The normal form of Borel sets
Part I : Borel sets of finite rank

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Résumé : Pour chaque ensemble borélien de réels A, de rang fini, nous donnons une "forme normale" pour A, en exhibant un ensemble borélien Ω de simplicité maximum tel que A et Ω se réduisent continuement l'un à l'autre. En termes plus techniques nous définissons des opérations boréliennes simples qui sont homomorphes aux opérations ordinales de somme, multiplication par un ordinal dénombrable, exponentiation de base ω_1, par la fonction qui envoie chaque ensemble borélien de rang fini A sur son degré de Wadge.

La forme normale des Boréliens de rang fini.

Abstract : For each Borel set of reals A, of finite rank, we obtain a "normal form" of A, by finding a Borel set Ω of maximum simplicity, such that A and Ω continuously reduce to each other. In more technical terms : we define simple Borel operations which are homomorphic to ordinal sum, to multiplication by a countable ordinal, and to ordinal exponentiation of base ω_1, under the map which sends every Borel set A of finite rank to its Wadge degree.

Version française abrégée. Par ensemble borélien de réels nous entendons un sous-ensemble A de Σ_A^ω, borélien pour la topologie produit de la topologie discrète sur Σ_A qui représente un alphabet au plus dénombrable. Etant donné deux tels ensembles A et B, A se réduit continuement à B (noté A$\leq_W$B) s'il existe une fonction continue f : $\Sigma_A^\omega \longmapsto \Sigma_B^\omega$ telle que $A = f^{-1}B$. Ce qui signifie que A est moins "compliqué" que B, strictement si B$\not\leq_W$A (noté A$<_W$B). Si $A \leq_W B \leq_W A$, on dit que A est <u>Wadge-équivalent</u> à B (noté A$\equiv_W$B). Lorsque A est non autodual, c'est-à-dire satisfait $A \equiv_W \bar{A}$ où $\bar{A}$ désigne le complémentaire de A, le degré de Wadge de A (noté d_W°A) est le plus grand ordinal α tel qu'il existe une suite $<_W$-croissante $\langle \Omega(\beta)/0 < \beta \leq \alpha \rangle$ de boréliens non autoduaux satisfaisant $\beta < \gamma \leq \alpha \Longrightarrow \Omega(\beta) <_W \Omega(\gamma) \leq_W A$; et pour A autodual c'est le plus petit degré de Wadge d'un ensemble non autodual B$>_W$A. Wadge et Martin ont montré que deux boréliens ont même degré de Wadge ssi ils sont Wadge-équivalents ou bien l'un à l'autre, ou bien l'un au complémentaire de l'autre. Dès lors la hiérarchie de Wadge constitue un raffinement de la hiérarchie borélienne puisque le type d'ordre qu'elle induit sur les boréliens de rang fini est ϵ^1 : le premier point fixe de l'opération ordinale d'exponentiation de base ω_1. Wadge tout d'abord, Louveau et Saint-Raymond ensuite, ont étudié le problème consistant à donner pour chaque degré de Wadge α la description d'un borélien $\Omega\,(\alpha)$ de degré précisément α. Pour le cas $\alpha < \epsilon^1$, nous donnons une nouvelle solution canonique en ce sens qu'elle délivre une définition borélienne de $\Omega\,(\alpha)$ qui est isomorphe à la description canonique de l'ordinal α lui-même. En fait, nous associons à chaque fonction ordinale suivante : la somme : $(\beta, \gamma) \longmapsto \beta + \gamma$, le produit : $(\beta, \nu) \longmapsto \beta.\nu$ (pour tout ν, $0 < \nu < \omega_1$) et l'exponentiation : $\beta \longmapsto \omega_1^\beta$, une opération

borélienne définie sur tout ensemble $A \subseteq \Sigma_A^{\leq\omega}$: somme : $(A,B) \longmapsto A \pm B$ - cf définition 1 -, multiplication : $(A,\nu) \longmapsto A \bullet \nu$ (ν, $0 < \nu < \omega_1$)- cf définition 2 - et exponentiation : $A \longmapsto (\pi, A)$- cf définition 3 -.
Ces opérations boréliennes sont en fait définies sur des sous-ensembles de $\Sigma_A^{\leq\omega}$, qui sont appelées conciliants par le fait qu'ils se prononcent non seulement sur l'acceptation ou le rejet d'ω-suites mais aussi sur ceux de suites finies. Nous définissons la notion de réduction conciliante ($\leq_C$) à l'aide de celle de jeu de Wadge : Wadge a en effet montré que A$\leq_W$B ssi il existe une stratégie gagnante pour le joueur II dans le jeu infini à information complète entre deux joueurs où chacun joue alternativement une lettre de l'alphabet, Σ_A pour I et Σ_B pour II, et pour lequel, après ω coups, I ayant produit $\alpha \in \Sigma_A^\omega$ et II $\beta \in \Sigma_A^\omega$, II gagne si $\alpha \in A \iff \beta \in B$, sinon I l'emporte. Dans la version conciliante de ce jeu, chaque joueur est autorisé à passer son tour aussi souvent qu'il le souhaite de sorte que les suites α,β obtenues en fin de partie peuvent être finies, les conditions d'acceptations restant les mêmes, seulement prolongées aux suites finies. On rejoint très facilement la hiérarchie de Wadge - cf theorem 1 - . En effet, $\forall A \subseteq \Sigma_A^{\leq\omega}$ le degré de A dans la hiérarchie basée sur ces réductions conciliantes est égal à d_W°A^b où b ("blanc") désigne une lettre qui n'est pas dans Σ_A, et $A^b = \{\alpha \in (\Sigma_A \cup \{b\})^\omega / \alpha \textit{ est une } \underline{\textit{variante}} \textit{ d'une suite } x \in A\}$; où α est une variante de x si c'est un mot infini se réduisant à x quand tous les "blancs" en sont ôtés.
Considérées du point de vue du joueur, les opérations boréliennes doivent être regardées de la manière suivante : la somme $(A \pm B)$ permet au joueur de débuter la partie avec B puis soit de continuer avec ce même ensemble, soit d'effacer ce qu'il a déjà joué et de choisir de jouer avec A ou $\bar{}$A, mais ce choix est irréversible. La multiplication dénombrable $(A \bullet \nu)$ est l'itération de cette opération appliquée à A, le joueur peut annuler son jeu et remplacer A par $\bar{}$A ou $\bar{}$A par A mais chaque remplacement peut être regardé comme nécessitant de décrémenter l'ordinal ν; ainsi un ordinal à chaque fois plus petit doit être produit lors de chaque remplacement ce qui limite ceux-ci au fini. L'exponentiation (π, A) est plus difficile à appréhender puisqu'elle diffère à la fin de la partie l'efficacité de la suite jouée, tout se passe comme si, jouant avec (π, A), le joueur jouait une infinité de jeux à la fois, chacun avec π, dont les résultats en fin de partie déterminent une suite qui sera lue (et acceptée ou non) par A.
Ces opérations permettent de définir pour chaque ordinal α, un ensemble $\Omega\,(\alpha)$ de degré α en considérant la "forme normale de Cantor itérée" de α et en évaluant ce polynome itéré d'ordinaux quand l'ordinal 1 est remplacé par l'ensemble vide et les opérations ordinales sont remplacées par leurs contreparties boréliennes - cf définition 4- . Le cas particulier $\Omega(\omega_1^{(n)})$, où $\omega_1^{(n)}$ est défini inductivement par $\omega_1^{(0)} = 1$ et $\omega_1^{(k+1)} = \omega_1^{\omega_1^{(k)}}$, est celui d'un ensemble $\mathbf{\Sigma}_{n+1}^\circ$-complet possèdant une définition explicite de forme $\{\beta \in 2^\omega / \exists v \exists ! i_1 \ldots \exists ! i_n \beta(< >) = 0\}$, de sorte que la suite de quantificateurs "$\exists \underbrace{\exists ! \ldots \exists !}_{n}$" révèle ainsi avoir le même pouvoir expressif que "$\exists \underbrace{\forall \exists \ldots Q}_{n}$".

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0. Introduction. By a <u>Borel set</u> we mean a Borel subset A of Σ_A^ω, where Σ_A is some atmost countable alphabet which does not depend on A, and the topology on Σ_A^ω is the product of the discrete topology on Σ_A. A <u>continuous reduction</u> of A to B is a continuous function f : $\Sigma_A^\omega \longrightarrow \Sigma_B^\omega$ such that $A = f^{-1}B$. It is denoted $A \leq_W B$; intuitively it means that A is less complicated than B -strictly if there is no continuous reduction of B to A; which is denoted $<_W$. If $A \leq_W B$ and $B \leq_W A$, in other words, A and B continuously reduce to each other, then A is <u>Wadge-equivalent</u> to B, denoted $A \equiv_W B$.
We recall the definition of a self-dual set : it is a set A that satisfies $A \equiv_W \bar{}A$ where $\bar{}$A stands for the complement of A. The Wadge degree of A, denoted d_W°A, for A non self-dual, is the

largest ordinal α such that there is an $<_W$-increasing sequence $\langle\Omega(\beta)/0<\beta\leq\alpha\rangle$ of Borel non self-dual sets which satisfies $\beta<\gamma\leq\alpha\Longrightarrow\Omega(\beta)<_W\Omega(\gamma)\leq_W A$; and for A self-dual, it is the least Wadge degree of non self-dual sets $<_W$-greater than A. Wadge and Martin proved that two Borel sets have same Wadge degree iff they are Wadge-equivalent or Wadge-equivalent to the complement of the other. Thus the Wadge degree of A is a measure of the complexity of A, just as the Borel rank of A. But the former is a much finer measure : to begin with, let ϵ^1 denote the first fixpoint of ordinal exponentiation of base ω_1 : $\epsilon^1=sup_{n<\omega}\omega_1^{(n)}$, where $\omega_1^{(0)}=1$, $\omega_1^{(n+1)}=\omega_1^{\omega_1^{(n)}}$. Then the degree of Borel sets of finite rank range from 1 to ϵ_1, while the Borel rank only goes from 0 to ω.

As a consequence, it is quite a problem to provide for each Wadge degree α a description of a Borel set $\Omega(\alpha)$ of degree precisely α. A solution was given essentially by Wadge [1, W84], and confirmed after ten years of work culminating with Louveau and Saint-Raymond [2, L 83][3, L-S 85]. We have given a new solution in the case $\alpha<\epsilon^1$, that is for Borel sets of finite rank (and we are preparing its extension to all Borel ranks). This solution is obviously canonical, and optimal in simplicity; its characteristic feature is : it gives a Borel definition of $\Omega(\alpha)$ which is isomorphic to the canonical description of the ordinal α itself.

1. Away with self-dual sets : the conciliating Wadge hierarchy. If $A\subseteq\Sigma_A^\omega$ et $B\subseteq\Sigma_B^\omega$, the Wadge game W(A,B) is designed so that A$\leq_W$B iff II has a winning strategy in the game where alternatively player I chooses a letter from Σ_A, player II a letter from Σ_B. I can't skip while II is allowed to skip even ω times, provided he also plays (a letter in Σ_B) ω times. Thus after ω moves, I chooses $\alpha\in\Sigma_A^\omega$ and II chooses $\beta\in\Sigma_B^\omega$; and II wins iff $\alpha\in A\longleftrightarrow\beta\in B$, otherwise it is I who wins. We define the conciliatory form C(A,B) of that game, which is this game where <u>both</u> players are allowed to skip, even indefinitely if they want so. Thus their final choices after ω moves : α,β, may be finite sequences. So we say that A, B are <u>conciliating</u> if they are considered as subsets of respectively $\Sigma_A^{\leq\omega}$, $\Sigma_B^{\leq\omega}$ and II wins the game C(A,B) between two such sets iff $\alpha\in A\longleftrightarrow\beta\in B$, otherwise it is I who wins. We write A$\leq_C$B iff II has a winning strategy in C(A,B); A$\equiv_C$B if A$\leq_C$B$\leq_C$A, and A$<_C$B if A$\leq_C$B <u>but</u> B$\not\leq_C$A. This defines the conciliatory hierarchy on Borel subsets of $\Sigma^{\leq\omega}$ for any at most countable Σ.

For the purpose of considering only non self-dual sets of finite rank, it is enough to replace sets of ω-sequences by sets of $\leq\omega$-sequences. This result is based on 3 facts :

1. In the conciliatory Wadge hierarchy, every conciliating set A is non self-dual : $A\not\equiv_C\bar{A}$.

2. The conciliatory Wadge hierarchy joins the previous one when sets are crammed with a specific letter "b" (for blanks) so as to turn them into sets of infinite sequences. That is, given a conciliating set $A\subseteq\Sigma_A^{\leq\omega}$, let "b" denote a letter not in Σ_A, and set $A^b=\{\alpha\in(\Sigma_A\cup\{b\})^\omega/\alpha$ *is a <u>variant</u> of some* $x\in A\}$; where α is a variant of x if it is, in this precise case, an infinite word reducing to x when all "blanks" are removed.
   Given A, B two conciliating sets, it is easy to see that a player wins C(A,B) iff the same player wins W(A^b,B^b); so, A$\leq_C$B iff $A^b\leq_W B^b$; and in particular, $A^b\not\equiv_W\bar{A}^b$

3. But we also get much more in a theorem which needs the whole present work to be proved :

   Theorem 1 *For every Borel set of finite rank* $A\in\Sigma_A^\omega$*, A is non self-dual iff there is an alphabet* Σ_B *and a conciliatory set* $B\subseteq\Sigma_B^{\leq\omega}$ *such that* $A\equiv_W B^b$.
   Furthermore, Σ_B *may be chosen finite or such that* Σ_B=Σ_A *and* $B\cap\Sigma_A^\omega=A$ *(so that B is just A extended on finite sequences).*

Now there's a trivial procedure which derives the structure of the self-dual classes from the non self-dual ones. So in the sequel, we concentrate on the conciliatory hierarchy and let self-dual classes take care of themselves.
We define the Wadge degree of a conciliating set A, denoted $d^\circ_C A$, as the largest ordinal α such that there is an $<_C$-increasing sequence $\langle \Omega(\beta)/0 < \beta \leq \alpha \rangle$ of conciliating sets which satisfies $\beta < \gamma \leq \alpha \Longrightarrow \Omega(\beta) <_c \Omega(\gamma) \leq_c A$. It is a consequence of theorem 1 that forall $B \subseteq \Sigma_B^{\leq\omega}$ if B^b is Borel of finite rank, then $d^\circ_C B = d^\circ_W B^b$.
Now since α is less than ϵ^1, α has an "iterated Cantor normal form of base ω_1" which describes α, using countable ordinals as parameters and only "arithmetical" operations on ordinals :
the sum : $(\beta,\gamma) \longmapsto \beta+\gamma$, the product : $(\beta,\nu) \longmapsto \beta.\nu$ (for each ν, $0<\nu<\omega_1$) and the exponentiation : $\beta \longmapsto \omega_1^\beta$. To each of these operations we associate an operation defined on conciliating sets : sum : $(A,B) \longmapsto A \pm B$, multiplication : $(A,\nu) \longmapsto A \bullet \nu$ (for each ν, $0<\nu<\omega_1$) and exponentiation : $A \longmapsto (\pi, A)$. Starting with $d^\circ_C \emptyset = 1$, we show :

Theorem 2 *Given* $A \subseteq \Sigma_A^{\leq\omega}$, $B \subseteq \Sigma_B^{\leq\omega}$,
1.) $d^\circ_C A \pm B = d^\circ_C A + d^\circ_C B$ 2.) $d^\circ_C A \bullet \nu = d^\circ_C A.\nu$ 3.) $d^\circ_C(\pi, A) = \omega_1^{d^\circ_C A+\varepsilon}$
Where $\varepsilon = -1, 0, 1$, *depending on the cofinality of* $\sup\{\lambda \leq d^\circ_C A/\lambda = 0 \text{ or } \lambda \text{ is limit}\}$.

For every $\alpha < \epsilon^1$, this yields the following way to define a Borel set $\Omega(\alpha)^b$ of degree α :
α has an iterated Cantor normal form : $\alpha = \sum_{i=o}^n \omega_1^{\xi_i}.\nu_i$, with $\epsilon^1 > \xi_0 > \ldots > \xi_n$ and $0 < \nu_i < \omega_1$; and where ξ_i itself is written in normal form, and so on. Then, in order to get $\Omega(\alpha)$, <u>evaluate</u> this "iterated polynomial of ordinals" when the ordinal 1 is replaced by the emptyset, and the ordinal operations are replaced by their above set theoretic counterparts. The above results (1), (2) and (3) guarantee that the resulting Borel set $\Omega(\alpha)^b$ is of degree α, provided that the correction introduced in (3) by the variable value $\varepsilon \in \{-1,0,1\}$ is duly taken into account.

2. The Borel counterpart of ordinal sum.

Definition 1 Let $A \subseteq \Sigma_A^{\leq\omega}$, $B \subseteq \Sigma_B^{\leq\omega}$, and $\forall i < \omega$, $A_i \subseteq \Sigma_{A_i}^{\leq\omega}$.

1. Define : $B \pm A = A \cup \Sigma_A^{<\omega}.\left(0.B^\triangleleft \cup 1.^-B^\triangleleft\right)$. Where 0 and 1 are letters not in Σ_A and $B^\triangleleft$ stands for the set of words which belong to B when every letter not in Σ_B is skipped.
2. Let $\forall i < \omega$, $A_i \subseteq \Sigma_{A_i}^{\leq\omega}$, choose two different letters 0,1. Define : $\sum_{i<\omega} \mathrm{A}_i = \bigcup_{i<\omega} 0^i.1.A_i^\triangleleft$.

Everything goes as if $B \pm A$ allows a player to begin the run in A and then to change one time his mind and go playing in B or $^-$B; but the choice is non reversible. While $\sum_{i<\omega} \mathrm{A}_i$ allows a player to choose the set A_i he wants to play with, unless he plays 0^ω which is rejected.

Theorem 3 *Let* $A \subseteq \Sigma_A^{\leq\omega}$, $B \subseteq \Sigma_B^{\leq\omega}$ *and* $\forall i < \omega$ $A_i \subseteq \Sigma_{A_i}^{\leq\omega}$ *and* $< A_i/i < \omega >$ *be* $<_C$-*unbounded.*
1.) $d^\circ_C \sum_{i<\omega} A_i = sup_{i<\omega} d^\circ_C A_i$ *2.)* $d^\circ_C A \pm B = d^\circ_C A + d^\circ_C B$

Given any countable alphabet $\Sigma_\emptyset$, remember that in our previous definition $d^\circ_C \emptyset$ is 1 so that $d^\circ_C(A \pm \emptyset) = d^\circ_C A + 1$.

3. The Borel counterpart of countable ordinal multiplication.

Definition 2 Let $A \subseteq \Sigma_A^{\leq\omega}$. Inductively define A$\bullet\nu$ for each $0 < \nu < \omega_1$:
A$\bullet$1=A, A$\bullet\xi+1$=A$\bullet\xi\pm$A and when λ is limit and λ=sup$_{n<\omega}\lambda_n$, A$\bullet\lambda$=$\sum_{n<\omega} A \bullet \lambda_n$.

The isomorphism between countable multiplication and $\bullet$ operation lays in next theorem :

Theorem 4 *Let* $A \subseteq \Sigma_A^{\leq\omega}$, $d^\circ_C A \bullet \nu = d^\circ_C A.\nu$

An important fact to note is the obvious following one : the operations specified above all preserve $\leq_C$. That is, for exemple, $A \leq_C A'$ and $B \leq_C B'$ implies $A \pm B \leq_C A' \pm B'$.
We now start straightaway enonciating weak separation theorems about the above operations.

Theorem 5 *Let* A, B, C *and all* A_i*'s be Borel sets,* $A \subseteq \Sigma_A^{\leq\omega}$, $B \subseteq \Sigma_B^{\leq\omega}$, $C \subseteq \Sigma_C^{\leq\omega}$, $\forall i < \omega$ $A_i \subseteq \Sigma_{A_i}^{\leq\omega}$ *and* $< A_i / i < \omega >$ *does not contain its* $<_C$*-upper bound; then :*

1. A *and* $\bar{}A \ \leq_C C \pm \emptyset \iff A \leq_C C$ *or* $\bar{}A \leq_C C$
2. $\emptyset <_C B$, $\ A$ *and* $\bar{}A \ \leq_C C \pm B \iff \exists D <_C B, A \leq_C C \pm D$
3. A *and* $\bar{}A \ \leq_C \sum_{i<\omega} A_i \iff \exists n < \omega,\ A \leq_C A_n$

4. The Borel counterpart of quasi-exponentiation of base ω_1. In order to complexify the sets we are soon going to build, we introduce an operation that reminds of automata reading words of length ω^2. In a different way it can be regarded as folded and unfolded games. Here comes the π-operation. It begins as a kind of a stretching of the $\leq \omega$-sequence played in order to obtain a ω^2-sequence, then each of its ω-subsequences of length ω is read by a π machine, who produces a letter, as to get at the end a more complicated ω-sequence than the initial one.

Definition 3 Given $\Sigma_A = \{a_i / i \in I\}$ with $|I| \leq \aleph_0$, let $\Sigma = \Sigma_A \cup \{\epsilon\}$ where ϵ is a letter not in Σ_A. For $\alpha \in \Sigma^{\leq\omega}$, if α is infinite, set $\alpha^\square = \alpha \circ K$, and if α is finite set $\alpha^\square = (\alpha \frown \epsilon^\omega) \circ K$, where K is the well-known bijection, $\omega^2 \longleftrightarrow \omega \ : \ K(n,m) = \frac{(n+m).(n+m+1)}{2} + n$ (any other bijection : $\omega^2 \longleftrightarrow \omega$ could be used). Let π denote the fonction : $\Sigma^\omega \longmapsto \Sigma$, that satisfies : $\pi(\beta) = a_i \iff \exists n < \omega,\ \beta(n) = a_i$ <u>and</u> $\forall m \neq n,\ \beta(m) = \epsilon$; and $\pi(\beta) = \epsilon$ otherwise. We define the ω-sequence $\pi[\alpha] \in \Sigma^\omega$ by : $\pi[\alpha] = \langle \pi\left(< \alpha \circ K(i,k) / k < \omega >\right) / i, \omega \rangle$. Finally we define $(\pi, A) \subseteq \Sigma^{\leq\omega}$:

$$\alpha \in (\pi, A) \iff \pi[\alpha] \in A^{\triangleleft} \ (\pi[\alpha] \textit{ belongs to } A \textit{ if the letters "}\epsilon\textit{" are skipped}).$$

In a certain sense, π is the least machine that always makes the folding and unfolding operations increase the strength of A. More precisely, it makes the Wadge conciliatory degree of A grow up to quasi exponentiation of base ω_1. The theorem below shows this matter of fact.

Theorem 6 *Let* $A \subseteq \Sigma_A^{\leq\omega}$ *and* $\forall i < \omega,\ A_i \subseteq \Sigma_{A_i}^{\leq\omega}$*:*

if $d^\circ_C A < \omega$ *:*	$d^\circ_C (\pi, A) = \omega_1^{d^\circ_C A - 1}$
if $d^\circ_C A = \lambda + n$ *(*λ *limit of cofinality* ω_1 *and* $n \in \omega$*) :*	$d^\circ_C (\pi, A) = \omega_1^{d^\circ_C A}$
if $d^\circ_C A = \lambda + n$ *(*λ *limit of cofinality* ω *and* $n \in \omega$*) :*	$d^\circ_C (\pi, A) = \omega_1^{d^\circ_C A + 1}$

The main result to prove this theorem is a device called the <u>remote-control theorem</u> which states that $(\pi, \_)$-operation preserves the conciliatory preorder.

Theorem 7 *Let* $A \subseteq \Sigma_A^{\leq\omega}$ *and* $B \subseteq \Sigma_B^{\leq\omega}$, $A \leq_C B \iff (\pi, A) \leq_C (\pi, B)$

It is named this way because given a winning strategy σ for player II in C(A,B), the proof goes providing a strategy σ_π winning for player II in C((π, A),(π, B)) which essentially carries σ so that in this game, the response of II using σ_π is very close to the one obtained in applying σ to what reads A. Applying σ_π to α gets essentially the same sequence as applying σ to $\pi[\alpha]$; thus σ_π comands the application of σ via operation π.

□

Theorem 7 has a relatively simple proof; but in order to prove theorem 6, one also has to prove a "remote control version" of the theorem 5 : A$<_C(\pi, B) \Longrightarrow$A$\leq_C(\pi, C)\bullet\nu$ for some C$<_C$B and $0 < \nu < \omega_1$. And the proof of that version combines the ideas and proofs from all results mentionned so far...
The combination of theorem 6 with theorem 3 proves the last point,(3), of the main result theorem 2. Thus it allows to define for every $\alpha < \epsilon_1$ a Borel set $\Omega(\alpha)$ of degree α, in the way explained in the introduction. Let us examine to what degree $\Omega(\alpha)$ is "canonical" in the sense of a "normal form" for Borel sets of degree α, by looking at the particular case $\alpha = \omega_1^{(n)}$: then $\Omega(\omega_1^{(n)}) = (\pi, \ldots, (\pi, \emptyset \bullet 2)...)$. It turns out that this $\mathbf{\Sigma}^\circ_{n+1}$-complete set is essentially the set $\{\beta \in 2^\omega / \exists v \exists! i_1 \ldots \exists! i_n \beta(< \ldots >) = 0\}$. To see this one simply has to make explicit the definitions of π and $\emptyset \bullet 2$ inside the definition of $(\pi, \ldots, (\pi, \emptyset \bullet 2)...)$. As a corollary the sequence of quantifiers "$\exists \underbrace{\exists! \ldots \exists!}_{n}$" has exactly the same expressive power as "$\exists \underbrace{\forall \exists \ldots Q}_{n}$" : it is $\mathbf{\Sigma}^\circ_{n+1}$-complete and the normal form of $\mathbf{\Sigma}^\circ_{n+1}$ sets which are the sets of degree $\omega_1^{(n)}$ provided by our set $\Omega(\omega_1^{(n)})$ is this "$\exists\exists! \ldots \exists!$" form - a canonical form indeed -. Moreover, the construction of $\Omega(\alpha)$ is uniform in α; so implicit in the definition of every $\Omega(\alpha)$,there is a similar canonical form of the Borel sets of degree exactly α.

5. The normal form of Borel sets of finite rank.

Definition 4 Let $0 < \alpha < \epsilon^1$, α admits a Cantor normal form of base ω_1. Which means :
$\alpha = \omega_1^{\alpha_k}.\nu_k + \ldots + \omega_1^{\alpha_0}.\nu_0$ with $\epsilon^1 > \alpha_k > \ldots > \alpha_0 \geq 0$ and $\forall i \leq k,\ 0 < \nu_i < \omega_1$
Define : $\Omega(\alpha) = \Omega(\omega_1^{\alpha_k}) \bullet \nu_k \pm \ldots\ldots \pm \Omega(\omega_1^{\alpha_0}) \bullet \nu_0$, where $\Omega(\omega_1^{\beta})$ with $0 \leq \beta < \epsilon^1$ is defined by :

1. $\beta = 0$: $\quad \Omega(\omega_1^{\beta})=\Omega(1)=\emptyset$
2. $\beta = n \in \omega$ and $n \neq 0$: $\quad \Omega(\omega_1^{\beta})=(\pi, \Omega(\beta+1))$
3. $\beta = \lambda + n$, λ limit of cofinality ω_1 and n$\in \omega$: $\quad \Omega(\omega_1^{\beta})=(\pi, \Omega(\beta))$
4. $\beta = \lambda + n$, λ limit of cofinality ω and $0 \neq n \in \omega$: $\quad \Omega(\omega_1^{\beta})=(\pi, \Omega(\beta - 1))$
5. $\beta = \lambda$, λ limit of cofinality ω and $\lambda = sup_{n<\omega}\lambda_n$: $\quad \Omega(\omega_1^{\beta}) = \sum_{n<\omega}(\pi, \Omega(\lambda_n))$

Theorem 8 *Let A$\subseteq \Sigma_A^\omega$, A non self-dual : $A \in \bigcup_{n<\omega} \mathbf{\Sigma}^\circ_n \iff \exists\alpha,\ 0 < \alpha < \epsilon^1$ A or $\bar{A}$ $\equiv_W$ $\Omega(\alpha)^b$*
Recall that if A is self-dual, either $\exists$ $B \subseteq \Sigma_B^\omega$ non self-dual A$\equiv_W 0.B \cup 1.\bar{B}$ or $\forall i < \omega$ $\exists B_i \subseteq \Sigma_{B_i}^\omega$ each $B_i <_C A$ and non self-dual and $A \equiv_W \bigcup_{i<\omega} i.B_i$.

Thus theorem 8 provides for every set A of finite Borel rank a canonical set $\Omega \equiv_W$A.

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