model-checking algorithms have been defined. For instance, model-checking for the modal μ-calculus over prefix-recognisable graphs is studied in [6], [17]. A further point of interest in this context is the bisimulation equivalence of such transition systems as considered in [22], [23].

Obviously, only restricted classes of infinite structures are suited for such an approach. In order to process a class $\mathcal{K}$ of infinite structures by algorithmic means two conditions must be met:

(i) Each structure $\mathfrak{A} \in \mathcal{K}$ must possess a *finite representation*.
(ii) The operations one would like to perform must be *effective* with regard to these representations.

One fundamental operation demanded by many applications is the evaluation of a query, that is, given a formula $\varphi(\bar{x})$ in some fixed logic and the representation of a structure $\mathfrak{A} \in \mathcal{K}$ one wants to compute a representation of the set $\varphi^{\mathfrak{A}} := \{\, \bar{a} \mid \mathfrak{A} \models \varphi(\bar{a}) \,\}$. Slightly simpler is the model-checking problem which asks whether $\mathfrak{A} \models \varphi(\bar{a})$ for some given $\bar{a}$. The class of tree-interpretable structures investigated in the present article has explicitly been defined in such a way that model-checking for MSO, monadic second order logic, is decidable. To the authors knowledge it is one of the largest natural classes with this property.

Several different notions of infinite graphs and structures have been considered in the literature:

- *Context-free graphs* [19], [20] are the configuration graphs of pushdown automata.
- *HR-equational graphs* [8] are defined by equations of hyperedge-replacement grammars.

- *Prefix-recognisable graphs* have been introduced in [7]. Several characterisations are presented in Section 3.
- *Automatic graphs* [15], [3], [5] are graphs whose edge relation is recognised by synchronous multihead automata.
- *Rational graphs* [15], [18] are graphs whose edge relation is recognised by asynchronous multihead automata.
- *Recursive graphs* [13] are graphs whose edge relation is recursive.

These classes of graphs form a strict hierarchy. The table to the right shows for which logic model-checking is still decidable for the various classes. $\mathrm{FO}(\exists^\kappa)$, $\mathrm{MSO}(\exists^\kappa)$, and $\mathrm{GSO}(\exists^\kappa)$ denote, respectively, first-order logic, monadic second-order logic, and guarded second-order logic extended by cardinality quantifiers. Σ_0 is the set of quantifier-free first-order formulae.

Class	Logic
context-free	$\mathrm{GSO}(\exists^\kappa)$
HR-equational	$\mathrm{GSO}(\exists^\kappa)$
prefix-recognisable	$\mathrm{MSO}(\exists^\kappa)$
automatic	$\mathrm{FO}(\exists^\kappa)$
rational	Σ_0
recursive	Σ_0

When investigating a class of finitely presented structures the question naturally arises which structures it contains. Usually it is quite simple to show that some structure belongs to the class by constructing a corresponding presentation. But the proof that such a presentation does not exists frequently requires more effort.

One possible approach consists in determining what additional information is needed in order to extract the presentation from a given structure. In the case of a tree-interpretable structure this information can be coded into a colouring of its elements and edges. A characterisation of these colourings amounts to one of the set of presentations of a structure. Besides determining whether a presentation exists such a characterisation can, for instance, be used to investigate the automorphism group of the structure.

In the present article we generalise the class of prefix-recognisable graphs to arbitrary relational structures and prove that each presentation corresponds to a $\mathrm{GSO}(\exists^k)$-definable colouring. This implies that each such structure is finitely axiomatisable in this logic. The outline of the article is as follows. Due to space constraints some parts had to be omitted. The full version appears in [4].

In Section 3 we review several characterisations of the class of prefix-recognisable graphs including characterisations in terms of languages, graph grammars, and interpretations.

The latter can be generalised to arbitrary relational structures most easily. The resulting class of tree-interpretable structures is defined in Section 4. After summarising some of its properties we also extend the characterisation via regular languages to this class.

Section 5 is devoted to the study of paths in tree-interpretable graphs. The presented results are mostly of a combinatorial nature and culminate in the proof that every connected component is spanned by paths with a certain property.

In Section 6 we prove our main theorem which states that all tree-interpretable structures are finitely axiomatisable in guarded second-order logic with cardinality quantifiers. We also show that the cardinality quantifiers are indeed needed.

Section 8 concludes the article with some lemmas about the orbits of the automorphism group of a tree-interpretable structure and the result that isomorphism is decidable for tree-interpretable structures of finite tree-width.

2 Preliminaries

Automata and trees. Let Σ be an alphabet. The complete tree over Σ is the structure $\mathfrak{T}_\Sigma := (\Sigma^*, (\mathrm{suc}_a)_{a\in\Sigma}, \preceq)$ where the suc_a denote the successor functions and $\preceq$ is the prefix-order. The longest common prefix of u and v is denoted by $u \sqcap v$. If $u = vw$ then we define $v^{-1}u := w$ and $uw^{-1} := v$.

For $u \in \Sigma^*$ and $k \in \mathbb{N}$ we write u/k for the prefix of u of length $|u| - k$, and $\mathrm{suf}_k\, u$ for the suffix of u of length k. In case $|u| < k$ we have $u/k = \varepsilon$ and $\mathrm{suf}_k\, u = u$. In particular, $(u/k)\,\mathrm{suf}_k\, u = u$ for all u and k.

Let $\leq_{\mathrm{lex}}$ be the lexicographic order and $\leq_{\mathrm{ll}}$ the length-lexicographic one defined by

$$\begin{aligned} x \leq_{\mathrm{lex}} y \quad &:\text{iff} \quad x \preceq y, \text{ or } wc \preceq x \text{ and } wd \preceq y \text{ for some } w \text{ and } c < d. \\ x \leq_{\mathrm{ll}} y \quad &:\text{iff} \quad |x| < |y|, \text{ or } |x| = |y| \text{ and } x \leq_{\mathrm{lex}} y. \end{aligned}$$

We denote automata by tuples $(Q, \Sigma, \Delta, q_0, F)$ with set of states Q, alphabet Σ, transition relation Δ, initial state q_0, and acceptance condition F.

Logic. Let us recall some basic definitions and fix our notation. Let $[n] := \{0, \dots, n-1\}$. We tacitly identify tuples $\bar{a} = a_0 \dots a_{n-1} \in A^n$ with functions $[n] \to A$ or with the set $\{a_0, \dots, a_{n-1}\}$. This allows us to write $\bar{a} \subseteq \bar{b}$ or $\bar{a} = \bar{b}|_I$ for $I \subseteq [n]$.

MSO, *monadic second-order logic*, extends first-order logic FO by quantification over sets. In *guarded second-order logic*, GSO, one can quantify over relations R of arbitrary arity with the restriction that every tuple $\bar{a} \in R$ is *guarded*, i.e., there is some relation S of the original structure that contains a tuple $\bar{b} \in S$ such that $\bar{a} \subseteq \bar{b}$. Note that every singleton a is guarded by $a = a$. For a more detailed definition see [14].

$\mathfrak{L}(\exists^\kappa)$ denotes the extension of the logic $\mathfrak{L}$ by *cardinality quantifiers* $\exists^\lambda$, for every cardinal λ, where $\exists^\lambda$ stands for "there are at least λ many".

A formula $\varphi(\bar{x})$ where each free variable is first-order defines on a given structure $\mathfrak{A}$ the relation $\varphi^{\mathfrak{A}} := \{\, \bar{a} \mid \mathfrak{A} \models \varphi(\bar{a}) \,\}$.

Definition 1. Let $\mathfrak{A} = (A, R_0, \dots, R_n)$ and $\mathfrak{B}$ be relational structures. A (one-dimensional) MSO-*interpretation* of $\mathfrak{A}$ in $\mathfrak{B}$ is a sequence

$$\mathcal{I} = \big\langle \delta(x),\ \varepsilon(x,y),\ \varphi_{R_0}(\bar{x}), \dots,\ \varphi_{R_n}(\bar{x}) \big\rangle$$

of MSO-formulae such that $\mathfrak{A} \cong \big(\delta^{\mathfrak{B}},\ \varphi^{\mathfrak{B}}_{R_0}, \dots,\ \varphi^{\mathfrak{B}}_{R_n}\big)/\varepsilon^{\mathfrak{B}}$. To make this expression well-defined we require that the relation $\varepsilon^{\mathfrak{B}}$ is a congruence of the structure $\big(\delta^{\mathfrak{B}},\ \varphi^{\mathfrak{B}}_{R_0}, \dots,\ \varphi^{\mathfrak{B}}_{R_n}\big)$. We denote the fact that $\mathcal{I}$ is an MSO-interpretation of $\mathfrak{A}$ in $\mathfrak{B}$ by $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{MSO}} \mathfrak{B}$ or $\mathfrak{A} = \mathcal{I}(\mathfrak{B})$.

The epimorphism $\big(\delta^{\mathfrak{B}},\ \varphi^{\mathfrak{B}}_{R_0}, \dots,\ \varphi^{\mathfrak{B}}_{R_n}\big) \to \mathfrak{A}$ is called *coordinate map* and also denoted by $\mathcal{I}$. If it is the identity function we say that $\mathfrak{A}$ is *definable* in $\mathfrak{B}$.

3 Prefix-recognisable graphs

Originally, the investigation of tree-interpretable structures was concerned only with transition systems. This subclass appears in the literature under several names using widely different definitions which all turned out to be equivalent. They are summarised in the next theorem. A more detailed description follows below.

Theorem 2. *Let $\mathfrak{G} = (V, (E_a)_{a \in A})$ be a graph. The following statements are equivalent:*

(1) *$\mathfrak{G}$ is prefix-recognisable.*
(2) *$\mathfrak{G} = h^{-1}(\mathfrak{T}_2)|_C$ for a rational substitution h and a regular language C.*
(3) *$\mathfrak{G}$ is the restriction to a regular set of the configuration graph of a pushdown automaton with ε-transitions.*
(4) *$\mathfrak{G}$ is* MSO*-interpretable in the binary tree $\mathfrak{T}_2$.*
(5) *$\mathfrak{G}$ is VR-equational.*

The equivalence of the first two items are due to Caucal [7], Stirling [23] mentioned the third characterisation, and Barthelmann [1] delivered the last two.

Definition 3. A graph is *prefix-recognisable* if it is isomorphic to a graph of the form $(S, (E_a)_{a \in A})$ where S is a regular language over some alphabet Σ and each E_a is a finite union of relations of the form

$$W(U \times V) := \{ (wu, wv) \mid u \in U,\ v \in V,\ w \in W \}$$

for regular languages U, V, $W \subseteq \Sigma^*$.

Actually in the usual definition the reverse order $(U \times V)W$ is used. The above formulation was chosen as it fits better to the usual conventions regarding trees.

Example. The structure $(\omega, \mathrm{suc}, \leq)$ is prefix-recognisable. If we represent the universe by a^* the relations take the form $\mathrm{suc} = a^*(\varepsilon \times a)$ and $\leq\, = a^*(\varepsilon \times a^*)$.

One can also characterise prefix-recognisable graphs via graph grammars. Using the notation of Courcelle [8], [10], [12] we consider the following operations on vertex-coloured graphs. Let C be a finite set of colours.

- $G + H$ is the disjoint union of G and H.
- $\varrho_\beta(G)$, for $\beta : C \to C$, changes the colour of the vertices from a to $\beta(a)$.
- $\eta^a_{b,c}(G)$ adds a-edges from each b-coloured vertex to all c-coloured ones.
- a denotes the graph with a single a-coloured vertex.

The clique-width of a graph $\mathfrak{G}$ is, by definition, the minimal number of colours one needs to write a term denoting $\mathfrak{G}$.

Definition 4. A countable coloured graph is *VR-equational* if it is the canonical solution of a finite system of equation of the form

$$x_0 = t_0, \quad \ldots, \quad x_n = t_n$$

where the t_i are finite terms build up from the above operations. Further, we require that none of the t_i equals a single variable x_k.

Proposition 1 (Barthelmann [1]). *A graph is prefix-recognisable if and only if it is VR-equational.*

Since only finitely many colours can be used in a finite system of equations it follows that the clique-width of each VR-equational graph is finite.

Corollary 1. *Each prefix-recognisable graph is of finite clique-width.*

Example. If we colour the first element by a and the other ones by b we can define $(\omega, \mathrm{suc}, <)$ by

$$x_0 = \eta^{<}_{a,b}(x_1), \qquad x_1 = \varrho_{c\to b}\eta^{\mathrm{suc}}_{a,c}(a + x_2), \qquad x_2 = \varrho_{a\to c}(x_0).$$

4 Tree-interpretable structures

The characterisation of prefix-recognisable graphs in terms of interpretations is the one most easily generalised to arbitrary relational structures.

Definition 5. A structure $\mathfrak{A}$ is called *tree-interpretable* iff $\mathfrak{A} \leq_{\mathrm{MSO}} \mathfrak{T}_2$.

From this definition one can immediately deduce some basic properties of the class of tree-interpretable structures.

Proposition 2. *The class of tree-interpretable structures is closed under* MSO-*interpretations. In particular, it is closed under*

(1) *isomorphisms,*
(2) *definable expansions,*
(3) *finite unions,*
(4) *expansion by finitely many constants,*
(5) *factorisation by definable congruences, and*
(6) *substructures with definable universe.*

Since MSO($\exists^\kappa$) model checking is decidable for $\mathfrak{T}_2$ and this property is conserved by MSO-interpretations we obtain a decidability result for all tree-interpretable structures.

Proposition 3. MSO($\exists^\kappa$) *model checking is decidable for every tree-interpretable structure.*

All tree-interpretable graphs are of finite clique-width. On the other hand, their tree-width can be unbounded as the example of the infinite clique $K_{\aleph_0}$ shows. A result of Courcelle [11] which was extended to tree-interpretable graphs by Barthelmann [2] shows that being of finite tree-width imposes a strong restriction on the structure of a tree-interpretable graph. Although stated only for graphs it also holds for arbitrary structures if one replaces $\mathfrak{G}$ by its Gaifman graph in (2)–(4).

Proposition 4 (Barthelmann [2], Courcelle [11]). *Let $\mathfrak{G}$ be a tree-interpretable graph. The following statements are equivalent:*

(1) *$\mathfrak{G}$ is HR-equational.*
(2) *$\mathfrak{G}$ has finite tree-width.*
(3) *$K_{n,n}$ is not a subgraph of $\mathfrak{G}$ for some $n < \aleph_0$.*
(4) *$\mathfrak{G}$ is uniformly sparse.*

This characterisation allows us to extend Proposition 3 to GSO($\exists^\kappa$).

Theorem 6. *Let $\mathfrak{A}$ be a tree-interpretable structure.* GSO *model checking is decidable for $\mathfrak{A}$ if and only if $\mathfrak{A}$ is of finite tree-width. The same holds for* GSO($\exists^\kappa$).

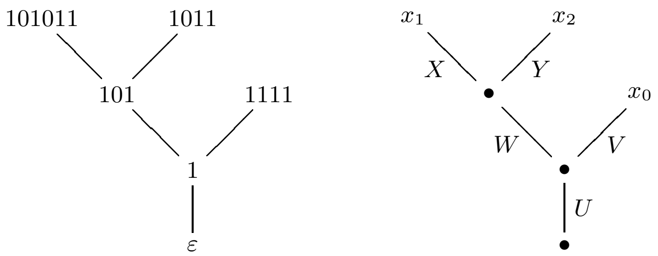

Figure 1. The branching structure of 1111, 1011, 101011 and its isomorphism type

Although the characterisation of tree-interpretable structures by interpretations is quite elegant, in actual proofs it is most of the time easier to work with a more concrete characterisation in terms of languages. Let us recall how automata are used to decide $\mathrm{MTh}(\mathfrak{T}_2)$ (see [24] for an overview).

Definition 7. For sets $X_0, \dots, X_{n-1} \subseteq \{0,1\}^*$ let $T_{\bar{X}}$ be the $\mathscr{P}([n])$-labelled binary tree with $T(w) := \{\, i < n \mid w \in X_i \,\}$ for $w \in \{0,1\}^*$. For singletons $X_i = \{x_i\}$ we also write $T_{\bar{x}}$.

With this notation we can now state Rabin's famous tree theorem in the following way:

Theorem 8. *For each* MSO*-formula* $\varphi(\bar{X}, \bar{x})$ *there is a tree-automaton* $\mathcal{A}$ *that recognises the language* $\{\, T_{\bar{X}\bar{x}} \mid \mathfrak{T}_2 \models \varphi(\bar{X}, \bar{x}) \,\}$.

Employing this correspondence we generalise the characterisation of prefix-recognisable graphs by relations of the form $W(U \times V)$ to arbitrary relational structures.

Definition 9. The *branching structure* of $x_0, \dots, x_{n-1} \in \Sigma^*$ is the partial order $(X, \preceq)$ where $X := \{\varepsilon\} \cup \{\, x_i \sqcap x_j \mid i,\ j < n \,\}$. The elements of X are called *branching points.*

Example. The branching structure of 1111, 1011, 101011 is depicted in Figure 1.

Note that for a fixed number of words there are only finitely many non-isomorphic branching structures.

Proposition 5 (Blumensath [4])**.** *An n-ary relation $R \subseteq (\{0,1\}^*)^n$ is* MSO*-definable in $\mathfrak{T}_2$ iff R is a finite union of relations R_i of the following form:*

(a) *All tuples $\bar{x} \in R_i$ have the same branching structure* (*up to isomorphism*)*.*

(b) *For each pair of adjacent branching points u, v there is a regular language $W_{u,v}$ such that $\bar{x} \in R_i$ if and only if for each such pair u, v the word $u^{-1}v$ belongs to $W_{u,v}$.*

Example. For the branching structure in Figure 1, a relation would be defined by five regular languages U, V, W, X, and Y with $R = U(V \times W(X \times Y))$.

Definition 10. Let $\mathfrak{A}$ be a tree-interpretable structure. Fixing an interpretation we can assume that the universe $A \subseteq \Sigma^*$ is regular and each relation R is specified by regular languages as in the preceding proposition. The *syntactic* *congruence* $\sim$ of $\mathfrak{A}$ (w.r.t. this interpretation) is the intersection of the syntactic congruences of all these languages. We denote the index of $\sim$ by I.

If some elements of a tree-interpretable structure are encoded by several words it becomes difficult to apply pumping arguments since the words obtained by pumping may encode the same element. Fortunately, for each tree-interpretable structure $\mathfrak{A}$ we can choose an interpretation where this does not happen.

Proposition 6 (Blumensath [4]). *If $\mathfrak{A} \leq_{\mathrm{MSO}} \mathfrak{T}_2$ then there is an interpretation $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{MSO}} \mathfrak{T}_2$ where the coordinate map is injective.*

This result allows us to identify the elements of a tree-interpretable structure with the unique word encoding them. We will do so tacitly in the remainder of the article. We conclude this section with a combinatorial lemma whose proof is based on a pumping argument.

Lemma 1. *Let $\mathfrak{A}$ be a tree-interpretable structure and $\varphi(x, y) \in \mathrm{MSO}(\exists^\kappa)$ such that, for every $a \in A$, there are only finitely many elements $b \in A$ with $\mathfrak{A} \models \varphi(a, b)$. There is a constant k such that $\varphi(a, b)$ implies $b/k \prec a$. In particular, $|\varphi(a, A)| \in \mathcal{O}(|a|)$.*

5 Paths in tree-interpretable graphs

In this section we consider a fixed tree-interpretable graph $\mathfrak{G} = (V, E_0, \dots, E_{r-1})$. By replacing each edge relation $E_a = \bigcup_i W_i(U_i \times V_i)$ by several relations $E_a^i := W_i(U_i \times V_i)$ we may assume that $E_a = W_a(U_a \times V_a)$ for regular languages $U_a, V_a, W_a \subseteq \Sigma^*$. We also add the relation $E_{a^-} := (E_a)^{-1}$ for each edge relation E_a. Note that these operations do not affect the syntactic congruence $\sim$.

Definition 11. (1) The *base-point* of an edge $(a, b) \in W(U \times V)$ is the longest word w contained in W such that $w^{-1}a \in U$ and $w^{-1}b \in V$. The *spine* of a path is the sequence of the base-points of its edges.

(2) A sequence $a_0, \dots, a_n$ is *k-increasing* if $|a_j| \geq |a_i| - k$ for all $i < j$.

(3) A path $a_0, \dots, a_n$ with spine $w_0, \dots, w_{n-1}$ is called *k-normal* if the path and its spine are k-increasing and $a_i/k \preceq a_j$ for all $j \geq i$.

Proposition 7 (Blumensath [4]). *Let $\mathfrak{G}$ be a tree-interpretable graph. There is a constant K such that each connected component of $\mathfrak{G}$ contains a vertex v, called its* root*, such that there are K-normal paths from v to all other vertices of the component.*

6 Axiomatisations

Equipped with the combinatorial lemmas of the previous section we can present the main result of this article. Each tree-interpretable structure $\mathfrak{A}$ is finitely $\mathrm{GSO}(\exists^\kappa)$-axiomatisable, i.e., there is a $\mathrm{GSO}(\exists^\kappa)$-sentence $\psi_{\mathfrak{A}}$ such that $\mathfrak{B} \models \psi_{\mathfrak{A}}$ if and only if $\mathfrak{B} \cong \mathfrak{A}$. Actually, we will prove the slightly stronger statement that, for each tree-interpretable structure, there is a colouring of the guarded tuples such that the coloured structure is $\mathrm{MSO}(\exists^\kappa)$-axiomatisable. That is, the axiom consists of a sequence of existential non-monadic second-order quantifiers followed by an $\mathrm{MSO}(\exists^\kappa)$-formula.

The congruence colouring. The axiomatisation uses colourings of elements and pairs of elements that are of the following form:

Definition 12. (a) Let $\approx \subseteq \Sigma^* \times \Sigma^*$ be a congruence of finite index and let $k \in \mathbb{N}$. The $(\approx, k)$-*congruence colouring* $\chi^k_\approx$ maps words $x \in \Sigma^*$ to the pair

$$\chi^k_\approx(x) := \big([x/k]_\approx,\ \mathrm{suf}_k\, x\big)$$

and pairs $(x, y) \in \Sigma^* \times \Sigma^*$ to

$$\chi^k_\approx(x, y) := \big(\chi^k_\approx(w^{-1}x),\ \chi^k_\approx(w^{-1}y)\big)$$

where $w := x \sqcap y$.

(b) A $(\approx', k')$-colouring χ' *refines* the $(\approx, k)$-colouring χ if $\approx' \subseteq \approx$ and $k' \geq k$. We denote this fact by $\chi' \geq \chi$. The *common refinement* of a $(\approx_0, k_0)$-colouring χ_0 and a $(\approx_1, k_1)$-colouring χ_1 is the $(\approx_0 \cap \approx_1,\ \max\{k_0, k_1\})$-colouring denoted by $\chi_0 \sqcup \chi_1$.

Definition 13. The χ-*expansion* $(\mathfrak{A}, \chi)$ of a structure $\mathfrak{A}$ expands $\mathfrak{A}$ by unary and binary relations for each colour class where the binary colour classes consists only of pairs (x, y) which are guarded.

The restriction to guarded pairs is essential since GSO allows only quantification over relations of this form. Below we frequently will need to obtain the value $\chi(x, y)$ for pairs (x, y) which are not guarded. These values must be computed explicitly from available data. This is where k-normal paths come into play.

Lemma 2. *Let $\mathfrak{A}$ be a tree-interpretable structure, $\approx$ a congruence of finite index, and k a constant. The $\chi^k_\approx$-expansion $(\mathfrak{A}, \chi^k_\approx)$ of $\mathfrak{A}$ is also tree-interpretable.*

We say that a set P of vertices codes a path between x and y if every element of P except for x and y is connected to exactly two other elements in P whereas x and y are connected to exactly one. Clearly, not every path can be coded in this way. Fortunately, for our purposes it is sufficient that, if there is a k-normal path between two vertices, then we can obtain a codable k-normal path between them by removing some vertices.

Lemma 3. *Let $\mathfrak{G}$ be a graph, χ a $(\approx, k)$-congruence colouring, and c a colour of χ. There is an* MSO*-formula $\varphi_c(P, x, y)$ such that $(\mathfrak{G}, \chi) \models \varphi_c(P, x, y)$ if and only if P codes a k-normal path from x to y and $\chi((x \sqcap y)^{-1}y) = c$.*

Forests. We start slowly by first showing that forests are finitely axiomatisable. We regard forests as partial orders such that the elements below any given one form a finite linear order. To axiomatise a forest it is sufficient to state, for each vertex, the number of its immediate successors of a a given colour. Note that these numbers only depend on the colour of the vertex.

Theorem 14. *Let $\mathfrak{T} := (T, \leq)$ be a tree-interpretable forest. The structure $(\mathfrak{T}, \chi)$ is finitely* FO$(\exists^\kappa)$*-axiomatisable for all $\chi \geq \chi^I_\sim$.*

Partial-orders. The next step consists in extending the result to tree-interpretable partial orders $\mathfrak{A} := (A, \leq)$ for which there is a constant $n \in \mathbb{N}$ such that $x \leq y$ implies $x/n \preceq y/n$ for all $x, y \in A$. To do so we have to define a forest in $\mathfrak{A}$. When speaking of paths we always consider undirected paths in this section, i.e., we ignore the direction of the edges.

Definition 15. Let $x \sqsubseteq y$ iff $x/n \preceq y/n$ and there is an undirected $\leq$-path $z_0, \dots, z_m$ from x to y with $x/n \preceq z_i/n$ for all $i \leq m$. Further, define $x \equiv y$ iff $x \sqsubseteq y$ and $y \sqsubseteq x$.

It is easy to show that $(A, \sqsubseteq)/{\equiv}$ is a forest and, thus, axiomatisable. This fact can be used to prove the following result.

Proposition 8. *There is a congruence colouring χ_0 such that $(A, \sqsubseteq, \chi)$ is finitely* MSO($\exists^\kappa$)*-axiomatisable for every $\chi \geq \chi_0$.*

In order to transfer the axiomatisability result from $(A, \sqsubseteq)$ to $\mathfrak{A}$, we have to show that each of the structures is definable in the other one.

Lemma 4. (a) *$(A, \sqsubseteq, \chi)$ is* MSO*-definable in $(\mathfrak{A}, \chi)$ for all $\chi \geq \chi_\sim^K$.*
(b) *$(\mathfrak{A}, \chi)$ is* MSO*-definable in $(A, \sqsubseteq, \chi)$ for all colourings $\chi \geq \chi_\sim^n$.*

These results allow us to transfer the axiomatisability from $(A, \sqsubseteq, \chi)$ to $(\mathfrak{A}, \chi)$.

Theorem 16. *Let $\mathfrak{A} := (A, \leq)$ be a tree-interpretable partial-order and let $n \in \mathbb{N}$ be a constant such that $x \leq y$ implies $x/n \preceq y/n$ for all $x, y \in A$. There is a congruence colouring χ_0 such that $(\mathfrak{A}, \chi)$ is finitely* MSO($\exists^\kappa$)*-axiomatisable for every $\chi \geq \chi_0$.*

The general case. Finally, we consider an arbitrary tree-interpretable structure $\mathfrak{A}$. For the reduction to the previous case we define, as above, a partial order $\leq$ and show that the structures $(A, \leq)$ and $\mathfrak{A}$ are definable within each other.

Definition 17. Let $x \vdash y$ if $x/I \preceq y/I$ and the pair (x, y) is guarded. Let $\leq$ be the reflexive and transitive closure of $\vdash$.

Lemma 5. *$(A, \leq, \chi)$ is* MSO*-definable in $(\mathfrak{A}, \chi)$ for all colourings $\chi \geq \chi_\sim^I$.*

The proof of the converse is more involved and requires an investigation of the branching structure of a tuple.

Definition 18. Let $\bar{a}, \bar{b} \in A^n$. We say that $\bar{a}$ is a *reduct* of $\bar{b}$ iff

(1) the branching structures of $\bar{a}$ and $\bar{b}$ are the same,
(2) $\inf_\preceq \bar{a} \sim \inf_\preceq \bar{b}$,
(3) $(a_i \sqcap a_j)^{-1}(a_k \sqcap a_l) \sim (b_i \sqcap b_j)^{-1}(b_k \sqcap b_l)$ for all indices such that $a_i \sqcap a_j \preceq a_k \sqcap a_l$,
(4) $|a_i| < |\inf_\preceq \bar{a}| + nI$ for all $i < n$.

A tuple is called *reduced* if it is a reduct of itself.

Lemma 6. *If $\bar{a}$ is a reduct of $\bar{b}$ and $\bar{b} \in R$ then $\bar{a} \in R$.*

To check whether a tuple $\bar{a}$ belongs to a relation R we use the characterisation of Proposition 5. Hence we need to compute the $\sim$-class of $u^{-1}v$ for branching points u and v of $\bar{a}$.

Definition 19. Let $\bar{a} \in A^n$. The elements $b_{ik} \in A$, for $i, k < n$, *code the branching structure* of $\bar{a}$ if

(1) $b_{ii} = a_i$ for $i < n$,
(2) $b_{ik}/nI \prec a_i \sqcap a_k \preceq b_{ik}$ for all i, k, and
(3) if $a_i \sqcap a_k \prec a_i \sqcap a_l$ then $b_{ik} \vdash b_{il}$ for $i, k, l < n$.

Given b_{ik} and b_{il} we can compute the $\sim$-class of $(a_i \sqcap a_k)^{-1}(a_i \sqcap a_l)$. Hence, if we can show that such elements always exists and that they are definable, then we are almost done.

Lemma 7. (a) *For each branching structure $\mathfrak{X}$ there is a formula $\beta_{\mathfrak{X}}(\bar{x}, \bar{y})$ such that $(A, \leq, \chi) \models \beta(\bar{a}, \bar{b})$ if and only if the branching structure of $\bar{a}$ is $\mathfrak{X}$ and it is coded by $\bar{b}$.*

(b) *Let R be an n-ary relation of $\mathfrak{A}$ and $\bar{a} \in R$. There are elements $b_{ik} \in A$, $i, k < n$, coding the branching structure of $\bar{a}$.*

At last, we are able to prove the other direction.

Lemma 8. *The structure $(\mathfrak{A}, \chi)$ is* MSO*-definable in $(A, \leq, \chi)$ for every $\chi \geq \chi^{nI}_{\sim}$ where n is the maximal arity of relations of $\mathfrak{A}$.*

Theorem 20. *Let $\mathfrak{A}$ be a tree-interpretable structure. There is a congruence colouring χ_0 such that $(\mathfrak{A}, \chi)$ is finitely* $\mathrm{MSO}(\exists^\kappa)$*-axiomatisable for all $\chi \geq \chi_0$.*

The proof is completely analogous to the one of Theorem 16. Since $\mathrm{GSO}(\exists^\kappa)$ allows quantification over colourings χ we obtain as immediate corollary the following result.

Theorem 21. *Every tree-interpretable structure is finitely* $\mathrm{GSO}(\exists^\kappa)$*-axiomatisable.*

7 Lower bounds

We have shown that every tree-interpretable structure is finitely $\mathrm{GSO}(\exists^\kappa)$-axiomatisable. Of course, the question arises if we can do better. In this section we show that at least the quantifiers $\exists^{\aleph_0}$ and $\exists^{\aleph_1}$ are needed. Since all tree-interpretable structures are countable we obviously can do without the ones for higher cardinalities.

For a logic $\mathfrak{L}$ let $\mathfrak{L}_m$ denote the set of $\mathfrak{L}$-formulae of quantifier rank at most m where we count both first- and second-order quantifiers. The following statements about the expressivity of MSO_m and $\mathrm{MSO}_m(\exists^{\aleph_0})$ can easily be proved using the corresponding versions of the Ehrenfeucht-Fraïssé game.

Lemma 9. (a) *For every $m \in \mathbb{N}$ there exists a constant k such that two sets A and B are* MSO_m*-equivalent if and only if either $|A| = |B|$ or $|A|, |B| \geq k$.*

(b) *For every m there is some k such that two sets A and B are* $\mathrm{MSO}_m(\exists^{\aleph_0})$*-equivalent iff either $|A| = |B|$, or $k \leq |A|, |B| < \aleph_0$, or $|A|, |B| \geq \aleph_0$.*

(c) *Any two infinite sets are* $\mathrm{MSO}(\exists^{\aleph_0})$*-equivalent.*

Since $\mathrm{GSO}(\exists^\kappa)$ collapses to $\mathrm{MSO}(\exists^\kappa)$ on trees, this lemma implies that $K_{1,\aleph_0}$ and $K_{1,\aleph_1}$ are $\mathrm{GSO}(\exists^{\aleph_0})$-equivalent. But the former structure is tree-interpretable while the latter obviously is not.

Theorem 22. *There exists tree-interpretable trees which are not* $\mathrm{GSO}(\exists^{\aleph_0})$*-axiomatisable.*

This shows that we cannot do without all cardinality quantifiers even if we allow infinitely many axioms. But do we really need non-monadic second-order quantifiers?

Open Problem. Are there tree-interpretable structures which are not (finitely) $\mathrm{MSO}(\exists^\kappa)$-axiomatisable?

8 Automorphisms of tree-interpretable structures

As mentioned in the introduction the axiomatisation of a tree-interpretable structure can be used to investigate its automorphism group.

Lemma 10. *Let* $\mathfrak{A}$ *be a tree-interpretable structure and* $a \in A$*. The orbit* O *of* a *under automorphisms is* $\mathrm{GSO}(\exists^\kappa)$*-definable.*

Proof. If $\mathfrak{A}$ is tree-interpretable then so is $(\mathfrak{A}, a)$. Let $\varphi(x)$ be the $\mathrm{GSO}(\exists^\kappa)$-formula obtained from the axiom of $(\mathfrak{A}, a)$ by replacing every occurrence of the constant a by the variable x. It follows that

$$b \in O \quad \text{iff} \quad (\mathfrak{A}, b) \cong (\mathfrak{A}, a) \quad \text{iff} \quad \mathfrak{A} \models \varphi(b). \qquad \square$$

Lemma 11 (Pélecq [21])**.** *Let* $\mathfrak{A}$ *be a tree-interpretable structure of finite tree-width and let* O *be the orbit of* $a \in A$ *under automorphisms. Then* $(\mathfrak{A}, O)$ *is tree-interpretable.*

Proof. O is $\mathrm{GSO}(\exists^\kappa)$-definable by the preceding lemma. Since $\mathfrak{A}$ is of finite tree-width it follows that O is even $\mathrm{MSO}(\exists^\kappa)$-definable and, therefore, $(\mathfrak{A}, O) \leq_{\mathrm{MSO}} \mathfrak{T}_2$. □

We conclude this article with a simple application to the isomorphism problem.

Theorem 23 (Courcelle [9])**.** *Given two tree-interpretable structures* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of finite-tree width one can decide whether* $\mathfrak{A} \cong \mathfrak{B}$*.*

Proof. Although not explicitly stated, the construction of the axiom in the previous section is effective. Thus, in order to determine whether $\mathfrak{A} \cong \mathfrak{B}$ one can construct the $\mathrm{GSO}(\exists^\kappa)$-formula $\varphi_{\mathfrak{A}}$ which axiomatises $\mathfrak{A}$ and check whether $\mathfrak{B}$ satisfies $\varphi_{\mathfrak{A}}$. □

Open Problem. Is isomorphism decidable for all tree-interpretable structures?

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