The normal form of Borel sets
Part II : Borel sets of infinite rank*

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La forme normale des Boréliens de rang infini.

Résumé : Pour chaque ensemble borélien de réels A, de rang infini, nous obtenons une "forme normale" pour A en donnant un ensemble borélien Ω tel que A et Ω se réduisent continuement l'un à l'autre. Cela s'effectue en définissant des opérations boréliennes homomorphes aux ω_1 premières opérations ordinales de Veblen de base ω_1, opérations nécessaires au calcul du degré de Wadge d'un ensemble borélien.

Abstract : For each Borel set of reals of infinite rank A we obtain a "normal form" of A by finding a Borel set Ω such that A and Ω continuously reduce to each other. We do so by defining simple Borel operations which are homomorphic to the ω_1 first Veblen ordinal operations of base ω_1 required to compute the Wadge degree of a Borel set.

Version française abrégée. Cet article est la suite de la Note [1] concernant la forme normale des Boréliens. Nous rappelons que nous considérons les ensembles $A \subseteq \Lambda_A^\omega$, qui sont boréliens pour la topologie produit de la topologie discrète sur un ensemble dénombrable Λ_A. Pour deux tels ensembles $A \subseteq \Lambda_A^\omega$ et $B \subseteq \Lambda_B^\omega$ la relation $A \leq_w B$ signifie qu'il existe une fonction continue de Λ_A^ω dans Λ_B^ω tel que A soit l'image inverse de B par cette fonction. La relation d'équivalence $A \equiv_w B$ désigne le fait qu'à la fois $A \leq_w B$ et $B \leq_w A$ sont vérifiés. La relation stricte $A <_w B$ dit que $A \leq_w B$ est vérifié sans que $B \leq_w A$ le soit. Martin et Wadge ont montré que cette dernière relation restreinte aux ensembles boréliens constituait un pré-bon ordre. Plus exactement la relation $<_w$, quotientée par la relation d'équivalence $\approx$ définie par $A \approx B$ ssi $(A \equiv_w B \ \vee \ A \equiv_w {}^-B)$, est un bon ordre sur les ensembles boréliens.
Le calcul de la longueur de ce bon ordre requiert l'utilisation des ω_1 premières opérations de la hiérarchie de Veblen de base ω_1. La première de ces fonctions, V^0, est à peu de choses près l'exponentiation de base ω_1 : pour λ limite et n entier, $V^0(\lambda + n)$ vaut ${\omega_1}^{\lambda+n+\epsilon}$ où ϵ prend la valeur -1 lorsque λ vaut 0, la valeur 0 lorsque λ est de cofinalité ω_1 et la valeur $+1$ dans les autres cas. Les fonctions V^ν pour ν non nul sont définies comme énumérant, en débutant à 2, les ordinaux de cofinalité ω_1 qui sont points fixes de chacunes des fonctions de Veblen de rang inférieur : $V^\nu(1+\alpha)$ est le α^{eme} ordinal de cofinalité ω_1 vérifiant $V^\mu(\alpha) = \alpha$ pour chaque ordinal $\mu < \nu$. Le type d'ordre de la relation $<_{w/\approx}$ restreinte aux ensembles boréliens de réels est l'ordinal

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$\sup_{\nu<\omega_1} V^\nu(2) = V^{\omega_1}(2)$.
Nous rappelons également que le degré de Wadge (noté $d_w^\circ A)$)d'un ensemble borélien non auto dual (c'est-à-dire n'étant l'image inverse de son complémentaire par aucune fonction continue) est défini comme étant le plus grand ordinal α tel qu'il existe une suite $\langle \Omega_\beta / 0 < \beta \leq \alpha \rangle$ de boréliens non auto duaux satisfaisant $\beta < \delta < \alpha \Longrightarrow \Omega_\beta <_w \Omega_\gamma \leq_w A$. Pour chaque ordinal α inférieur à $V^{\omega_1}(2)$ nous produisons un ensemble non auto dual $\Omega(\alpha)$ dont le degré de Wadge est précisément α, à partir de la forme normale de l'ordinal α en remplaçant l'ordinal 1 par l'ensemble vide et chacune des opérations ordinales impliquées dans cette description par sa contrepartie ensembliste.

En fait nous travaillons avec les ensembles dénommés conciliants en [1], ensembles de la forme $A \subseteq \Lambda_A^{\leq\omega}$. A partir d'un ensemble conciliant $A \subseteq \Lambda_A^{\leq\omega}$, nous retrouvons les ensembles de réels en considérant $A^b = \{z \in (\Lambda_A \cup \{b\})^\omega / \ z' \in A\}$ où z' désigne la suite obtenue à partir de z une fois toute occurence de la lettre b effacée - b étant une lettre quelconque qui n'est pas dans Λ_A. Clairement aucun ensemble de la forme A^b ne se réduit à son complémentaire $(\Lambda_A \cup \{b\})^\omega \setminus A^b$. Inversement une conséquence de notre étude est que tout ensemble borélien non auto dual est Wadge équivalent à un ensemble de cette forme.
Nous complétons ici les opérations ensemblistes de somme : $(A, B) \longmapsto A \pm B$, multiplication dénombrable : $(A, \nu) \longmapsto A \bullet \nu$ et quasi exponentiation de base ω_1 : $A \longmapsto (\pi, A)$, définies en [1], en produisant pour chaque ordinal dénombrable ν une opération $\mathbf{V}_c^\nu$ portant sur les ensembles conciliants et telle que : $d_w^\circ(\mathbf{V}_c^\nu(A))^b = V^\nu(d_w^\circ(A^b))$. La première de ces opérations ($\mathbf{V}_c^0$) remplace l'opération $A \longmapsto (\pi, A)$ de [1]. Les suivantes sont obtenues à partir d'elle à l'aide de jeux dépliés en itérant cette opération le long de la Forme Normale de Cantor de base ω des ordinaux dénombrables impliqués.
Les clés de ce travail reposent d'une part sur l'utilisation de la version du jeu de Wadge pour les ensembles conciliants introduite en [1] et la généralisation de ce que nous dénommions théorème de télécommande, ainsi qu'une notion "d'opération inverse" pour chacune des opérations $\mathbf{V}_c^\nu$. Cela permet pour chaque ordinal non nul $\alpha < V^{\omega_1}(2)$ de donner de manière canonique un ensemble $\Omega(\alpha)$ de degré α en remplaçant dans la forme normale de α, l'ordinal 1 par l'ensemble vide et les opérations ordinales par leurs contreparties ensemblistes.

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0. Introduction. This paper is a continuation of [1]. We recall that we deal with Borel subsets A of Λ_A^ω where Λ_A is some alphabet of countable cardinality and Λ_A^ω is equipped with the usual topology: the product of the discrete topology on Λ_A. The relation $A \leq_w B$ between $A \subseteq \Lambda_A^\omega$ and $B \subseteq \Lambda_B^\omega$ denotes the existence of a continuous function $f \ : \ \Lambda_A^\omega \longmapsto \Lambda^\omega$ such that $x \in A$ iff $f(x) \in B$. Relations $A \equiv_w B$ and $A <_w B$ respectively stand for $A \leq_w B \leq_w A$ and $A \leq_w B \not\leq_w A$. By results of Wadge and Martin [7] $<_w$ is a pre-well-ordering on Borel sets. More precisely, let $\approx$ be the equivalence relation on Borel sets defined by: $A \approx B$ iff $A \equiv_w B$ or $A \equiv_w {}^-B$ (where ${}^-B$ denotes $\Lambda_B^\omega \setminus B$), then $<_{w/\approx}$ well orders Borel sets.
A set is self dual if it satisfies $A \equiv_w {}^-A$. The Wadge degree of A, denoted d_w°A, for A non self-dual, is the largest ordinal α such that there is an $<_W$-increasing sequence $\langle \Omega(\beta)/0 < \beta \leq \alpha \rangle$ of Borel non self-dual sets which satisfies $\beta < \gamma \leq \alpha \Longrightarrow \Omega(\beta) <_W \Omega(\gamma) \leq_W A$; and for A self-dual, it is the least Wadge degree of non self-dual sets $<_W$-greater than A. Self dual sets are easily describable on the basis of non self dual ones.
So in order to focus on non self dual sets we introduce in [1] the notion of **conciliatory sets**. They are sets of the form $A \subseteq \Lambda^{\leq\omega}$. We do not put any topology on $\Lambda^{\leq\omega}$ but consider that $B \subseteq \Lambda^{\leq\omega}$ is Borel iff the set B^b is Borel in $(\Lambda \cup \{b\})^\omega$ where b is an extra letter and B^b denotes all sequences α of $(\Lambda \cup \{b\})^\omega$ which belong to B once every occurrence of the letter b has been erased. Clearly any conciliatory set A gives rise to a non self dual set since $A^b \equiv_w {}^-(A^b)$ always fails. Therefore they

are non self dual. In the other direction one has the following theorem which is a consequence of the whole study.

Theorem 1 *Let $A \subseteq \Lambda_A^{\omega}$ be some non self dual Borel set, there exists $B \subseteq \Lambda_{\bar{A}}^{\leq\omega}$ such that $A \equiv_w B^b$*

In addition we change $\leq_w$ into $\leq_c$by mean of a suitable version of the Wadge game when dealing with conciliatory sets. Given $A \subseteq \Lambda_{\bar{A}}^{\leq\omega}$ and $B \subseteq \Lambda_{\bar{B}}^{\leq\omega}$, the game $\mathbf{C}(A, B)$ is a two player infinite game with perfect information where player I (resp. player II) plays letters in Λ_A(resp. Λ_B), but both players are allowed to skip, indefinitely if they want to; so that they may produce sequences α for I and β for II that can be finite. We say that II wins this game iff $(\alpha \in A \longleftrightarrow \beta \in B)$ and we denote by $A \leq_c B$ the fact II has a winning strategy; and wrote $A \equiv_c B$ and $A <_c B$ when respectively $A \leq_c B \leq_c A$ and $A \leq_c B \not\leq_c A$ hold. in the game $\mathbf{C}(A, B)$. Since a player has a winning strategy in the game $\mathbf{C}(A, B)$ iff the same player has a w.s. in the Wadge game $\mathbf{W}(A^b, B^b)$, provided A^b and B^b are both Borel, it is a consequence of Borel (Wadge) Determinacy ([5] or [4]) that :

- the game $\mathbf{C}(A, B)$ is determined,
- Wadge's lemma holds: $A \leq_c B$ or ${}^{-}B \leq_c A$,
- $\leq_c$, up to $\equiv_c$ and complementation, is a well ordering on Borel conciliatory sets.

The conciliatory degree is defined by: $d_c^{\circ} A = d_w^{\circ} A^b$.

In [1] we introduced operations on conciliatory sets $\pm$, $\sum$, $\bullet$ and $(\pi,\ )$ that preserve the conciliatory ordering $\leq_c$ and form the set theoretic counterpart of ordinal operations in the sense that, given A, B and $(A_i)_{i \in I}$ conciliatory Borel sets with $|I| < \omega_1$:

- $d_c^{\circ}(A \pm B) = d_c^{\circ}(A) + d_c^{\circ}(B)$
- $d_c^{\circ}(\sum_{n\in\omega} A_n) = \sum_{n\in\omega} d_c^{\circ}(A_n)$ (when $\{d_c^{\circ}(A_n)/n \in \omega\}$ does not contain its upper bound)
- $d_c^{\circ}(A \bullet \nu) = d_c^{\circ}(A).\nu$ (for $0 < \nu < \omega_1$)
- $d_c^{\circ}(\pi, A) = {\omega_1}^{d_c^{\circ}(A)+\epsilon}$ (where $\epsilon \in \{-1, 0, 1\}$ depending on the of the least limit ordinal $\lambda \leq d_c^{\circ}(A)$).

Then we used these operations to define for every non null ordinal α less than the first fixpoint of cofinality ω_1 of the exponentiation of base ω_1, a "canonical" Borel set $\Omega(\alpha)$ of degree precisely α by replacing ordinal 1 by the empty set and ordinal operations involved in the canonical description of α by their set-theoretic counterpart.
Here we modify operation $\sum$ in order not to involve countable choice and call this new version **sup**. It still preserves $\leq_c$ and satisfies $d_w^{\circ}(\mathbf{sup}_{i\in I} A_i)^b = \sup_{i\in I} d_w^{\circ}(A_i)^b$ as long as the sequence $(A_i)_{i\in I}$ does not contain its $<_c$-upper bound.

Definition 2 *Let I be a set of elements from Λ_A and, for each i in I, A_i be a subset of $\Lambda_{\bar{A}}^{\leq\omega}$,*

$$\mathbf{sup}_{i\in I} A_i = \{ < i > {}^\frown \alpha : \ i \in I, \ \alpha \in A_i \}$$

We extend this work to Borel sets of any rank. For that purpose we need:

1. Veblen hierarchy.

Definition 3 *The Veblen hierarchy of base* ω_1 *is the sequence of functions* $(V^\xi)_{\xi\leq\omega_1}$, $V^\xi : \omega_2 \setminus \{0\} \longmapsto \omega_2$ *defined by:* $V^\xi(1) = 1 \quad (\forall \xi \leq \omega_1)$ *and:*

- ◦ ⋄ $V^0(2) = \omega_1$
  - ⋄ $V^0(\alpha+1) = V^0(\alpha)\ .\omega_1$.
  - ⋄ *if* α *is limit of cofinality* ω_1, *then* $V^0(\alpha) = \omega_1^\alpha$
  - ⋄ *if* α *is limit of countable cofinality, then* $V^0(\alpha) = \omega_1^{\alpha+1}$

  (V^0 *is almost the exponentiation of base* ω_1*:* $\alpha \longmapsto {\omega_1}^\alpha$*; except that there is a shift to avoid* ω_1^α *when* α *is limit of countable cofinality, and also that it starts at 2 instead of 1.)*

- ◦ *For* $\lambda > 0$, V^λ *is the function that, starting at 2, enumerates ordinals with cofinality* ω_1 *that are fixpoints of all Veblen functions of lesser degree (*V^ξ *for* $\xi < \lambda$*). i.e.* $V^\lambda(1+\nu)$ *is the* ν^{th} *fixpoint of cofinality* ω_1 *of all* $V^\xi \quad (\xi < \lambda)$.

Theorem 4 (Wadge) *The order type of the Wadge ordering on Borel sets of reals is* $V^{\omega_1}(2)$

Note that for each limit ordinal of countable cofinality $\alpha < V^{\omega_1}(2)$, one may define - without using the Axiom of Choice - a countable set I_α of ordinals that are cofinal in α [6][2]. With the help of that result we intend to give - **without AC** - a normal form for Borel sets that be an extension of our work on Borel sets of finite rank in [1]. To achieve this goal we need, for each countable ordinal ν, to define conciliatory operations "$\mathbf{V}_c^\nu$" which preserve the conciliatory ordering and with the property that given any Borel conciliatory set A the Wadge degree of the resulting set $(\mathbf{V}_c^\nu(A))^b$ is exactly $V^\nu(d_w^\circ(A^b))$.

Definition 5 *Let* Λ *be any alphabet,* $\{\twoheadleftarrow_{(\theta)} : 0 < \theta < \omega_1\}$ *a set of different arrows indexed by* **non zero** *ordinals such that* $\Lambda \cap \{\twoheadleftarrow_{(\theta)} : 0 < \theta < \omega_1\} = \emptyset$.
Given $z \in (\Lambda \cup \{\twoheadleftarrow_{(\theta)} : 0 < \theta < \omega_1\})^{\leq\omega}$ and a countable ordinal ξ (possibly 0) we successively define $z^{\hookleftarrow\xi}$, $z^{\circlearrowleft\xi} \in \Lambda \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\})^{\leq\omega}$ and $z^{\circlearrowleft} \in \Lambda^{\leq\omega}$.

Definition of $\alpha^{\hookleftarrow\xi}$

- ◦ $<>^{\hookleftarrow\xi} = <>$
- ◦ *in case* z *is finite but non empty,* $z = y^\frown < a >$*:*

$$\begin{aligned}(y^\frown < a >)^{\hookleftarrow\xi} &= y^{\hookleftarrow\xi\frown} < a > && \text{if } a \neq \twoheadleftarrow_{(\xi)}\\ &= <> && \text{if } a = \twoheadleftarrow_{(\xi)} \text{ and } lh(y^{\hookleftarrow\xi}) \leq 1\\ &= y^{\hookleftarrow\xi} \restriction (k-1) && \text{if } a = \twoheadleftarrow_{(\xi)} \text{ and } lh(y^{\hookleftarrow\xi}) = k > 1\end{aligned}$$

- ◦ *for* z *infinite:* $z^{\hookleftarrow\xi} = \bigcup\left\{u :\ \exists m \forall n \geq m \quad u \subseteq \left((\alpha \restriction n)^{\hookleftarrow\xi}\right)\right\}$.

In simple words, $z^{\hookleftarrow\xi}$ *is just* z *when every occurrence of the letter* $\twoheadleftarrow_{(\xi)}$ *is translated into an eraser. This means that, beginning from left to right and imagining* $\twoheadleftarrow_{(\xi)}$ *represents the Back Space key (the one familiar to your keyboard), the sequence* $z^{\hookleftarrow\xi}$ *is the one left after the whole erasing work is done.*

Definition of $z^{\circlearrowleft\xi}$ *The sequence* $z^{\circlearrowleft\xi}$ *is an element of* $(\Lambda \cup \{\twoheadleftarrow_{(\theta)} : 0 < \theta \leq \xi\})^{\leq\omega}$ *defined by*

- ◦ *if* $z = <>$, *then* $z^{\circlearrowleft\xi} = <>$, *if* $z \neq <>$, *depending on* $z(0)$:

- *if* $z(0) \in \Lambda$, *then* $z^{\circlearrowleft\xi}$ *is just* z *where all letters of the form* $\twoheadleftarrow_{(\theta)}$ *for* $\theta > \xi$ *are removed.*
- *if* $z(0) = \twoheadleftarrow_{(\nu)}$
  - *if* $\nu \leq \xi$, *then* $z^{\circlearrowleft\xi}$ *is just* z *where all letters of the form* $\twoheadleftarrow_{(\theta)}$ *for* $\theta > \xi$ *are removed.*
  - *if* $0 < \xi < \nu = \mu + 1$, *then* $z^{\circlearrowleft\xi} = ((y)^{\looparrowleft\nu})^{\circlearrowleft\xi}$, *where* y *is* z *after all letters of the form* $\twoheadleftarrow_{(\theta)}$ *for* $\theta > \nu$ *are removed.*
  - *if* $0 \leq \xi < \nu$ *is* **limit**, *then* $z^{\circlearrowleft\xi} =< {(y)_i}^{\circlearrowleft\xi}(0)/i \in \omega >$ *where* y *is* z *where all letters of the form* $\twoheadleftarrow_{(\theta)}$ *for* $\theta \geq \nu$ *are removed and* $y_i =< y(K(i,j))/j \in \omega >$ *with* K *the Cantor pairing function:* $K(i,j) = \frac{(i+j)(i+j+1)}{2} + j$.

Definition of $z^{\circlearrowleft}$ *(in any case* $z^{\circlearrowleft}$ *is a sequence that belongs to the space* $\Lambda^{\leq\omega}$*)*

- *if* $z =<>$, *then* $z^{\circlearrowleft} =<>$,
- *if* $z(0) \in \Lambda$, *then* $z^{\circlearrowleft}$ *is the sequence* z *where all letters of the form* $\twoheadleftarrow_{(\theta)}$ *for* $\theta > 0$ *are removed.*
- *if* $z(0) = \twoheadleftarrow_{(\delta)}$
  - *if* $\delta = \nu + 1$, *then* $z^{\circlearrowleft} = (z^{\circlearrowleft\nu})^{\circlearrowleft}$.
  - *if* δ *is limit and admits the following Cantor Normal Form of base* ω:
    $\delta = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0$ *with* $\omega_1 > \xi_k > \ldots > \xi_0 > 0$ *and* $\omega > n_i > 0$ *each* $i \leq k$
    then $z^{\circlearrowleft} = (z^{\circlearrowleft\omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.(n_0 - 1)})^{\circlearrowleft}$.

Operation $^{\circlearrowleft}$ on finite or infinite sequence is required to define conciliatory Veblen operations:

Definition 6 *Let* $A \subseteq \Lambda_A^{\leq\omega}$, $\Lambda \cap \{\twoheadleftarrow_{(\theta)} : \; 0 < \theta < \omega_1\} = \emptyset$, $0 \leq \nu < \omega_1$,

$$\mathbf{V}_c^{\nu}(A) = \{\alpha \in (\Lambda \cup \{\twoheadleftarrow_{(\theta)} : \; 0 < \theta < \omega^{\nu}\})^{\leq\omega} : \; \alpha^{\circlearrowleft} \in A\}$$

Although operation $\mathbf{V}_c^0$ is not similar to operation π we used in [1] it has the same strength, i.e. for any conciliatory set A, $\mathbf{V}_c^0(A) \equiv_c (\pi, A)$ holds. Besides for every countable ν, operation $\mathbf{V}_c^{\nu}$ preserves the conciliatory ordering, an extension of the "remote-control lemma" in [1]:

Lemma 7 *Let* $A \subseteq \Lambda_A^{\leq\omega}$, $B \subseteq \Lambda_B^{\leq\omega}$, $0 \leq \nu < \omega_1$,

$$A \leq_c B \iff \mathbf{V}_c^{\nu}(A) \leq_c \mathbf{V}_c^{\nu}(B)$$

2. A Normal Form for Borel Sets

We use the Veblen hierarchy to define the following "canonical" conciliatory Borel sets.

Definition 8 *Let* $0 < \alpha < V^{\omega_1}(2)$, α *admits a unique Cantor Normal Form of base* ω_1:

$$\alpha = {\omega_1}^{\alpha_n}.\nu_n + \ldots + {\omega_1}^{\alpha_0}.\nu_0 \text{ with } V^{\omega_1}(2) > \alpha_n > \ldots > \alpha_0 \geq 0 \text{ and } \forall i \leq n, \quad 0 < \nu_i < \omega_1$$

define:

$$\Omega(\alpha) = \Omega({\omega_1}^{\alpha_n}) \bullet \nu_n \pm \ldots\ldots \pm \Omega({\omega_1}^{\alpha_0}) \bullet \nu_0$$

where $\Omega({\omega_1}^{\alpha_i})$ *with* $0 \leq i \leq n$ *is defined by:*

$\alpha_i = 0$:	$\Omega({\omega_1}^{\alpha_i}) = \emptyset$
α_i *limit of countable cofinality* :	$\Omega({\omega_1}^{\alpha_i}) = \mathbf{sup}_{\theta \in I_{\alpha_i}} \Omega(\omega_1^{\theta})$
$\left.\begin{array}{l}\alpha_i \text{ successor} \\ \alpha_i \text{ limit of cofinality } \omega_1\end{array}\right\}$	$\Omega(\omega_1^{\beta}) = \mathbf{V}_c^{\nu}(\gamma)$

where ν *is the least ordinal such that there exists* $\gamma < {\omega_1}^{\alpha_i}$ *satisfying* ${\omega_1}^{\alpha_i} = V^{\nu}(\gamma)$. *And* I_{α_i} *is any countable set of ordinals that verifies* $\sup I_{\alpha_i} = \alpha_i$.

Those canonical sets constitute a hierarchy which describes (up to complementation) the whole Wadge hierarchy on Borel non self dual sets:

Theorem 9
Let $A \subseteq \Lambda^\omega$ be Borel and non self dual then there exists a unique ordinal $0 < \alpha < V^{\omega_1}(2)$ such that

$$^{-}A \text{ or } A \ \equiv_w \Omega(\alpha)^b \quad \text{and} \quad d_w^\circ(\Omega(\alpha)^b) = \alpha$$

The proof of this main theorem is based on the following three lemmas:

Lemma 10 *Let $A \subseteq \Lambda_A^{\leq\omega}$ such that A^b is $\mathbf{\Sigma}_1^\circ$-complete and $0 \leq \nu < \omega_1$,*

$$(\mathbf{V}_c^\nu(A))^b \text{ is } \mathbf{\Sigma}^\circ_{1+\omega^\nu}\text{-complete}$$

Lemma 11 *Let $0 < \beta < \alpha < V^{\omega_1}(2)$,*

$$\Omega(\beta) \leq_c \Omega(\alpha) \text{ and } {}^{-}\Omega(\beta) \leq_c \Omega(\alpha)$$

Lemma 12 (weak separation lemma) *Let A be non self dual and $0 < \alpha < V^{\omega_1}(2)$,*

$$\left({}^{-}A \leq_w \Omega(\alpha)^b \ \wedge \ A \leq_w \Omega(\alpha)^b\right) \Longrightarrow \exists\, \beta < \alpha \quad \left({}^{-}A \leq_w \Omega(\beta)^b \ \vee \ A \leq_w \Omega(\beta)^b\right)$$

This lemma shows that the structure $\left\langle (\Omega(\alpha))_{0<\alpha<V^{\omega_1}(2)} \ , \ \leq_c \right\rangle$ is a well ordering because every conciliatory set is non self dual.

proof of 9: Since A is Borel there exists some countable ordinal ν such that $A \in \mathbf{\Sigma}^\circ_{1+\omega^\nu}$. Therefore, by lemma 10, since $\Omega(2)^b$ is $\mathbf{\Sigma}_1^\circ$-complete), one gets $A \leq_w (\mathbf{V}_c^\nu(\Omega(2)))^b$ which is by definition $A \leq_w (\Omega(V^\nu(2)))^b$. Let α be the least ordinal such that ${}^{-}A \leq_w \Omega(\alpha)^b$ or $A \leq_w \Omega(\alpha)^b$. By lemma 12 and minimality of α only one possibility holds. By symmetry assume that ${}^{-}A \leq_w \Omega(\alpha)^b$ fails. By determinacy this leads to $\Omega(\alpha)^b \leq_w A$; thus $A \equiv_w \Omega(\alpha)^b$. The other case leads to ${}^{-}A \equiv_w \Omega(\alpha)^b$. At last, lemma 11 gives unicity of α. Finally lemmas 11 and 12 combined together show that $d_w^\circ(\Omega(\alpha))^b = \alpha$. 9□

Proofs of the three lemmas involve lemma 7, a generalization of what was called in [1] the remote control lemma because it allows to carry a strategy through the Veblen conciliatory operations: if $\mathcal{T}$ is a strategy for *II* in $\mathbf{C}(A,B)$ it provides a strategy $\sigma_\mathcal{T}^\nu$ for *II* in $\mathbf{C}(\mathbf{V}_c^\nu(A), \mathbf{V}_c^\nu(B))$ with the property that $\sigma_\mathcal{T}^\nu$ applied to α is almost $\mathcal{T}$ applied to $\alpha^{\circlearrowleft}$. The second main key is a kind of inverse operation for $\mathbf{V}_c^\nu$: $\mathbf{V}_c^{\nu-1}$, which, by use of auxiliary games, affects the topology of the space in a way that respects the conciliatory Veblen operations. Then proofs can be conducted by induction on the ordinal α since operation $\mathbf{V}_c^{\nu-1}$ allows to collapse the part of the hierarchy we are dealing with to a smaller initial segment of it where the issues are settled.

References

[1] **Duparc J.** (1995) *The normal form of Borel sets. Part I: Borel sets of finite rank.* Cr. Acad. Sci. Paris, t. 320, Série I, p.651-656, 1995

[2] **Girard J.-Y., Vauzeilles J.** [1984] *Functors and ordinal notations I: A functorial construction of the Veblen hierarchy* The Journal of Symbolic Logic, vol 49 (1984) 713-729.

[3] **Louveau A.** (1983) *Some Results in the Wadge Hierarchy of Borel Sets.* Cabal Sem 79-81, Lecture Notes in Mathematics (1019) 28-55.

[4] **Louveau A.,St.Raymond J.** (1985) *The strength of Borel Wadge determinacy.* Cabal Sem 81-85, Lecture Notes in Mathematics (1333) 1-30.

[5] **Martin D.A.** (1975) *Borel Determinacy.* Ann. Math. vol.102 (1975) 363-371.

[6] **Veblen O.** [1908] *Continuous increasing fonctions of finite and transfinite ordinals.* Transactions of the American Mathematical Society vol 9 (1908) 280-292.

[7] **Wadge W.W.** (1984) *Reducibility and determinateness on the Baire space.* Ph.D. Thesis, Berkeley.