Wadge hierarchy and Veblen hierarchy. Part II: Borel sets of infinite rank*

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Abstract

We consider Borel sets of the form $A \subseteq \Lambda^\omega$ (with usual topology) where cardinality of Λ is less than some uncountable regular cardinal κ. We obtain a "normal form" of A, by finding a Borel set $\Omega(\alpha)$ such that A and $\Omega(\alpha)$ continuously reduce to each other. We do so by defining Borel operations which are homomorphic to the κ first Veblen ordinal functions of base κ required to compute the Wadge degree of the set A: the ordinal α.

1 Introduction

This paper is a continuation of [1]. We recall that we are dealing with **Borel** subset A of Λ_A^ω where Λ_A is some well ordered alphabet of cardinality strictly less than some given uncountable **regular** cardinal κ. The topology on Λ_A^ω is the usual one: the product topology of the discrete topology on the alphabet. For two such subsets $A \subseteq \Lambda_A^\omega$ and $B \subseteq \Lambda_B^\omega$ the relation $A \leq_w B$ stands for the existence of a continuous function $f : \Lambda_A^\omega \longmapsto \Lambda_B^\omega$ such that $A = f^{-1}B$. The notation $A \equiv_w B$ means that both reductions $A \leq_w B$ and $B \leq_w A$ hold, and $A <_w B$ that $A \leq_w B$ is satisfied but not $B \leq_w A$.
It is a consequence of **Borel Determinacy** ([6]) that $\leq_w$ is a well founded relation on Borel sets. Moreover, Wadge's lemma holds: for any two subsets A and B one has either $A \leq_w B$ or $^-B \leq_w A$ (where ^-B stands for the complement of B).
We recall that a **self dual set** A is a set that satisfies $A \equiv_w {}^-A$. The Wadge degree of a Borel set A - denoted $d_w^\circ A$ - is defined depending on if A is self dual or not: in case A is self dual it is defined as the least Wadge degree of non self dual sets $<_w$-greater than A. And in case A is non self dual it is defined by:

- $d_w^\circ(A) = 1$ iff $A \equiv_w \ominus$ or $\oplus$; where $\ominus$ stands for the emptyset and $\oplus$ for its complement, i.e. the whole space.
- $d_w^\circ(A) = \sup\{d_w^\circ(B) :\ B$ non self dual $\wedge\ B <_w A\}$ in case $\ominus <_w A$.

*Supported by Lavoisier Grant from the French Ministry of Foreign Affairs.

Therefore two non self dual Borel sets have same Wadge degree iff they are Wadge equivalent or one is equivalent to the complement of the other. In [1] we studied the Wadge hierarchy of Borel sets of finite rank (in fact even the one of $\mathbf{\Delta}^\circ_\omega$ sets). We gave for each Borel Wadge degree α a definition of a Borel set $\Omega(\alpha)$ of degree precisely α which was isomorphic to the canonical description of the ordinal α. Our main tool was to replace sets $A \subseteq \Lambda^\omega$ by what we called **conciliatory** sets of the form $B \subseteq \Lambda^{\leq\omega}$, allowing this way finite sequence in addition of infinite ones. We did not put any topology on $\Lambda^{\leq\omega}$ but considered that $B \subseteq \Lambda^{\leq\omega}$ was Borel iff the set B^b was Borel in $(\Lambda \cup \{b\})^\omega$ where b was an extra letter and B^b denoted all sequences α of $(\Lambda \cup \{b\})^\omega$ which belonged to B once every occurrence of the letter b had been erased. Clearly any conciliatory set is non self dual, or in other words any set of the form B^b does not reduce to its complement.
In this paper we extend to Borel sets of any rank the result involving the other direction of the equivalence between non self dual sets and conciliatory ones that we previously stated only for Borel sets of finite rank:

Theorem 1 *Let $A \subseteq \Lambda^\omega_A$ be some non self dual Borel set,*

$$\textit{there exists } S(A) \subseteq \Lambda_A^{<\omega} \textit{ such that } A \equiv_w (A \cup S(A))^b$$

We postpone the proof of this theorem to the last section of this paper. In addition to conciliatory sets we introduced in [1] the **Wadge conciliatory ordering** ($\leq_c$). This binary relation was defined as a suitable version of the Wadge game when dealing with conciliatory sets. Given $A \subseteq \Lambda_A^{\leq\omega}$ and $B \subseteq \Lambda_B^{\leq\omega}$, the game $\mathbf{C}(A, B)$ is a two player infinite game with perfect information where player *I* (resp. player *II*) plays letters in Λ_A(resp. Λ_B), but both players are allowed to skip, indefinitely if they want to; so that they may produce sequences α for *I* and β for *II* that can be finite. We say that *II* wins this game iff ($\alpha \in A \longleftrightarrow \beta \in B$) and we denote by $A \leq_c B$ the fact *II* has a w.s. in the game $\mathbf{C}(A, B)$. Since a player has a w.s. in the game $\mathbf{C}(A, B)$ iff the same player has a w.s. in the Wadge game $\mathbf{W}\left(A^b, B^b\right)$, provided A^b and B^b are Borel, it is a consequence of Borel Determinacy that :

- the game $\mathbf{C}(A, B)$ is determined,
- Wadge's lemma holds: $A \leq_c B$ or ${}^{\neg}B \leq_c A$
- $\leq_c$ is a prewellordering on Borel conciliatory sets.
  ($\leq_c$ is a well ordering up to $\equiv_c$ and complementation).

We defined the conciliatory degree on a conciliatory Borel set A by referring to the Wadge degree of its embedding in the Wadge hierarchy: $d^\circ_c A = d^\circ_w A^b$.

In [1] we introduced operations on conciliatory sets $+$, $\mathbf{sup}$, $\bullet$ and $^\sim$ that preserve the conciliatory ordering $\leq_c$ and form the set theoretic counterpart of ordinal operations in the sense that, given A, B and $(A_i)_{i\in I}$ conciliatory Borel sets with $|I| < \kappa$:

- $d^\circ_c(A + B) = d^\circ_c(A) + d^\circ_c(B)$
- $d^\circ_c(\mathbf{sup}_{i\in I} A_i) = \sup_{i\in I} d^\circ_c(A_i)$ (when $\{d^\circ_c(A_i)/i \in I\}$ does not contain its upper bound)
- $d^\circ_c(A \bullet \nu) = d^\circ_c(A).\nu$ (for $0 < \nu < \kappa$)

- $d_c^\circ(A^\sim) = \kappa^{d_c^\circ(A)+\epsilon}$ (where $\epsilon \in \{-1,0,1\}$ depending on the cofinality of the least limit ordinal $\lambda \leq d_c^\circ(A)$).

Then we used these operations to define for every non null ordinal α less than the first fixpoint of cofinality κ of the exponentiation of base κ, a "canonical" Borel set $\Omega(\alpha)$ of degree precisely α by replacing the ordinal 1 by the emptyset and ordinal operations involved in the canonical description of α by their set theoretic counterpart.

2 Iteration of the "Back Space" operation

Here we want to extend that process to the whole hierarchy of Borel sets. So we need to know how high the Wadge degree of a Borel set can be.

Theorem 2 *The order type of the Wadge hierarchy on Borel sets is*

$$OT_{\mathcal{B}} = \sup_{\nu<\omega_1} {}^{\kappa}V^{\nu}(2)$$

i.e. let $A \subseteq \Lambda_A^\omega$,

$$A \text{ is Borel } \textit{ iff } 0 < d_w^\circ(A) < \sup_{\nu<\omega_1} {}^{\kappa}V^{\nu}(2)$$

Where the functions ${}^{\kappa}V^{\nu}$ are the Veblen functions of base κ defined below. (For any ordinal λ the notation λ^* stands for $\lambda \smallsetminus \{0\}$).

Definition 3
The Veblen hierarchy of base κ consists of the functions $({}^{\kappa}V^{\xi})_{\xi\leq\kappa}$, ${}^{\kappa}V^{\xi} : \kappa^{+*} \longmapsto \kappa^{+}$ defined by:
First of all, none of them is defined on 0.
Second each of them takes value 1 at 1: ${}^{\kappa}V^{\xi}(1) = 1 \quad (\forall \xi \leq \kappa)$ and:

- *${}^{\kappa}V^0$ is almost the exponentiation of base κ: $\alpha \longmapsto \kappa^\alpha$; except that there is a shift to avoid κ^α when α is limit of cofinality strictly less than κ, and also that it starts at 2 instead of 1:*
  - ${}^{\kappa}V^0(2) = \kappa$
  - ${}^{\kappa}V^0(\alpha+1) = {}^{\kappa}V^0(\alpha).\kappa$; *i.e. if ${}^{\kappa}V^0(\alpha) = \kappa^\beta$ then ${}^{\kappa}V^0(\alpha+1) = \kappa^{\beta+1}$.*
  - *if α is limit of cofinality κ, then ${}^{\kappa}V^0(\alpha) = \kappa^\alpha$*
  - *if α is limit of cofinality strictly less than κ, then ${}^{\kappa}V^0(\alpha) = \kappa^{\alpha+1}$*

  In other words, ${}^{\kappa}V^0$ is the function that, starting at 2, enumerates the ordinals of cofinality κ that are closed under the ordinal addition.
  Other Veblen operation can be regarded as define in a similar way but by iterating this process along the fixpoints, namely as the ones that enumerates (starting at 2 instead of 1) the ordinals of cofinality κ that are closed under each Veblen function previously defined:

- *For $\lambda > 0$, ${}^{\kappa}V^{\lambda}$ is the function that, starting at 2, enumerates ordinals with cofinality κ that are fixpoints of all Veblen functions of lesser degree (${}^{\kappa}V^{\xi}$ for $\xi < \lambda$). i.e. ${}^{\kappa}V^{\lambda}(2)$ is the first fixpoint of cofinality κ of all ${}^{\kappa}V^{\xi}$ $(\xi < \lambda)$, ${}^{\kappa}V^{\lambda}(3)$ is the second,...,${}^{\kappa}V^{\lambda}(1+\nu)$ is the ν^{th}, etc.*

The way we defined the Veblen hierarchy may seem to be weird since it is not obviously the most simple way to do it. However, those complications in the presentation will be very useful when coming to the conciliatory operations that form the set theoretic counterpart of the ordinal operations. We complicated the presentation of the Veblen hierarchy to notably simplify the one of the Wadge hierarchy.
We want to make the two hierarchies fit together since for each countable ordinal ν we wish to define conciliatory operations "${}^{\kappa}V_c^{\nu}$" which preserve the conciliatory ordering and with the property that given any Borel conciliatory set A the Wadge degree of the resulting set ${}^{\kappa}V_c^{\nu}(A)$ is exactly ${}^{\kappa}V^{\nu}(d_c^{\circ}(A))$.

To begin with, let us properly define the notion of erasing from a sequence α every occurrence of letters among a given set Γ. This manipulation of removing extra letters before applying either an unfolding operation or an erasing one (by identification of one of these letters to the Back Space key) will be used a lot in the fore coming definition 5.

Definition 4 *Let $\alpha \in \Lambda^{\leq\omega}$, $\Gamma \subseteq \Lambda$, we define $\alpha(\ /\Gamma) = \alpha$ where all letters from Γ are removed by:*

- *in case α is finite*
  - *if $\alpha = <>$ then $\alpha' = <>$*
  - *if $\alpha = \beta^\frown < a >$, $a \in \Gamma$ then $\alpha(\ /\Gamma) = \beta(\ /\Gamma)$ where $\beta(\ /\Gamma)$ is β where all letters from Γ are removed.*
  - *if $\alpha = \beta^\frown < b >$, $b \notin \Gamma$ then $\alpha(\ /\Gamma) = \beta(\ /\Gamma)^\frown b$.*
- *in case α is infinite* $\alpha(\ /\Gamma) = \lim_n \Big((\alpha \restriction n)(\ /\Gamma)\Big) = \bigcup_n \Big((\alpha \restriction n)(\ /\Gamma)\Big)$

In order not to overload the scripture, in case $\Gamma = \{\twoheadleftarrow_{(\theta)} :\ 0 < \xi < \theta\}$ we write

$$\alpha(\ /\xi <) \text{ instead of } \alpha(\ /\{\twoheadleftarrow_{(\theta)} :\ 0 < \xi < \theta\});$$

and in case $\Gamma = \{\twoheadleftarrow_{(\theta)} :\ 0 < \xi \leq \theta\}$ we write

$$\alpha(\ /\xi \leq) \text{ instead of } \alpha(\ /\{\twoheadleftarrow_{(\theta)} :\ \xi \leq \theta\}).$$

Given any letter p, q on any infinite sequence, we write $\alpha(p/q)$ for the sequence α where every occurrence of the letter q is turned into an occurrence of the letter p. Therefore the notation $\alpha(\ /q)$ can be understood as the result of replacing every occurrence of q in α by nothing at all (which literally means erasing it).

We introduce an operation on finite and infinite sequences: $\alpha \longmapsto \alpha^{\circlearrowleft}$ based on the priority argument used in [1] to define the $\sim$ operation and is concerned with iterating this operation by unfolding.

Definition 5 *Let Λ be any alphabet, $\{\twoheadleftarrow_{(\theta)} : 0 < \theta < \omega_1\}$ a set of different arrows indexed by* **non zero** *ordinals such that , $\Lambda \cap \{\twoheadleftarrow_{(\theta)} : 0 < \theta < \omega_1\} = \emptyset$.*
As a convention, given $a \in \Lambda \cup \{\twoheadleftarrow_{(\theta)} : 0 < \theta < \omega_1\}$ we define $\mathcal{D}(a)$ the **depth** *of a letter in $\Lambda \cup \{\twoheadleftarrow_{(\theta)} : 0 < \theta < \omega_1\}$ by*

$$\begin{array}{lcl} \mathcal{D}(a) = 0 & \iff & a \in \Lambda \\ \mathcal{D}(a) = \theta > 0 & \iff & a = \twoheadleftarrow_{(\theta)} \end{array}$$

Given $\alpha \in (\Lambda \cup \{\twoheadleftarrow_{(\theta)} : 0 < \theta < \omega_1\})^{\leq\omega}$ and a countable ordinal ξ (possibly 0) we successively define $\alpha^{\hookleftarrow\xi}$, $\alpha^{\circlearrowleft\xi} \in \Lambda \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\})^{\leq\omega}$ and $\alpha^{\circlearrowleft} \in \Lambda^{\leq\omega}$. *In simple words, the first of them, $\alpha^{\hookleftarrow\xi}$, is just α when every occurrence of the letter $\twoheadleftarrow_{(\xi)}$ is translated into an eraser. This means that, beginning from left to right and imagining $\twoheadleftarrow_{(\xi)}$ represents the Back Space key (the one familiar to your keyboard), the sequence $\alpha^{\hookleftarrow\xi}$ is the one left after the whole erasing work is done; formally:*

Definition of $\alpha^{\hookleftarrow\xi}$

- $<>^{\hookleftarrow\xi} = <>$
- *in case α is finite but non empty, $\alpha = \beta^\frown < a >$:*

$$\begin{array}{lll} (\beta^\frown < a >)^{\hookleftarrow\xi} & = \beta^{\hookleftarrow\xi\frown} < a > & \textit{if } a \neq \twoheadleftarrow_{(\xi)} \\ & = <> & \textit{if } a = \twoheadleftarrow_{(\xi)} \textit{ and } lh(\beta^{\hookleftarrow\xi}) \leq 1 \\ & = \beta^{\hookleftarrow\xi} \upharpoonright (k-1) & \textit{if } a = \twoheadleftarrow_{(\xi)} \textit{ and } lh(\beta^{\hookleftarrow\xi}) = k > 1 \end{array}$$

- *for α infinite:*

$$\alpha^{\hookleftarrow\xi} = \lim_{n\in\omega}\Big((\alpha \upharpoonright n)^{\hookleftarrow\xi}\Big) = \cup\{u :\ \exists m \forall n \geq m \ \ u \subseteq \Big((\alpha \upharpoonright n)^{\hookleftarrow\xi}\Big)\}.$$

This operation $\alpha \longmapsto \alpha^{\hookleftarrow\xi}$ was the first step, now we use this tool to define $\alpha^{\circlearrowleft\xi}$. This next operation consists in an iteration of the previous one in a downward way that stops when arriving at level ξ.

Definition of $\alpha^{\circlearrowleft\xi}$ *Let $0 \leq \xi < \omega_1$,*
The sequence $\alpha^{\circlearrowleft\xi}$ is an element of $(\Lambda \cup \{\twoheadleftarrow_{(\theta)} : 0 < \theta \leq \xi\})^{\leq\omega}$ defined by

- *if $\alpha = <>$, then $\alpha^{\circlearrowleft\xi} = <>$,*

if $\alpha \neq <>$, depending on the depth of $\alpha(0)$:

- *if $\mathcal{D}(\alpha(0)) = 0$, then $\alpha^{\circlearrowleft\xi} = \alpha(\ /\xi <)$,*
  where $\alpha(\ /\xi <)$ is just α where all letters of the form $\twoheadleftarrow_{(\theta)}$ for $\theta > \xi$ are removed, (in particular if $\xi = 0$ then $\alpha^{\circlearrowleft\xi} = \alpha^{\circlearrowleft 0} = \alpha(\ /\{\twoheadleftarrow_{(\iota)} : 0 < \iota < \omega_1\})$)
- *if $0 < \mathcal{D}(\alpha(0)) \leq \xi$ i.e. $\alpha(0) = \twoheadleftarrow_{(\nu)}$ for some $\nu \leq \xi$, then $\alpha^{\circlearrowleft\xi} = \alpha(\ /\xi <)$,*
- *if $0 < \xi < \mathcal{D}(\alpha(0))$ and $\mathcal{D}(\alpha(0))$ is successor, i.e. $\alpha(0) = \twoheadleftarrow_{(\nu+1)}$ for some $\nu \geq \xi$, then $\alpha^{\circlearrowleft\xi} = ((\alpha(\ /\nu+1 <))^{\hookleftarrow\nu+1})^{\circlearrowleft\xi}$,*
- *if $0 \leq \xi < \mathcal{D}(\alpha(0)) = \delta$ **limit**, then*

$$\alpha^{\circlearrowleft\xi} =< (\alpha(\ /\delta \leq))_i{}^{\circlearrowleft\xi}(0)/i \in \omega >$$

where:

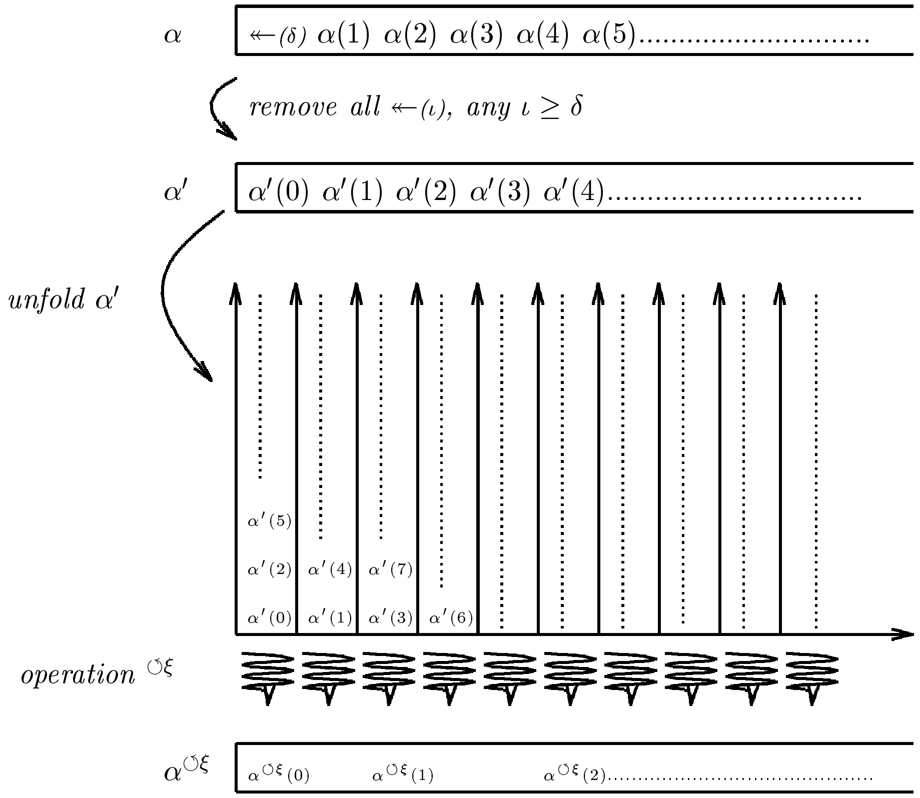
  - ◇ $(\alpha(\ /\delta \leq))$ *is α where all letters of the form $\twoheadleftarrow_{(\theta)}$ for $\theta \geq \delta$ are removed and*

◇ *for any sequence α' the sequence α'_i is defined by unfolding α' into ω sequences and picking up the one labeled i: $\alpha'_i = <\alpha'(\phi(i,j))/j \in \omega>$ for some bijection ϕ between ω^2 and ω, for example the Cantor function $K : \omega \times \omega \longleftrightarrow \omega$ defined by*

$$K(n,m) = \frac{(n+m)(n+m+1)}{2} + m$$

So to get $\alpha^{\circlearrowleft\xi}$ in case $\mathcal{D}(\alpha(0)) = \delta$ is limit greater than ξ (see picture below), one has first to erase every letter of depth at least δ which gives a new sequence α', then to unfold this sequence α' into ω sequences (the α_i's) and apply the $^{\circlearrowleft\xi}$ operation independently to each of them. And finally one has to pick up the first value of each resulting sequence and concatenate those letters:

$$\alpha^{\circlearrowleft\xi} = (\alpha'^{\circlearrowleft\xi}_0 \upharpoonright 1)^\frown(\alpha'^{\circlearrowleft\xi}_2 \upharpoonright 1)^\frown \ldots {}^\frown(\alpha'^{\circlearrowleft\xi}_n \upharpoonright 1)^\frown(\alpha'^{\circlearrowleft\xi}_{n+1} \upharpoonright 1)^\frown \ldots$$



The last step consist in iterating operation $\alpha \longmapsto \alpha^{\circlearrowleft\xi_i}$ along a descending "well chosen" chain of variables ξ_i.

Definition of $\boldsymbol{\alpha}^{\circlearrowleft}$ *(in any case $\alpha^{\circlearrowleft}$ is a sequence that belongs to the space $\Lambda^{\leq\omega}$)*

- *if $\alpha = <>$, then $\alpha^{\circlearrowleft} = <>$,*
- *if $\mathcal{D}(\alpha(0)) = 0$ i.e. $\alpha(0) \in \Lambda$, then $\alpha^{\circlearrowleft} = \alpha^{\circlearrowleft 0}$ which is by definition the sequence α where all letters of the form $\twoheadleftarrow_{(\theta)}$ for $\theta > 0$ are removed, so that it is exactly α after all "arrows" are thrown away.*

- *if $\mathcal{D}(\alpha(0)) = \nu + 1$ i.e. $\alpha(0) = \twoheadleftarrow_{(\nu+1)}$, then $\alpha^{\circlearrowleft} = (\alpha^{\circlearrowleft \nu})^{\circlearrowleft}$, which roughly speaking means $\alpha^{\circlearrowleft} = (\alpha(\ /\nu + 1 <)^{\looparrowleft \nu+1})^{\circlearrowleft}$ where $\alpha(\ /\nu + 1 <)^{\looparrowleft \nu+1}$ is just α where first all letters of the form $\twoheadleftarrow_{(\theta)}$ for $\theta > \nu + 1$ are removed and after that the "Back Space key" is identified with $\twoheadleftarrow_{(\nu+1)}$.*
- *if $\mathcal{D}(\alpha(0)) = \delta$ limit, it admits a Cantor Normal Form of base ω:*

  $$\delta = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0$$

  with

  $$\omega_1 > \xi_k > \ldots > \xi_0 > 0 \text{ and } \omega > n_i > 0 \text{ for each } i \leq k$$

  then

  $$\alpha^{\circlearrowleft} = (\alpha^{\circlearrowleft \delta - \omega^{\xi_0}})^{\circlearrowleft}$$

  where by convention $\delta - \omega^{\xi_0} = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.(n_0 - 1)$. (Remark it is possible to have $\delta - \omega^{\xi_0} = 0$).

So for a finite or infinite sequence α which letters are among the alphabet $\Lambda \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota < \omega_1\}$, the decoding process of α that we note $\alpha^{\circlearrowleft}$ consists in: first of all looking at the first letter of the sequence. If α is the empty sequence then there is no such letter and then $\alpha^{\circlearrowleft}$ is just the empty sequence. If $\alpha(0)$ exists the first thing to do is to throw away every arrow indexed by a greater ordinal than $\mathcal{D}(\alpha(0))$ (this is to insure the finiteness of the process) which gives rise to a new sequence α'. Then the depth of $\alpha(0)$ determines the way it has to be read:

- if $\alpha'(0)$ is a letter that belongs to Λ then $\alpha^{\circlearrowleft}$ is just α'. Nothing else can be done and the decoding process stops.

- If $\alpha'(0)$ is a letter of the form $\twoheadleftarrow_{(\nu+1)}$ then the next step is to apply the erasing operation to α' with $\twoheadleftarrow_{(\nu+1)}$ used as an eraser. Namely this gives α'' which is exactly α' when the letter $\twoheadleftarrow_{(\nu+1)}$ is replaced by the Back Space key and operates from left to right inside α' (considering the sequence α' is also written from left to right: $\alpha' = \alpha(0).\alpha(1)\ldots.$). Then to obtain $\alpha^{\circlearrowleft}$ one has to once again iterate this process but this time to the sequence α'' instead of α'. So we get $\alpha^{\circlearrowleft} = \alpha''^{\circlearrowleft}$.

- Last but not least is what happens when $\alpha'(0) = \twoheadleftarrow_{(\delta)}$ for some δ limit. (This is really the interesting case since the previous one is interesting only in case ν is finite). The goal is to unfold α' into ω sequences and apply the very same operation to each of them so to finally pick up one information from each of them, every such different information being gathered into a new sequence. But the sequence α' still contains letters of the form $\twoheadleftarrow_{(\delta)}$ and we want the decoding process to stop, so we need first to eliminate all letters of the form $\twoheadleftarrow_{(\delta)}$ and to unfold the resulting sequence into ω sequences $(\alpha'_i)_{i \in \omega}$ by use of the Cantor pairing function. Then it is like the operation $^{\circlearrowleft}$ is applied to each of them; except that in order to sufficiently make complex the resulting sequence $\alpha^{\circlearrowleft}$, we need to repeat many times this unfolding operation. So for each ${\alpha'}_i$ we need the decoding process to stop at some point in order to gather the information and then to start again with the resulting sequence, and to do that many times. Precisely this border line we have to stop at is given by referring to the Cantor Normal Form of base ω of the ordinal δ. We are going to stop at the ordinal $\delta' = \omega^{\delta_k}.n_k + \ldots + \omega^{\delta_0}.(n_0 - 1)$

when the C.N.F. of δ is $\omega^{\delta_k}.n_k + \ldots + \omega^{\delta_0}.n_0$. That means that any ${\alpha'}_i$ is decoded by use of this iterating process until a sequence α''_i is reached that is either the empty sequence or contains a first letter of depth less or equal to δ'. Then, picking up the first letter of each α''_i (no letter at all when α''_i is the empty sequence) one gathers this information into a new sequence β which necessarily contains letters of depth at most δ': $\beta =< \alpha''_i(0)/i < \omega >$. At last we repeat the whole process but this time we apply it to β, so that this stops after a finite number of steps and gives $\alpha^{\circlearrowleft} = \beta^{\circlearrowleft}$.

We could have chosen another definition for α that avoids referring to the Cantor Normal Form of countable ordinals simply by incorporating the C.N.F. in the definition itself: we could have considered arrows of the form $\twoheadleftarrow_{((\delta_k,n_k),\ldots,(\delta_0,n_0))}$ where the sequence $(\delta_i)_{i\leq k}$ is strictly increasing and each n_i is some non null integer.

Remarks 6 Since we defined $^{\circlearrowleft}$ by use of a set of "arrows" disjoint from Λ, we may iterate this process and consider another set of arrows disjoint from both Λ and $\{\twoheadleftarrow_{(\iota)} : 0 < \iota < \omega_1\}$, and repeat it again. so that going on this way we may define sets or arrows of different type (or color) or more precisely of different strengths, indicating the order in which they were created; or in other words, making a (finite) hierarchy of their relative strength.
Given $0 < \xi, \nu < \omega_1$, let $\{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}$ be a set of arrows indexed by ordinals, disjoint from Λ. Set $\Gamma = \Lambda \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}$ and get $\{\twoheadleftarrow'_{(\iota)} : 0 < \iota \leq \nu\}$ a set of "prime" arrows disjoint from Γ. Assume the prime operations $^{\circlearrowleft'\theta}$ and $^{\circlearrowleft'}$ are defined the same way $^{\circlearrowleft\theta}$ and $^{\circlearrowleft}$ were defined except that they operate on "prime arrows" regarding the other arrows as simple letters just as if they were letters from Λ. Moreover assume that ξ admits the following Cantor Normal Form of base ω:

$$\xi = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0$$

and assume also that the ordinal ν is strictly lesser than ω^{ξ_0+1} which means $\nu \leq \omega^{\xi_0}.n$ holds for some integer n.

We claim that in both following games player *II* has a winning strategy. (These games are particular conciliatory Wadge games).

(a) At each move *I* plays a letter in $\Lambda \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi + \nu\}$ and *II* plays a letter in $\Gamma \cup \{\twoheadleftarrow'_{(\iota)} : 0 < \iota \leq \nu\}$. So at the end of a run *I* has produced α and *II* has played β:

$$\textit{II} \text{ wins iff } \alpha^{\circlearrowleft} = (\beta^{\circlearrowleft'})^{\circlearrowleft}$$

(b) At each move *I* plays a letter in $\Gamma \cup \{\twoheadleftarrow'_{(\iota)} : 0 < \iota \leq \nu\}$ and *II* replies with a letter in $\Lambda \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi + \nu\}$. So at the end of the game *I* and *II* have respectively produced α and β:

$$\textit{II} \text{ wins iff } (\alpha^{\circlearrowleft'})^{\circlearrowleft} = \beta^{\circlearrowleft}$$

Obviously, a w.s. for *II* in the first game consists in playing exactly the same letters as *I* except when *I* plays $\twoheadleftarrow_{(\theta)}$, with $\theta > \xi$, in which case there exists a unique $\delta \leq \nu$ such that $\theta = \xi + \delta$ and *II* plays $\twoheadleftarrow'_{(\theta)}$. According to this strategy, if α is the empty sequence then β is also the empty sequence; and if $\alpha(0)$ exists then it is tedious but straightforward to establish the result by induction

on $\mathcal{D}(\alpha(0))$. The very same idea *mutatis mutandis* works in the second game.
So, in general we will always switch between

$$\Lambda \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi + \nu\} \text{ and } \Lambda \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\} \cup \{\twoheadleftarrow'_{(\iota)} : 0 < \iota \leq \nu\}.$$

Also, generalizing that property, we will be authorized to decompose a set of arrows into different sets of different types of arrows, each type being related to a special "strength" (relatively to the other ones).

Nevertheless we have to be careful when juggling with arrows of different "strengths". In the above remark it is essential that ν is not too big: we mentioned the condition $\nu < \omega^{\xi_0+1}$. We will realize at the end of this preliminary work that more just $\nu < \omega^{\xi_0+1}$, what is really essential is the condition $\nu < \omega^{\xi_k+1}$. The property holds with $\nu < \omega^{\xi_k+1}$ but fails as soon as $\nu \geq \omega^{\xi_k+1}$.

Remarks 7 In [1] we defined operation $^{\hookleftarrow}$ without any ordinal index since we were only dealing with one arrow. For any sequence $\alpha \in (\Lambda \cup \{thla\})^{\leq\omega}$ the sequence $\alpha^{\hookleftarrow}$ was defined simply as the resulting sequence after identifying the arrow with the Back Space key. It corresponds to our version of a sequence $\alpha' \in (\Lambda \cup \{thlavar1\})^{\leq\omega}$ when $\alpha' = \Big(< \twoheadleftarrow_{(1)} > {}^\frown(\alpha(\twoheadleftarrow_{(1)}/\twoheadleftarrow))\Big)^{\circlearrowleft}$. According to that, in the sequel, when we will need only one extra arrow we will write $\twoheadleftarrow$ instead of $\twoheadleftarrow_{(1)}$; and same shortening will hold in case of arrows of different type (e.g. $\twoheadleftarrow'$ will mean $\twoheadleftarrow'1$).

Our main tool in that work is an extension of the "remote-control lemma" stated in [1] (lemma 21) . This generalization says that given a strategy $\mathcal{T}$ for *II* in a conciliatory Wadge game, when *I* plays letters in Λ_A and *II* plays letters in Λ_B, player *II* has a strategy $\sigma_{\mathcal{T}}^{\xi}$ in the conciliatory game where *I* plays letters in $\Lambda_A \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}$ and *II* replies with letters in $\Lambda_B \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}$; this strategy $\sigma_{\mathcal{T}}^{\xi}$ being a "remote-control" of $\mathcal{T}$ in the sense that the sequence obtained by applying $\sigma_{\mathcal{T}}^{\xi}$ to *I*'s run and then "decoding" it - i.e. after the $^{\circlearrowleft}$ operation involved by the arrows is taken into account - is almost the result of the strategy $\mathcal{T}$ applied to the decoding result of *I*'s. "Almost" means here that it is $\mathcal{T}$ applied to an s-variant of *I*'s run once decoded. In other words it is a sequence that is equivalent to it when forgetting the skips (see picture below).

$$
\begin{tikzcd}[column sep=large]
 & \left(\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}\right)^{\omega} & \left(\Lambda_B \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}\right)^{\omega} \\
 & \alpha \arrow[r, "\sigma_{\tau}^{\xi}"] \arrow[d, bend right] & \beta \arrow[d, bend right] \\
\text{"removing" the Skips} & \alpha(\ /s) \arrow[d, squiggly] & \beta(\ /s) \arrow[d, squiggly] \\
\text{applying the } {}^{\circlearrowleft} \text{ operation} & \alpha(\ /s)^{\circlearrowleft} & \beta(\ /s)^{\circlearrowleft} \\
\text{"removing" the Skips} & \alpha' \arrow[r, "\tau"] \arrow[u, bend left] & \beta' \arrow[u, bend left] \\
 & \left(\Lambda_A \cup \{s\}\right)^{\omega} & \left(\Lambda_B \cup \{s\}\right)^{\omega}
\end{tikzcd}
$$

Precisely let us introduce the relation $\preceq_\tau$ between countable ordinals. We will be concerned with that relation until the end of this section. What we mentioned above can be express by $\xi \preceq_\tau \xi$ holds for any countable ξ and any strategy τ.

Definition 8 (remote control of a strategy) *Given* ζ, ξ *two countable ordinals and a strategy*τ*:*

$$\tau \ : \ (\Lambda_A \cup \{s\})^{<\omega^*} \longmapsto \Lambda_B \cup \{s\}$$

we write:

$$\boldsymbol{\zeta \preceq_\tau \xi}$$

if there exists a strategy:

$$\sigma_\tau^{\zeta,\xi} \ : \ (\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\})^{<\omega^*} \longmapsto \Lambda_B \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}$$

such that $\forall \alpha \in (\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\})^\omega$*:*

- *(a)* $(\sigma_\tau^{\zeta,\xi} * \alpha)(0) = \twoheadleftarrow_{(\xi)}$ *(by convention in case* $\xi = 0$ *there is no condition (a) at all)*

  and

- *(b)* $\exists \alpha' \in \Lambda_A \cup \{s\} \quad \alpha(\ /s)^{\circlearrowleft} = \alpha'(\ /s)$ *and* $(\sigma_\tau^{\zeta,\xi} * \alpha)(\ /s)^{\circlearrowleft} = (\tau * \alpha')(\ /s)$

We write $\boldsymbol{\sigma_\tau^\xi}$ *instead of* $\sigma_\tau^{\xi,\xi}$*. And we use the notation* $\boldsymbol{\zeta \preceq \xi}$ *when* $\zeta \preceq_\tau \xi$ *holds for any strategy* τ*. If we regard the strategy* $\sigma_\tau^{\zeta,\xi}$ *as a strategy for player II in a conciliatory game, condition (a) says that player II must play at least one letter and its first move has to be* $\twoheadleftarrow_{(\xi)}$*. Condition (b) says that the strategy remote-controls the strategy* τ*.*

Remarks 9

(a) If $\xi \preceq \xi$ holds and ζ is an ordinal strictly less than ξ then $\zeta \preceq \xi$ holds too because when proving $\xi \preceq \xi$, in the underlying game, I may play $\alpha(0) = \twoheadleftarrow_{(\zeta)}$.

(b) We have already shown that $1 \preceq 1$ holds in [1] (see lemma 21). This can be generalized and give:

Lemma 10 (remote-control lemma) *For all countable ξ,*

$$\xi \preceq \xi$$

$\square$ **lem. 10** - The proof goes by induction on $\xi < \omega_1$.

$\xi = 0$ is obvious since $\sigma^0_\tau = \tau$ works.

$\xi = 1$ was proved in [1] (see lemma 21), except that in the actual version player $I\!I$ has to play $\twoheadleftarrow_{(1)}$ at first move, previously to the run stated in [1]. This is the one and only difference between the two versions.

$\xi = \zeta + 1$ Given $\tau \ : \ (\Lambda_A \cup \{s\})^{<\omega^*} \longmapsto \Lambda_B \cup \{s\}$ we define σ^ξ_τ that fits conditions of definition 8. We regard σ^ξ_τ as a strategy for $I\!I$ in a conciliatory game where I plays α and $I\!I$ replies with β according to the strategy σ^ξ_τ.

First, $I\!I$ plays $\beta(0) = \twoheadleftarrow_{(\xi)}$. Then:

(a) as long as I skips, $I\!I$ plays directly applying strategy τ. So, if player I indefinitely skips, then player $I\!I$ will have produced $\beta = \twoheadleftarrow_{(\xi)}{}^\frown \tau * s^\omega$; which gives $\alpha(\ /s)^{\circlearrowleft} = \alpha(\ /s) =<>$ and $\beta(\ /s)^{\circlearrowleft} = (\tau * s^\omega)(\ /s)$.

(b) If I plays first real move $\alpha(0)$, then $I\!I$ erases the initial segment of $\tau * s^\omega$ he has already played at that moment, using only the letter $\twoheadleftarrow_{(\xi)}$ for this task. So, after this is done everything goes as if he has only played $\beta(0) = \twoheadleftarrow_{(\xi)}$ and nothing else.

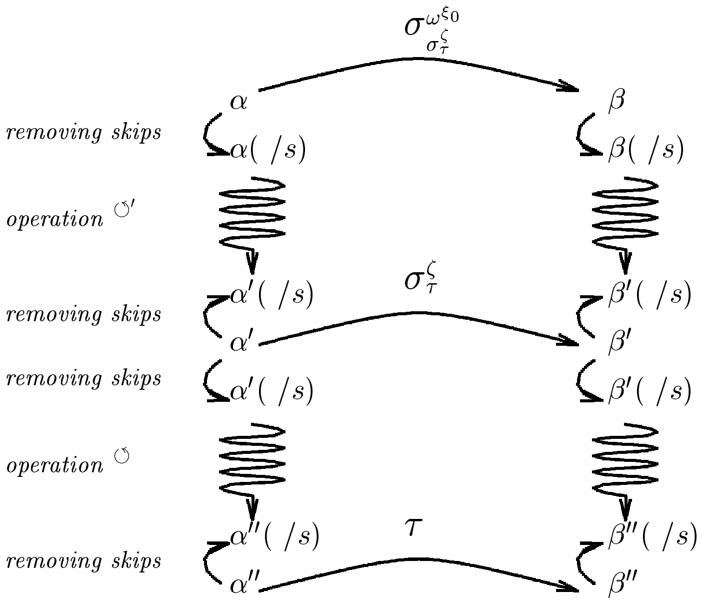
(i) if $\alpha(0) \in \Lambda_A$, then $I\!I$ simply applies τ,

(ii) if $\alpha(0) = \twoheadleftarrow_{(\theta)}$ with $\theta < \xi$, then $I\!I$ applies strategy σ^θ_τ that θ-remote controls τ, taking care of turning every letter $\twoheadleftarrow_{(\theta')}$ for $\theta' > \theta$ played by I into a skip.

(iii) If $\alpha(0) = \twoheadleftarrow_{(\xi)}$, then $I\!I$ 1-remote controls the strategy σ^ζ_τ (where $\twoheadleftarrow_{(\xi)}$ stands for the letter $\twoheadleftarrow'$ induced by the 1-remote control process). This means $I\!I$ applies $\sigma^1_{\sigma^\zeta_\tau}$. More precisely, we may regard the game as a game where

- I plays letters among $\Lambda_A \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota < \xi\} \cup \{\twoheadleftarrow'\}$ and
- $I\!I$ replies with letters among $\Lambda_B \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota < \xi\} \cup \{\twoheadleftarrow'\}$

with the prime arrow being of greater strength than all the others. Player I instead of playing α plays α' which differs from α by the fact every occurrence of $\twoheadleftarrow_{(\xi)}$ is replaced by $\twoheadleftarrow'$, and $I\!I$ replies with $\beta' = \sigma^1_{\sigma^\zeta_\tau} * \alpha'$. Then the sequence $\beta = \beta'(\twoheadleftarrow_{(\xi)}/\twoheadleftarrow')$ is as required because, by remark 6, one gets $(\beta'^{\circlearrowleft'})^{\circlearrowleft} = \beta^{\circlearrowleft}$ and $(\alpha'^{\circlearrowleft'})^{\circlearrowleft} = \alpha^{\circlearrowleft}$.



ξ limit The Cantor Normal Form of ξ is $\xi = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0$ with $\omega_1 > \xi_k > \ldots > \xi_0 > 0$ and $\omega > n_i > 0$ for all $i \leq k$.

$k > 0$ or $n_0 > 1$ Set $\zeta = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.(n_0 - 1)$. By induction hypothesis, $\zeta \preceq_{\tau} \zeta$ holds too by induction hypothesis because $\omega^{\xi_0} < \xi$ is satisfied. So there exists a w.s. for *II* in the correlated game:

$$\sigma^{\zeta}_{\tau} \ : \ (\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\})^{<\omega^*} \longmapsto \Lambda_B \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\}$$

And $\omega^{\xi_0} \preceq_{\sigma^{\zeta}_{\tau}} \omega^{\xi_0}$ holds; so their exists a w.s. for *II* in the correlated game:

$$\sigma^{\omega^{\xi_0}}_{\sigma^{\zeta}_{\tau}} \ : \ (\Lambda'_A \cup \{s\} \cup \{\twoheadleftarrow'_{(\iota)} : 0 < \iota \leq \omega^{\xi_0}\})^{<\omega^*} \longmapsto \Lambda'_B \cup \{s\} \cup \{\twoheadleftarrow'_{(\iota)} : 0 < \iota \leq \omega^{\xi_0}\}$$

where $\Lambda'_A = \Lambda_A \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\}$ and $\Lambda'_B = \Lambda_B \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\}$. But we saw in remark 6 that we may as well rename the prime arrows by $\twoheadleftarrow_{(\iota)}$ for $\zeta < \iota \leq \zeta + \omega^{\xi_0}$. Therefore this simple fact shows that when using these new labels, the following strategy works

$$\sigma^{\omega^{\xi_0}}_{\sigma^{\zeta}_{\tau}} \ : \ \Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\} \longmapsto \Lambda_B \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}.$$

$k = 0$ & $n_0 = 1$ i.e. $\xi = \omega^{\xi_0}$. *II* plays $\beta(0) = \twoheadleftarrow_{(\omega^{\xi_0})}$, which "opens an array" meaning that the way β will be read to get $\beta^{\circlearrowleft}$ consists in unfolding $\beta(\ /\twoheadleftarrow_{(\omega^{\xi_0})})$ into ω sequences $(\beta^i)_{i\in\omega}$ so that $\beta^{\circlearrowleft} =< {\beta^i}^{\circlearrowleft}(0)/i \in \omega >$. And we regard β^i as being the content of the column of the array labeled i. So we say that *II* plays so and so in column i instead of speaking of β^i. Now as long as *I* skips, *II* plays so that in column i, at the end of the game, if *I* has always been skipping, there is $\twoheadleftarrow_{(1)}{}^\frown \left(\tau(s^{i+1})\right)^{\omega}$ if $\tau(s^{i+1})$ is not a skip and $\twoheadleftarrow_{(1)}{}^{\omega}$ otherwise.

If *I* plays $\alpha(0)$, then *II* kills every column he has already played something in. By

"killing" a column we mean that *II* plays something that column so that the decoding of its content will lead to the empty sequence. We say that the column is killed because in the decoding process it is as if it does not exist any more.
And the way we want *II* to kill each column in which something has already been played in, is of course by use of the eraser $\twoheadleftarrow_{(1)}$ (for example playing $\twoheadleftarrow_{(1)}$ in that column until the end of the game). Then *II* relabels the columns that are not killed: from now on, the first column not killed is labeled 0, the second is labeled 1, and so on. We also say that *II* plays β_0 in the first column, β_1 in the second,etc.

(a) if $\alpha(0) \in \Lambda_A$ then everything is easy.

(b) if $\alpha(0) = \twoheadleftarrow_{(\theta)}$ with $\theta < \omega^{\xi_0}$; get $\zeta < \xi_0$ and n a non null integer such that $\omega^{\xi_0} > \omega^{\zeta}.n > \theta$. As $\theta \preceq_{Id} \omega^{\zeta}.n$ holds by induction hypothesis, we may without loss of generality assume that *I* played $\alpha(0) = \omega^{\zeta}.n$. We need the following claim.

Claim 11 - *Given a letter "r" which is not in $\Lambda_A \cup \{s, \twoheadleftarrow\}$, there is a w.s for II in the conciliatory game where I plays letters in Λ_A, II plays letters in $\Lambda_A \cup \{r, \twoheadleftarrow\}$, both players are allowed to skip. So that at the end of the run I has constructed $\alpha \in \Lambda_A^{\leq\omega}$ and II $\beta \in (\Lambda_A \cup \{r, \twoheadleftarrow\})^{\leq\omega}$ and II wins iff*

- $\beta^{\circlearrowleft} = \alpha$ *in the case* α *is infinite*
- $\beta^{\circlearrowleft} = \alpha.r^{\omega}$ *in the case* α *is finite*

And I wins otherwise.

$\square$ **cl. 11** - The strategy consists for *II* in playing $\beta_0 = \twoheadleftarrow$ and then the letter r every time *I* skips,providing that when *I* plays a letter $a \in \Lambda_A$, *II* erases his greatest (for some integer p) terminal sequence of the form r^p and plays this same letter a. This gives the result, for if *I* played an infinite sequence, *II* played β such that $\beta^{\circlearrowleft} = \alpha$; and if *I* played a finite sequence, it is because he always skipped after a certain move thus *II* played β such that $\beta^{\circlearrowleft} = \alpha.r^{\omega}$.

11$\square$

Let δ be the strategy defined in claim 11. We may think as well that δ is a strategy from $(\Lambda_A \cup \{s\})^{<\omega^*}$ into $\Lambda_A \cup \{r\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\zeta}\}$ - instead of $\Lambda_A \cup \{r, \twoheadleftarrow\}$ - so that $(\delta * \alpha)^{\circlearrowleft} = \alpha(\ /s)$ when $\alpha(\ /s)$ is infinite and $(\delta * \alpha)^{\circlearrowleft} = \alpha(\ /s)^\frown r^{\omega}$ when $\alpha(\ /s)$ is finite. By induction hypothesis and referring to remark 6, there exists a strategy that $\omega^{\zeta}.n$-remote controls this strategy δ. We may view this strategy as:

$$\sigma_{\delta}^{\omega^{\zeta}.(n+1)} : \Big(\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\zeta}.n\}\Big)^{<\omega^*} \longmapsto \Lambda_A \cup \{r\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\zeta}.(n+1)\}.$$

Now, consider the strategy γ_i that picks up the initial segment of length $i+1$ from a sequence and applies τ to it, turning every occurrence of the letter r into skips:
$\gamma_i : (\Lambda_A \cup \{s, r\})^{<\omega^*} \longmapsto \Lambda_A \cup \{s\}$ defined by:

- $\gamma_i(u) = s$ if $lh\Big(u(\ /s)\Big) \neq i+1$
- $\gamma_i(u) = \tau(u(\ /s)(s/r))$ if $lh\Big(u(\ /s)\Big) = i+1$

and $\sigma_{\gamma_i}^{\omega^{\zeta}.(n+1)}$ a strategy that $\omega^{\zeta}.(n+1)$-remote controls the strategy γ_i:

$$\sigma_{\gamma_i}^{\omega^{\zeta}.(n+1)} : (\Lambda_A \cup \{s, r\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\zeta}.(n+1)\} \longmapsto (\Lambda_A \cup \{s, r\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\zeta}.(n+1)\}$$

II plays $\left(\sigma_{\gamma_i}^{\omega^\zeta.(n+1)} \circ \sigma_\tau^{\omega^\zeta.(n+1)}\right) * \alpha$ in column labeled i. So, this leads to a sequence β that satisfies:

$$\beta^{\circlearrowleft} =< \beta_i^{\circlearrowleft}(0)/i \in \omega >$$

with

$$\begin{array}{rcll} < \beta_i^{\circlearrowleft}(0)/i \in \omega > & = & \delta * (\alpha(\ /s)^{\circlearrowleft}) & \text{if } \alpha(\ /s)^{\circlearrowleft} \text{ is infinite} \\ < \beta_i^{\circlearrowleft}(0)/i \in \omega > & = & \delta * (\alpha(\ /s)^{\circlearrowleft\frown} s^\omega) & \text{otherwise.} \end{array}$$

(c) If $\alpha(0) = \twoheadleftarrow_{(\omega^{\xi_0})}$, this letter "opens an array". Without loss of generality we may assume that

(i) *I* never skips anymore after he played $\alpha(0)$ (note that playing once again the letter $\twoheadleftarrow_{(\omega^{\xi_0})}$ is the same as skipping). Moreover when *I* plays α_i in column labeled i we want each α_i to be infinite. This is clearly possible because of two different points

- it is easy to fill up a column that will be decoded as the empty sequence, therefore it is as if it does not exist at all. So assume for example that player *I* waits indefinitely and so never write anything in all columns labeled $i + k$ with $k \in \omega$; one can replace that run by another equivalent one where something is written in every column but of course almost all of them, once decoded, give rise to the empty sequence.
- When at least one letter has been written by *I* in a column (so that α_i will not be the empty sequence) one can continuously get an infinite sequence α'_i that will be the same as α_i after both have being decoded.

(ii) For each $i \in \omega$, $\alpha_i(0) = \omega^{\zeta_i}.n_i$ for some $\zeta_i < \xi_0$ and $0 < n_i < \omega$ (we simply use the fact that for any $\theta < \omega^{\xi_0}$ there exists some ordinal $\omega^{\zeta_i}.n_i$ which satisfies $\theta < \omega^{\zeta_i}.n_i < \omega^{\xi_0}$ and $\theta \preceq \omega^{\zeta_i}.n_i$ holds by induction hypothesis, so the identity strategy can be $\omega^{\zeta_i}.n_i$-remote controlled).

(iii) If $i < j$ holds then $\omega^{\zeta_i}.n_i < \omega^{\zeta_j}.n_j$ is satisfied. (This is only required for convenience in the explanation of the proof).

Same argument as developed in (b) gives a strategy:

$$\sigma_\delta^{\omega^{\zeta_i}.(n_i+1)} : \left(\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\zeta_i}.n_i\}\right)^{<\omega^*} \longmapsto \Lambda_A \cup \{r\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\zeta_i}.(n_i+1)\}$$

which is a $\omega^{\zeta_i}.(n_i)$-remote control of the strategy δ defined in claim 11, that we regard as a strategy from $(\Lambda_A \cup \{s\})^{<\omega^*}$ into $\Lambda_A \cup \{r\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\zeta_i}\}$ - instead of $\Lambda_A \cup \{r, \twoheadleftarrow\}$ - and such that $(\delta * \alpha')^{\circlearrowleft} = \alpha'(\ /s)$ when $\alpha'(\ /s)$ is infinite and $(\delta * \alpha')^{\circlearrowleft} = \alpha'(\ /s)^\frown r^\omega$ when $\alpha'(\ /s)$ is finite. Given by induction hypothesis, $\sigma_\tau^{\omega^{\zeta_i}.(n_i+1)}$ is the strategy that $\omega^{\zeta_i}.n_i$-remote controls δ. So *I* plays α_i in column i. Remark that $\sigma_\tau^{\omega^{\zeta_i}.(n+1)} * \alpha_i = \alpha^i$ is a sequence that verifies the following property:

(i) $\alpha^{i\circlearrowleft}$ is infinite

(ii) $\alpha_i^{\circlearrowleft} = \alpha^{i\circlearrowleft}(\ /r)$ (the letter r witnesses the possible skips).

(iii) $\alpha_i^{\circlearrowleft} =<>$ iff $\alpha^{i\circlearrowleft}(0) = r$.

What we want to do is to consider $\alpha^{\circlearrowleft}(0)$ and play in $I\!I$'s first available column a sequence β_0 which is given by a $\omega^{\xi_0}.n_0$-remote control of the strategy that just considers $\alpha^{0^{\circlearrowleft}} \restriction 1$ and apply $\mathcal{T}$ to that 1-sequence (previously turning r into s if necessary). This will continue so that in $I\!I$'s $(i+1)^{th}$ available column a sequence β_i which would belong to $(\Lambda_B \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\xi_i} n.i\})^\omega$ with the property that $\beta_i^{\circlearrowleft}(0)$ is exactly $\mathcal{T}(< \alpha^{0^{\circlearrowleft}}(0), \alpha^{1^{\circlearrowleft}}(0), \ldots, \alpha^{i^{\circlearrowleft}}(0) > (s/r))$. If one could do that, it is clear that such a strategy would be winning for $I\!I$. So, what we just need is the following:

Corollary 12 For all $n \in \omega,\ \xi < \omega_1$ and $\mathcal{T}$ any strategy:

$$\mathcal{T}\ :\ \Big((\Lambda_{A_0} \cup \{s\}) \times (\Lambda_{A_1} \cup \{s\}) \times \ldots \times (\Lambda_{A_n} \cup \{s\}) \Big)^{<\omega^*} \longmapsto \Lambda_B \cup \{s\}$$

there exists a strategy:

$$\sigma_{\mathcal{T}}^{(n,\xi)}\ :\ \Big(\prod_{j \leq n} \big(\Lambda_{A_j} \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}\big) \Big)^{<\omega^*} \longmapsto \Lambda_B \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}$$

such that:

If $\forall i \leq n \quad \alpha_i \in (\Lambda_{A_i} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\})^\omega$ satisfies $\boldsymbol{\alpha_i\ (\ /s)^{\circlearrowleft}}$ **is infinite** then

$$\mathcal{T}\!\ast \big(\alpha_0\,(\ /s)^{\circlearrowleft}, \ldots, \alpha_n\,(\ /s)^{\circlearrowleft}\big) = \Big(\sigma_{\mathcal{T}}^{(n,\xi)} \ast (\alpha_0, \ldots, \alpha_n)\Big)\,(\ /s)^{\circlearrowleft}$$

$\square$ **cor. 12** - This is by induction on ξ.

The case $\xi = 0$ is obvious.

The case $\xi = 1$ is still easy: as every $\alpha_i(\ /s)^{\hookleftarrow}$ is infinite, for all $k < \omega$ there exists a stage $\phi(k)$ such that **every** $\alpha_i(\ /s)^{\hookleftarrow} \restriction k$ will remain un-erased until the end of the game. First agree that $I\!I$ is always waiting until each column of I contains an **effective** sequence of length at least k to play its k^th^ move which is not a skip. So, this agreement settles the problem of I's skips. Now $I\!I$ can apply $\mathcal{T}$ as long as I goes on playing letters which are not $\twoheadleftarrow$. When player I erases a part of the content of one column i.e. the **effective** current run in the column i was u_i of length q and is now $u \restriction p$ for some $p < q$, then $I\!I$ erases just what is enough of his final moves so to keep only $\mathcal{T}\!\ast(u_0 \restriction p, \ldots, u_n \restriction p)$.

Assume now that ξ is strictly greater than 1 and admits the following C.N.F. of base ω:

$$\xi = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0$$

- **Assume $\boldsymbol{n_0 > 1}$ or $\boldsymbol{k > 0}$**. Set $\zeta = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.(n_0 - 1)$. By induction hypothesis there exists a strategy

  $$\sigma_{\mathcal{T}}^{(n,\zeta)}\ :\ \Big(\prod_{j \leq n} \big(\Lambda_{A_j} \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\}\big) \Big)^{<\omega^*} \longmapsto \Lambda_B \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\}$$

  that ζ remote-controls the strategy $\mathcal{T}$. And also by induction hypothesis, there exists a strategy $\sigma_{\sigma_{\mathcal{T}}^{(n,\zeta)}}^{(n,\omega^{\xi_0})}$ from

  $$\Big(\prod_{j \leq n} \Big(\Lambda_{A_j} \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\} \cup \{\twoheadleftarrow'_{(\iota)} : 0 < \iota \leq \omega^{\xi_0}\}\Big) \Big)^{<\omega^*}$$

into

$$\Lambda_B \cup \{s\} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \zeta\} \cup \{\twoheadleftarrow'(\iota) : 0 < \iota \leq \omega^{\xi_0}\}$$

that ω^{ξ_0} remote-controls the strategy $\sigma_\tau^{(n,\zeta)}$.
Now remark that for any $i \leq n$, if the sequence $\alpha_i \in (\Lambda_{A_i} \cup \{s\} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \zeta\})^\omega$ satisfies $\alpha_i(\ /s)^\circlearrowleft$ is infinite then it necessarily verifies α_i is infinite. And also by remark 6 we may rename the prime arrows and view the strategy $\sigma_{\sigma_\tau^{(n,\zeta)}}^{(n,\omega^{\xi_0})}$ as a strategy from

$$\Big(\prod_{j \leq n} \big(\Lambda_{A_j} \cup \{s\} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \xi\}\big)\Big)^{<\omega^*}$$

into

$$\Lambda_B \cup \{s\} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \xi\}$$

And it is the desired strategy.

- **Assume $n_0 = 1$ and $k = 0$**, with $\xi_0 > 0$ so $\xi = \omega^{\xi_0}$ is limit. By what precedes, without loss of generality we may assume that $\alpha_i(0) = \omega^{\xi_0}$ so that each sequence α_i will be unfolded into ω different sequences $\alpha_{i,k}$ to get the result $\alpha_i^\circlearrowleft =< \alpha_{i,k}^\circlearrowleft(0)/k \in \omega >$. Moreover, applying to the identity strategy the remote-control lemma that we are actually proving, we may also assume that $\alpha_{i,k}$ is a sequence that belongs to $(\Lambda_{\mathrm{A}_i} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\theta_k}.n_k\})^\omega$ (k does not depend on i, it is the same for all i) with the additional property that the sequence $(\omega^{\theta_k}.n_k)_{k\in\omega}$ is increasing.
Now by use of claim 11 and remark 6 we may "**continuously**" replace every sequence $\alpha_{i,k} \in (\Lambda_{\mathrm{A}_i} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\theta_k}.n_k\})^\omega$ by a sequence

$$\alpha'_{i,k} \in (\Lambda_{\mathrm{A}_i} \cup \{r\} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\theta_k}.(n_k+1)\})^\omega$$

that satisfies

$$\begin{aligned} \alpha'^{\circlearrowleft}_{i,k} &= \alpha^{\circlearrowleft}_{i,k} \text{ when } \alpha_{i,k} \text{ is infinite} \\ &= \alpha^{\circlearrowleft}_{i,k}{}^\frown r^\omega \text{ when } \alpha_{i,k} \text{ is finite} \end{aligned}$$

Remark that for every i and k the sequence $\alpha'^{\circlearrowleft}_{i,k}$ is infinite so satisfies the condition of the claim we are now proving. Also we have

$$\alpha'^{\circlearrowleft}_{i,k}(0) = r \longleftrightarrow \alpha^{\circlearrowleft}_{i,k} =<> \text{ and } \alpha'^{\circlearrowleft}_{i,k}(0) \neq r \longleftrightarrow \alpha'_{i,k}(0)^\circlearrowleft = \alpha_{i,k}(0)^\circlearrowleft$$

The only remaining problem to define $\sigma_\tau^{(n,\xi)}$ is to find the right strictly increasing sequence of integers $(i_p)_{p<\omega}$ that satisfies

$$k = i_p \iff \alpha'_{i,k}(0) \neq r.$$

So for each integer p, let γ_p be the following strategy from

$$\Big(\underbrace{\Lambda_{\mathrm{A}_0} \cup \{r,s\} \times \ldots \times \Lambda_{\mathrm{A}_0} \cup \{r,s\}}_{p+1} \times \ldots\ldots \times \underbrace{\Lambda_{\mathrm{A}_n} \cup \{r,s\} \times \ldots \times \Lambda_{\mathrm{A}_n} \cup \{r,s\}}_{p+1}\Big)$$

into $\Lambda_B \cup \{s\}$ defined by:

$\diamond$ if for all $i \leq n$ and $j \leq p$ the letter $a_{i,j}$ belongs to $\Lambda_{A_i} \cup \{r, s\}$ then

$$\gamma_p(a_{0,0}, \ldots, a_{0,p}, \ldots, a_{n,0}, \ldots, a_{n,p}) = \tau(< b_{0,0}, \ldots, b_{0,q} >, \ldots, < b_{n,0}, \ldots, b_{n,q} >)$$

$$\Longleftrightarrow$$

$$\begin{cases} \forall i \leq n \ \ \forall j \leq p \ \ a_{i,j} \neq s \\ \\ \inf_{i \leq n} lh((a_{i,0}, \ldots, a_{i,p})(\ /r)) = q + 1 \\ \\ \exists i \leq n \Big(lh((a_{i,0}, \ldots, a_{i,p})(\ /r)) = q + 1 \wedge a_{i,p} \neq r \Big) \\ \\ \forall i \leq n \quad < b_{i,0}, \ldots, b_{j,q} > = < a_{i,0}, \ldots, a_{i,p} > (\ /r) \end{cases}$$

$\diamond$ And on any other argument γ_p takes value s.

Now by induction hypothesis let $\sigma_{\gamma_p}^{((p+1).n, \omega^{\theta_p}.n_p)}$ be a strategy that $\omega^{\theta_p}.n_p$ remote-controls the strategy γ_p.

At last we may define the rewarded strategy $\sigma_\tau^{(n,\xi)}$. We write $\beta = \sigma_\tau^{(n,\xi)} * (\alpha_0, \ldots, \alpha_k)$. First of all $\beta(0) = \twoheadleftarrow_{(\xi)}$ so that $\beta^{\circlearrowleft}$ is the concatenation of the first letters of sequences $\beta_i^{\circlearrowleft}$: $\beta^{\circlearrowleft} = < \beta_i^{\circlearrowleft}(0)/i \in \omega >$. To get the result it suffices to set:

$$\beta_p = \sigma_{\gamma_p}^{((p+1).n, \omega^{\theta_p}.n_p)} * (\alpha'_{0,0}, \ldots, \alpha'_{0,p}, \ldots, \alpha'_{n,0}, \ldots, \alpha'_{n,p}).$$

12$\square$

So we come back to the end of the proof of the remote-control lemma. For each integer i consider the strategies:

$$\mu_i \ : \ \underbrace{\Big((\Lambda_A \cup \{s, r\}) \times \ldots \times (\Lambda_A \cup \{s, r\}) \Big)}_{i+1 \ times} \longmapsto \Lambda_B \cup \{s\} \text{ defined by:}$$

- $\mu_i(a_0, \ldots, a_k) = \tau(< a_0(s/r), \ldots, a_i(s/r) >$ for $a_0, \ldots, a_k \in \Lambda_A \cup \{s, r\}$
- $\mu_i(u_0, \ldots, u_k) = s$ when $lh(u_0) = \ldots = lh(u_k) > 1$

By above claim, for each integer i there exists a strategy $\sigma_{\mu_i}^{(i, \omega^{\zeta_i}.(n_i+1))}$ which $\omega^{\zeta_i}.(n_i+1)$-remote controls μ_i. Then it is enough to play in column labeled i the sequence β_i given by:

$$\beta_i = \sigma_{\mu_i}^{(i, \omega^{\zeta_i}.(n_i+1))} * (\alpha^0, \ldots, \alpha^i) = \sigma_{\mu_i}^{(i, \omega^{\zeta_i}.(n_i+1))} * (\sigma_\tau^{\omega^{\zeta_i}.(n_i+1)} * \alpha_0, \ldots, \sigma_\tau^{\omega^{\zeta_i}.(n_i+1)} * \alpha_i)$$

Since the sequence β satisfies $\beta^{\circlearrowleft} = < \beta_i^{\circlearrowleft}/i \in \omega > = \tau * < \alpha_i^{\circlearrowleft}(0)(s/r)/i \in \omega >$.

10$\square$

We proved the main result called the remote control lemma, namely that $\zeta \preceq \xi$ holds whenever $\zeta \leq \xi$. We improve our study of the $^{\circlearrowleft}$ operation. A question arises: what happens when $\zeta > \xi$? Briefly speaking, the result still holds in many cases, and fails in many others; as we'll see ζ can't be too big compared to ξ. To be more precise we define a new relation between ordinals.

Definition 13 *Let $0 \leq \zeta, \xi < \omega_1$, we write*

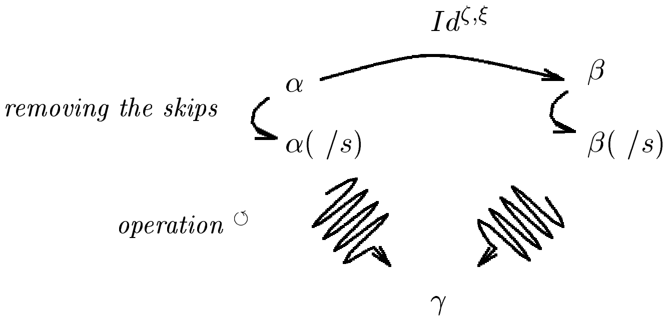
$$\zeta \overset{\prec}{\approx} \xi$$

if there is a strategy:

$$Id^{\zeta,\xi} \ : \ (\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\})^{<\omega^*} \longmapsto \Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}$$

such that $\forall \alpha \in (\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\})^{\omega}$:

$$\alpha(\ /s)^{\circlearrowleft} = (Id^{\zeta,\xi} * \alpha)(\ /s)^{\circlearrowleft}$$



Remark that this is a weak version of $\zeta \preceq_{Id} \xi$ where Id stands for the identity strategy, for here, we don't ask player *II* in the underlying game to play $\twoheadleftarrow_{(\xi)}$ at first move - i.e. we don't need $(Id^{\zeta,\xi} * \alpha)(0) = \twoheadleftarrow_{(\xi)}$. This is the one and only difference between $\overset{\prec}{\approx}$ and $\preceq_{Id}$.
Given $0 \leq \zeta \leq \xi < \omega_1$, the fact $\zeta \overset{\prec}{\approx} \xi$ is immediate for the identity is a w.s. for player *II* in the underlying game. So we concentrate on the cases where $\zeta > \xi$. One can easily see that $\xi + 1 \overset{\prec}{\approx} \xi$ fails when ξ is an integer. But it holds as soon as ξ is transfinite. This result is just the successor step in the induction proof of the general result concerning this operation.

Lemma 14 *Let $\omega \leq \xi < \omega_1$,*

$$\xi + 1 \overset{\prec}{\approx} \xi$$

$\square$ **lem. 14** - The proof goes by induction on ξ:

$\boldsymbol{\xi = \omega}$ the w.s. $\sigma_{Id}^{\omega+1,\omega}$ for player *II* consists in:

(a) as long as *I* skips, *II* skips,

(b) if *I* plays $\alpha(0) \neq \twoheadleftarrow_{(\omega+1)}$, then *II* applies the identity strategy, taking care of replacing in α every occurrence of $\twoheadleftarrow_{(\omega+1)}$ by a skip.

(c) if *I* plays $\alpha(0) = \twoheadleftarrow_{(\omega+1)}$, then *II* plays $\beta(0) = \twoheadleftarrow_{(\omega)}$ and then as long as *I* skips or plays $\twoheadleftarrow_{(\omega+1)}$, *II* skips. We call β_i the content of the column labeled i of the array opened by *II* when playing $\beta(0) = \twoheadleftarrow_{(\omega)}$. Now, the precise thing to know that is meaningful is what $\alpha(\ /s)^{\circlearrowleft\omega}(0)$ will be. Using $\twoheadleftarrow_{(\omega+1)}$, *I* may choose $\alpha(\ /s)^{\circlearrowleft\omega}(0) = \twoheadleftarrow_{(\omega)}$ by playing $\twoheadleftarrow_{(\omega)}$ as first letter which is not $\twoheadleftarrow_{(\omega+1)}$. But later on, *I* may change his mind, he may erase that letter with the help of $\twoheadleftarrow_{(\omega+1)}$ and choose another one - e.g. $\twoheadleftarrow_{(n)}$ for some integer n. Assume first that I when erasing what could be $\alpha(\ /s)^{\circlearrowleft\omega}(0)$, was always choosing $\twoheadleftarrow(\omega)$ again and never some $\twoheadleftarrow(n)$ for any integer n. (We'll see below that one can consider this particular case as the general one since II can always replace I's game by an equivalent one that carries this property).

So, player II views I's play as the opening of an array that he begins to fill up and all of a sudden partially or completely erases it - which means he erases a final segment of a column (without deleting it's first letter that defines the way it has to be read), or he erases one or more column(s) or even the whole array - and keep on playing and erasing until the end of the game.

So, II plays as follows: the first time I chooses to open the array - i.e. plays a letter $\twoheadleftarrow(\omega)$ which, if never erased later, will be $\alpha(\ /s)^{\circlearrowleft\omega}(0)$, II plays in his array **almost** the same as I does in his own one; and this holds as long as I does not use the eraser $\twoheadleftarrow(\omega+1)$. "Almost the same" means that when I is playing the sequence α_i in column i with $\mathcal{D}(\alpha_i(0)) = n$, II requires an additive eraser: he plays $\beta_i = \twoheadleftarrow(n+1)^\frown\alpha'_i$ in his column labeled i, where α'_i is just α_i where every occurrence of $\twoheadleftarrow(n+1)$ is turning into $\twoheadleftarrow(n+2)$(=something that counts as a skip).

If I uses the eraser $\twoheadleftarrow(\omega+1)$ but not to erase the letter $\twoheadleftarrow(\omega)$ which opens the array, this action generates the deleting of some final segments of the content of some columns, but does not delete any complete column (in the sense it does not erase the first letters). Then II uses his additive eraser - played as first letter in each of his column - to erase the final corresponding segment.

If I erases at least column, i.e. if using $\twoheadleftarrow(\omega+1)$ he erases $\alpha_i(0)$ for instance, then remark that he erases $\alpha_{i+p}(0)$ for any such letters already played. So, in that situation, II kills the same corresponding columns with the help of the specific additive eraser attached to each of them. This means II plays $\beta_i(0)^\omega$ as final segment to kill his column labeled i.

Then, as I refills up his previously erased columns, II of course can't reproduce the same action, for he is now killing these columns. But he can go further in the array to redo the same: if I erased with $\twoheadleftarrow(\omega+1)$ his columns labeled $i+1,\ldots,i+k$ but not his columns $0,\ldots,i$, then the columns $i+1,\ldots,i+k$ are killed in II's array, and from now on, II's columns labeled $i+1+k+j$ correspond to I's columns $i+j$ and II's columns $0,\ldots,i$ still corresponds to I's columns $0,\ldots,i$; until some new erasing from I happens.

If I erases the letter that opens the array, then II kills all the columns he played something in before and relabels the ones remaining alive taking care of preserving the ordering: if he kills columns $0,\ldots,p$, then the actual column i is now what the column $i+p+1$ was,etc.

We see that

- if $\alpha(\ /s)^{\circlearrowleft\omega}(0)$ does not exist, then II's array contains killed columns or columns in which nothing has been played. So $\beta(\ /s)^{\circlearrowleft} = \langle\rangle = \alpha(\ /s)^{\circlearrowleft}$;
- if $\alpha(\ /s)^{\circlearrowleft\omega} = \twoheadleftarrow(\omega)$, then II's array looks like the same as I's array except that many columns are killed and the others contain an additive eraser. Furthermore there is a strictly increasing function $\phi\ :\ \omega \longmapsto \omega$, such that $\beta^{\circlearrowleft}_{\phi(i)} = \alpha^{\circlearrowleft}_i$ for all integer i, and for

any integer j which is not in the range of the function ϕ, one has $\beta_j^{\circlearrowleft} =<>$. Therefore

$$\alpha(\ /s)^{\circlearrowleft} =< \alpha_i^{\circlearrowleft}/i \in \omega >=< \beta_i^{\circlearrowleft}/i \in \omega >= \beta(\ /s)^{\circlearrowleft}.$$

We consider now the general case. We have been dealing with the particular case where I when reaching a position u that satisfies $lh(u^{\circlearrowleft\omega})$=1, it was always $u^{\circlearrowleft\omega}(0) = \twoheadleftarrow(\omega)$. If I, at some moment of the game, reaches a position u such that $u^{\circlearrowleft\omega} = \twoheadleftarrow(n)$ then things don't get much more complicated because at that precise moment II's play looks like an array in which some columns are killed, others are partially filled and still ω others remain "virgin". So what II does, is first to kill the columns he used to play something in, and then in the remaining columns he plays according to the following fact: At that point, as long as I does not reach a position v extending u such that $v^{\circlearrowleft\omega} =<>$ - i.e. a position that kills the "opening letter" $\twoheadleftarrow(n)$ - everything goes as if the role of $\twoheadleftarrow(\omega+1)$ was exactly the same as any eraser stronger than $\twoheadleftarrow(n)$, for example $\twoheadleftarrow(n+1)$, and any letter $\twoheadleftarrow(m)$ for $m > n$ has the same impact as a skip. This remark holds as long as we don't get $v^{\circlearrowleft\omega} =<>$. I's play may be replaced by an equivalent one with only letters among $\{\twoheadleftarrow(\iota) : 0 < \iota \leq n+1\}$. And secondly, such a play can also be replaced by another one involving letters among $\{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega\}$ with the additional property that the first letter is $\twoheadleftarrow(\omega)$. (Simply because $n+1 \preceq \omega$ holds).
Therefore this double replacement makes us reach the particular case we studied and thus appears to be the general one.

$\boldsymbol{\xi = \zeta + 1}$ By induction hypothesis $\zeta + 1 \overset{\preceq}{\approx} \zeta$ holds. So there exists a strategy $Id^{\zeta+1,\zeta}$ that works in that case. Now simply get a strategy that 1 remote controls $Id^{\zeta+1,\zeta}$, *mutatis mutandis* this strategy proves $(\zeta+1)+1 \overset{\preceq}{\approx} \zeta+1$

ξ **limit** This case is very similar to the case $\xi = \omega$ except it is a little bit more tedious. The very same ideas work.

14□

After the first step, we state the general case.

Lemma 15 *Let* $0 < \xi_0 < \ldots < \xi_k$ *and* $\forall i \leq k \quad 0 < n_i < \omega$,

$$\omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0 \overset{\preceq}{\approx} \omega^{\xi_k}.n_k$$

□ **lem. 15** - By induction on $\xi = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0 \overset{\preceq}{\approx} \omega^{\xi_k}.n_k$:

(a) $k = 0$ is obvious.

(b) $k > 1$ comes from $\omega^{\xi_k}.n_k + \ldots + \omega^{\xi_1}.n_1 \overset{\preceq}{\approx} \omega^{\xi_k}.n_k$ and $\omega^{\xi_k}.n_k + \omega^{\xi_0}.n_0 \overset{\preceq}{\approx} \omega^{\xi_k}.n_k$.

(c) $k = 1$

(i) $n_0 > 1$ is carried out by induction:

$$\omega^{\xi_1}.n_1 + \omega^{\xi_0}(n_0 - 1) \overset{\preceq}{\approx} \omega^{\xi_1}.n_1 \text{ and } \omega^{\xi_1}.n_1 + \omega^{\xi_0} \overset{\preceq}{\approx} \omega^{\xi_1}.n_1.$$

(ii) $n_0 = 1$ and $n_1 > 1$ is also straightforward. The relation $\omega^{\xi_1} + \omega^{\xi_0} \precapprox \omega^{\xi_1}$, holds by induction. Therefore for any set Λ' there exists a strategy:

$$Id^{\omega^{\xi_1}+\omega^{\xi_0},\omega^{\xi_1}} : (\Lambda'\cup\{s\}\cup\{\twoheadleftarrow'(\iota) : 0 < \iota \leq \omega^{\xi_1}+\omega^{\xi_0}\})^{<\omega^*} \longmapsto \Lambda'\cup\{s\}\cup\{\twoheadleftarrow'(\iota) : 0 < \iota \leq \omega^{\xi_1}\}$$

Set $\Lambda' = \Lambda \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\xi_1}(n_1 - 1)\}$ where the prime arrows are stronger than the plain ones. By remark 6 and renaming the prime arrows, one gets a strategy $Id^{\omega^{\xi_1}.n_1+\omega^{\xi_0},\omega^{\xi_1}.n_1}$ that fits the conditions:

$$Id^{\omega^{\xi_1}.n_1+\omega^{\xi_0},\omega^{\xi_1}.n_1} : (\Lambda\cup\{s\}\cup\{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\xi_1}.n_1+\omega^{\xi_0}\})^{<\omega^*} \longmapsto \Lambda\cup\{s\}\cup\{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\xi_1}.n_1\}$$

(iii) The one and only interesting case is $n_1 = n_0 = 1$: Case $\xi_0 = 1$ is lemma 14. Only remains $\xi_0 > 1$. So, we show that $\omega^{\xi_1} + \omega^{\xi_0} \precapprox \omega^{\xi_1}$, by defining a w.s. for player *II* in the underlying game.
Without loss of generality we may assume the following.

- *I* never skips,
- *I* plays $\alpha(0) = \twoheadleftarrow(\omega^{\xi_1} + \omega^{\xi_0})$.
- *I* plays α such that $\alpha^{\circlearrowleft\omega^{\xi_1}}(0) = \twoheadleftarrow(\omega^{\xi_1})$,
- for all $k \in \omega$ $\alpha'_k(0) = \twoheadleftarrow(\omega^{\nu_k}.n_k)$ for some ordinal ν_k that satisfies $\xi_1 > \nu_k \geq \xi_0$ and $n_k \in \omega^*$ (n_k and ν_k both depend on k). Where $\alpha' = \alpha^{\circlearrowleft\omega^{\xi_1}}$ and $\alpha'_k =< \alpha^{\circlearrowleft\omega^{\xi_1}}(\ /\twoheadleftarrow(\omega^{\xi_1}))(K(k,p))/p \in \omega >$.

Let $(\theta_i)_{i\in\omega}$ enumerates the set of finite and non empty sequences of elements among

$$\{\mathcal{D}(\alpha(p)) :\ p \in \omega\ \wedge\ \mathcal{D}(\alpha(p)) \text{ is of the form } \omega^\nu.n \text{ with } \omega^\nu.m < \omega^{\xi_1}\}.$$

With the additional property that $(\theta_i)_{i\in\omega}$ preserves the sequence ordering (i.e. if $\theta_i \subsetneq \theta_j$ is satisfied then $i < j$ holds). Now *II* plays $\beta_0 = \twoheadleftarrow(\omega^{\xi_1})$ which "opens and array" in which he plays in column i is related to $\theta_i =< \omega^{\nu_0}.m_0, \ldots, \omega^{\nu_p}.m_p >$ betting that *I* will play α that will satisfy

$$\alpha'_0(0) = \omega^{\nu_0}.m_0, \ldots, \alpha'_p(0) = \omega^{\nu_p}.m_p$$

Remark that there is an obvious w.s. $\mathcal{T}_i$ for *II* in the conciliatory game where *I* plays letters in $\Lambda_A \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\xi_1}\}$ and *II* plays letters in $\Lambda_A \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\nu_p}.m_p\}$; and the winning conditions for *II* are:
I's first move is not $\twoheadleftarrow(\omega^{\xi_1})$ or in case *I*'s first move is $\twoheadleftarrow(\omega^{\xi_1})$: this letter played by *I* opens an array in which:

- if the first letters of the $k{+}1$ first columns of that *I*'s array are exactly $\twoheadleftarrow(\omega^{\nu_0}.m_0), \ldots, \twoheadleftarrow(\omega^{\nu_p}.m_p)$ (what corresponds to θ_i), then *II* must produce some infinite sequence δ that satisfies $\delta^{\circlearrowleft} = \gamma_p^{\circlearrowleft}$; where γ_p is the content of the $p + 1^{\text{th}}$ column of *I*'s array.
- And otherwise δ must verify $\delta^{\circlearrowleft} =<>$.

So $\mathcal{T}_i$ is a strategy

$$(\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\xi_1}\})^{<\omega^*} \longmapsto \Lambda_A \cup \{s\} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^{\nu_p}.m_p\}.$$

Get $\sigma_{\mathcal{T}_i}^{\omega^{\xi_0}}$ a strategy that ω^{ξ_0}-remote controls $\mathcal{T}_i$. This new strategy maps

$(\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\xi_1} + \omega^{\xi_0}\})^{<\omega^*}$ into $(\Lambda_A \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\nu_p}.m_p + \omega^{\xi_0}\})$.

Finally note that $\omega^{\nu_p}.m_p + \omega^{\xi_0}$ is less than ω^{ξ_1} and we are done.

So we finally come back to our pigeons. After playing $\beta(0) = \twoheadleftarrow_{(\omega^{\xi_1})}$ *II* plays in column labeled i exactly the sequence $\sigma_{\mathcal{T}_i}^{\omega^{\xi_0}} * \alpha$.

15□

Corollary 16

(a) Let $0 < \xi_0 < \ldots < \xi_k$ and $\forall i \leq k \;\; 0 < n_i < \omega$,

$$\omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0 \preceq \omega^{\xi_k}.n_k$$

(b) Let $\xi,\ \zeta$ be countable non null ordinals,

$$\xi + \zeta \overset{\preceq}{\approx} \zeta + \xi \quad \text{and} \quad \xi + \zeta \preceq \zeta + \xi$$

□ **cor. 16** -
(a) is a direct consequence of previous lemma for one gets both

$$\omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0 \overset{\preceq}{\approx} \omega^{\xi_k}.n_k \text{ and } \omega^{\xi_k}.n_k \preceq \omega^{\xi_k}.n_k.$$

(b) is straightforward. 16□

Now we are done with the $^{\circlearrowleft}$ operation. We saw that it carries an interest only when the set of indexed arrows involved in the decoding process is of the form $\{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\xi}.n\}$. Because all other cases may be lead back to this special one. Nevertheless we have not shown for the moment that this operation was relevant at all in providing help in building complicated sets on the basis of simple ones. For that purpose we introduce the $^{\sim\nu}$ operation. In simple words it consists in enriching an alphabet with indexed arrows and looking at the sequences in that new alphabet that belong to a given set once applied the $^{\circlearrowleft}$ operation.

3 Transfinite iteration of exponentiation on sets

Definition 17 *Let $A \subseteq \Lambda_A^{\leq\omega}$, $\lambda < \omega_1$, $\{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \lambda\} \cap \Lambda_A \neq \emptyset$*

$$A^{\sim\lambda} = \{\alpha \in (\Lambda_A \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \lambda\})^{\leq\omega} :\ \alpha^{\circlearrowleft} \in A\}$$

Note that $^-(A^{\sim\lambda}) = (^-A)^{\sim\lambda}$ holds because by definition one has $\alpha \notin A^{\sim\lambda}$ iff $\alpha^{\circlearrowleft} \notin A$. Moreover, as a consequence of the study of $^{\circlearrowleft}$ operation, $A \longmapsto A^{\sim\lambda}$ preserves the conciliatory ordering:

Proposition 18 (remote-control) *Let $A \subseteq \Lambda_A^{\leq \omega}$, $B \subseteq \Lambda_B^{\leq \omega}$, $\lambda \leq \delta$*

$$A \leq_c B \implies A^{\sim\lambda} \leq_c B^{\sim\delta}$$

$\square$ **prop.18** - This is immediate from lemma 10 and remark 9(a). Because given a strategy τ that is winning for *II* in the game $\mathbf{C}(A, B)$, one gets σ_τ^δ that δ-remote control τ and is a winning strategy for *II* in the game $\mathbf{C}\left(A^{\sim\delta}, B^{\sim\delta}\right)$, which shows $A^{\sim\delta} \leq_c B^{\sim\delta}$. Besides the fact $A^{\sim\lambda} \leq_c A^{\sim\delta}$ is obvious.

18 $\square$

But since the $^{\sim\nu}$ operation relies on $^{\circlearrowleft}$ it is useless when the ordinal involved is not of the form $\omega^\xi.n$. For this reason we define a kind of suitable finite iteration of $^{\sim\nu}$ along the Cantor Normal Form of the countable ordinal involved: the $^{\approx\nu}$ operation.

Notation 19 *Let $A \subseteq \Lambda_A^{\leq \omega}$ and λ a countable ordinal which Cantor Normal Form of base ω is*

$$\lambda = \omega^{\lambda_k}.n_k + \ldots + \omega^{\lambda_0}.n_0$$

the notation $A^{\approx\lambda}$ stands for $\left(\left(\left(A^{\sim\omega^{\lambda_0}.n_0}\right)^{\cdots}\right)^{\sim\omega^{\lambda_k}.n_k}\right)$.
And for $\alpha \in \Lambda \cup \{\twoheadleftarrow_{(\iota_p)} : 0 < \iota_p \leq \omega^{\lambda_p}.n_p, \quad 0 \leq p \leq k\}$ we write

$$\alpha^{\odot} \text{ for } \left(\left(\ldots\left(\alpha^{0\circlearrowleft}\right)\ldots\right)^{k-1\circlearrowleft}\right)^{k\circlearrowleft}$$

where $\beta^{p\circlearrowleft}$ is just $\beta^{\circlearrowleft}$ when the $^{\circlearrowleft}$-operation only deals with the arrows of type p. So that

$$\alpha \in A^{\approx\lambda} \text{ iff } \alpha^{\odot} \in A.$$

We also write $\mathcal{A}_\xi$ for the set of arrows $\{\twoheadleftarrow_{(\iota_p)} : 0 < \iota_p \leq \omega^{\lambda_p}.n_p, \quad 0 \leq p \leq k\}$.

The set $A^{\sim 1}$ is exactly what we called $A^{\sim}$ in [1]. So $A^{\sim\lambda}$ is just an extension or iteration by use of folding and unfolding games of this very simple operation.
Remark that $\ominus^{\sim\lambda} = \ominus$ and also $\oplus^{\sim\lambda} = \oplus$. But as soon as A is not the empty set nor the whole space, the resulting set $A^{\sim\lambda}$ is much more complicated than A. And in general, the larger the ordinal λ is, the more complicated the set $A^{\sim\lambda}$ becomes. More precisely, as an example of this phenomenon, if A is $\mathbf{\Sigma}_1^\circ$-complete the set $A^{\approx\lambda}$ will be a $\mathbf{\Sigma}_{1+\lambda}^\circ$-complete set.

4 A kind of inverse for $^{\approx\lambda}$

As we did in [1] in the context of $^{\sim}$, in order to be able to deal with $A^{\approx\lambda}$ we introduce a kind of inverse operation $^{\not\approx\lambda}$. Nevertheless $^{\approx\lambda}$ is defined on conciliatory sets when $^{\not\approx\lambda}$ is defined on sets of the form $A \subseteq \Lambda_A^\omega$. So $A^{\not\approx\lambda}$ will also be a set of **infinite** sequences and will be involved in **Wadge games** that have in particular the property that player *I* can't skip.

So as $^{\approx\lambda}$ is an iteration of $^{\sim}$, the "pseudo operation"$^{\not\approx\lambda}$ is also an iteration of $^{\not\sim}$, easy to understand but tedious to describe.

The main effect of $\stackrel{\sim\lambda}{}$ is to change the topology of the space we deal with, in a way that it turns some $\mathbf{\Sigma}^\circ_\lambda$ sets into $\mathbf{\Sigma}^\circ_1$. This is done in a way that fits with the game theoretic point of view that is central in the study of the Wadge hierarchy. So, changing the topology is done by answering auxiliary questions. The new space in which $\mathbf{\Sigma}^\circ_\lambda$ sets are turned into open ones is set up by use of auxiliary games.

Defining $\stackrel{\sim\lambda}{}$ goes in many steps. First we set a notation for a coding of integers that are related to the nodes of special trees in which the runs take place. Each of the λ-$\mathcal{T}$rees may be regarded as made of as a package of ξ-$\mathcal{T}$rees for any ordinal $\xi < \lambda$. Each node of a λ-$\mathcal{T}$ree is related to some couple of the form (ξ, n) with ξ an ordinal strictly lesser than λ and n an integer. There are also nodes of the form $(-1, n)$ where the information given by playing at this level (choosing a node in the tree) is just a letter from the original space (the space one starts from): the run relativized to the $(-1, n)$ nodes forms the run in the main game, all other moves are auxiliary moves saying something about this original one. The moves on positions (ξ, n) with $\xi \geq 0$ make a choice about whether or not the following run will remain in a closed subset or not. But, as the construction is made by induction on ξ, this (closed) subset is a subset of a complicated space implicitly described by the nodes of the λ-$\mathcal{T}$ree that are related to couples of the form (ζ, n) with $\zeta < \xi$. So inductively, this closed subset can easily be coding information about whether or not the part of the run that belongs to the original space will remain in a $\mathbf{\Sigma}^\circ_\xi$ set or not.

Since when looking at a λ-$\mathcal{T}$ree, we want to be able to see it as the result of an increasing chain of ξ-$\mathcal{T}$rees for any $\xi < \lambda$ (increasing in the sense that it codes more and more information), we begin with a notation about a suitable coding of integers.
For not to overloading the script we will write $\boldsymbol{\theta^-}$ **instead of** $\boldsymbol{\theta \cup \{-1\}}$ for θ any countable ordinal.

Notation 20 *Let* $0 \leq \lambda < \omega_1$,

Given some bijection $\begin{array}{ccc} \lambda^- \times \omega & \longleftrightarrow & \omega \\ (\zeta, n) & \longmapsto & \prec \zeta, n \succ^\lambda \end{array}$ *that satisfies the following condition* (C^*)

$$(C^*) \qquad \forall \zeta < \lambda \quad (n < m \Longrightarrow \prec \zeta, n \succ^\lambda \ < \ \prec \zeta, m \succ^\lambda)$$

For any ordinal ξ *such that* $0 \leq \xi < \lambda$, *we define the bijection* $\begin{array}{ccc} \xi^- \times \omega & \longleftrightarrow & \omega \\ (\zeta, n) & \longmapsto & \prec \zeta, n \succ^\xi \end{array}$ *by the collapse of the trace of* $\prec\ \succ^\lambda$ *on* $\xi^- \times \omega$.
Namely, $\prec\ \succ^\xi$ *is the one and only enumeration in increasing order of the following set of integer:*

$$\{\prec \zeta, n \succ^\lambda\colon\ \zeta < \lambda,\ n < \omega\}.$$

Or to say it in an inductive way, this enumeration takes the form:
$\prec \zeta, n \succ^\xi = 0$ iff $\forall \theta < \xi\ \forall m < \omega\ \ \prec \theta, m \succ^\lambda \geq \prec \zeta, n \succ^\lambda$;
assume now that have been determined $\prec \zeta_0, n_0 \succ^\xi = 0, \ldots, \prec \zeta_k, n_k \succ^\xi = k$, then
$k+1 = \prec \zeta_{k+1}, n_{k+1} \succ^\xi = inf\{\prec \theta, n \succ^\lambda\colon\ \theta < \xi,\ n < \omega\} \smallsetminus \{\prec \zeta_0, n_0 \succ^\xi, \ldots, \prec \zeta_k, n_k \succ^\xi\}$.

Remark that, by construction, the function $\prec\ \succ^\xi$ still satisfies the condition (C^*):

$$\forall \zeta < \xi \quad (n < m \Longrightarrow \prec \zeta, n \succ^\xi \ < \ \prec \zeta, m \succ^\xi)$$

Definition 21

Given $0 \leq \lambda < \omega_1$ *and a bijection* $\begin{array}{ccc} \lambda^- \times \omega & \longleftrightarrow & \omega \\ (\xi, n) & \longmapsto & \prec \xi, n \succ^\lambda \end{array}$ *that satisfies condition* (C^*):

$$\forall \xi < \lambda \quad (n \ < \ m \implies \prec \xi, n \succ^\lambda \ < \ \prec \xi, m \succ^\lambda)$$

we define a **λ-Tree** T *on* Λ *by induction on* λ. *It is a n.e.p. tree on a complicated alphabet that will be cleared out by the explanation of the kind of nodes involved in it. But first we must define for any* u *in such a* λ*-Tree and any* $\zeta < \lambda$ *what is* $\boldsymbol{u^\zeta}$ *by induction on the length of* u:

$$\begin{array}{rcll} <>^\zeta & = & <> & \\ (u^\frown \langle \alpha \rangle)^\zeta & = & u^{\zeta\frown} \langle \alpha \rangle & \text{if} \quad lh(u) = \prec \theta, n \succ^\lambda \text{ with } \theta < \zeta \\ & = & u^\zeta & \text{otherwise.} \end{array}$$

Or in other words,

$$u^\zeta = \{(\prec \theta, n \succ^\zeta, \alpha) : \ (\prec \theta, n \succ^\lambda, \alpha) \in u \ \wedge \ \theta < \zeta, \ n < \omega\}$$

We also define $\boldsymbol{T^\zeta}$ *that we regard as a "subTree" of* T *by*

$$T^\zeta = \{u^\zeta : \ u \in T\}$$

Essentially we only take into account codings that satisfy condition (C^*) *because we want to assure that given* $\zeta < \xi < \lambda$ *and* T *some* λ*-Tree the* ζ*-Tree* $(T^\xi)^\zeta$ *to be exactly the same as* T^ζ.

This preamble is closed, we come to the very definition of a λ*-Tree* T. *This definition is by induction on* λ *in the particular sense that for any ordinal* $\xi < \lambda$ *the "subTree"* $\boldsymbol{T^\xi}$ **must be a $\boldsymbol{\xi}$-Tree**. *Moreover, a* λ*-Tree is some* **maximal** *(for inclusion) tree* T *that satisfies: first of all,* $<>$ *is in* T *and for any* u *in* T *and* $k < lh(u)$ *the value of* $u(k)$ *- assume the coding of integers gives* $k = \prec \zeta, n \succ^\lambda$ *- is:*

if $\boldsymbol{\zeta = -1}$: *then* $u(\prec \zeta, n \succ^\lambda) \in \Lambda$ *(this corresponds to the main run); and*

if $\boldsymbol{0 \leq \zeta}$: *then this corresponds to the auxiliary runs and there are two options :*

Option $\boldsymbol{< 1 >}$: $u(\prec \zeta, n \succ^\lambda) = < 1 > {}^\frown v$ *with* $v \in T^\zeta$ *and* $\Big(u \restriction \prec \zeta, n \succ^\lambda)\Big)^\zeta \subseteq v$. *So* v *is some position in the subTree* T^ζ *that is extending the actual position* $\Big(u \restriction \prec \zeta, n \succ^\lambda)\Big)^\zeta$. *But then, any* w *in* T *that verifies* $u \restriction (k+1) \subseteq w$ *must satisfy* $w^\zeta \subseteq v \ v \ \subseteq w^\zeta$. *This means that further moves that are allowed are those restricted to* v.

Option $\boldsymbol{< 0 >}$: $u(\prec \zeta, n \succ^\lambda) = < 0 >$. *But then, any* w *in* T *that verifies* $u \restriction (k+1) \subseteq w$ *must satisfy*

$$v \subseteq w^\zeta \text{ fails for any } \boldsymbol{v} \text{ such that } (u \restriction k)^\frown \Big\langle < 1 > {}^\frown v \Big\rangle \in T.$$

In order to limit the choices in option $< 1 >$ *to only a unique possible one, we require* T *to satisfy the additional condition:*

$$\Big(u \in T \ \wedge \ u^\frown << 1 > {}^\frown v > \in T \ \wedge \ u^\frown << 1 > {}^\frown w > \in T\Big) \implies v \perp w.$$

Notation 22 *Given a λ-Tree T we write $[T]$ for the set of infinite branches of T and for $\alpha \in [T]$ and $\xi < \lambda$ the symbol α^ξ denotes the trace of α on $[T^\xi]$:*

$$\alpha^\xi = \{(\prec \zeta, n \succ^\xi, \alpha(\prec \zeta, n \succ^\lambda)) : \ \zeta < \xi, \ n < \omega\} = \bigcup \{u^\xi : \ u \in \alpha\}$$

Remarks 23

(a) When T is a λ-Tree $T^0 = (\{-1\} \times \Lambda)^{<\omega}$ is isomorphic to $\Lambda^{<\omega}$. So we identify T^0 and $\Lambda^{<\omega}$ and by the way, we regard $u \in T^0$ and $\alpha \in [T^0]$ as respectively a finite sequence of letters from Λ and an element of Λ^ω.

(b) When ζ is at least 0, one may think of $u(\prec \zeta, n \succ^\lambda)$ as a response to some question about the future of the run in T^ζ, it asks if it will remain in some closed subset $C_{u\restriction(\prec\zeta,n\succ^\lambda+1)}$ of $[T^\zeta]$: it is simply expressed by negativity (not choosing option $< 1 >$):

$$C_{u\restriction(\prec\zeta,n\succ^\lambda+1)} = \left\{\alpha \in [T^\zeta] : \ (u \restriction \prec \zeta, n \succ^\lambda)^\frown \left\langle < 1 > ^\frown v \right\rangle \in T \Longrightarrow \ v \not\subseteq \alpha\right\}$$

- When the answer is $< 0 >$ meaning **yes**, the player insures that the sequence $\alpha \in [T]$ he will have produced at the end of the game will satisfy the condition $\alpha^\xi \in C_{u\restriction(\prec\zeta,n\succ^\lambda+1)}$.
- When the answer is $< 1 > ^\frown v$ meaning **no**, the player (while choosing the $< 1 >$ option) swears that α^ζ will exit the closed set $C_{u\restriction(\prec\zeta,n\succ^\lambda+1)}$. And by playing the sequence v he indicates the way the run will effectively get out of $C_{u\restriction(\prec\zeta,n\succ^\lambda+1)}$.

In any cases, all moves in the future are restricted to the ones that agree with options previously taken.

Definition 24 *Given $A \subseteq \Lambda^\omega$, $\lambda < \omega_1$ and T a λ-Tree on Λ,*
by A^T we denote the set $\{\alpha \in [T] : \ \alpha^0 \in A\}$ enriched with all sequences of Λ_T^ω that exit T at some position whereas at the very previous position (the last time it is still in T) there is an extending sequence α remaining in $[T]$ and verifying $\alpha^0 \in A$.:
the space in which "lives" A^T is

$$\Lambda_T^\omega = \left(\{\Lambda_A \cup \{< 0 >\} \cup \{< 1 > ^\frown v : \ v \in T^\xi, 0 \leq \xi < \lambda\}\right)^\omega$$

Λ_T is what we called the "complicated alphabet" of the Tree T and A^T is the subset defined by

$$A^T = \{\alpha \in [T] : \ \alpha^0 \in A\} \cup \{u^\frown < a > ^\frown \beta \in \Lambda_T^\omega : \ u \in T, \ u^\frown < a > \notin T, \ \exists \alpha \ (u^\frown \alpha \in [T] \wedge (u^\frown \alpha)^0 \in A)\}$$

We defined A^T this way so that in the Wadge game $\mathbf{W}\left(A^T, A^T\right)$ player *II* has a w.s. that always remains in T. So from now on we will always assume that a run of a player being in charge of a set of the form A^T always takes place in T.

Remarks 25

(a) Assume T is a λ-Tree on an alphabet Λ, with $\lambda = \xi + \zeta$. One can regard T as a ζ-tree on the alphabet of T^ξ. More precisely, there exists a ζ-Tree T^ζ on Λ_{T^ξ} (the alphabet of T^ξ) such that for any set $A \subseteq \Lambda^\omega$ the relation $A^T \equiv_w (A^{T^\xi})^{T^\zeta}$ holds. This is obvious by induction on ζ. In the sequel we will speak of it saying that T^ζ is "the rest of T when T_ξ has been removed".

(b) Given T and T' two λ-*Trees* on the same alphabet Λ, one may easily define T'' a λ-*Tree* on Λ that codes both T and T', i.e. for all $\alpha \in [T]$ and $\alpha' \in [T']$ verifying $\alpha^0 = \alpha'^0$ there exists $\alpha'' \in [T'']$ such that $\alpha^0 = \alpha'^0 = \alpha''^0$ and for all ξ that satisfies $0 < \xi \leq \lambda$ there exist strictly increasing functions f_ξ and $f'_\xi : \omega \longmapsto \omega$ such that

$$\alpha^\xi = \alpha''^\xi \circ f_\xi \text{ and } \alpha'^\xi = \alpha''^\xi \circ f'_\xi.$$

Or in other words, for any set $A \subseteq \Lambda^\omega$ player *II* has a w.s. in the following Wadge games

$$\mathbf{W}\left(A^T, A^{T''}\right),\ \ \mathbf{W}\left(A^{T''}, A^T\right),\ \ \mathbf{W}\left(A^{T'}, A^{T''}\right),\ \ \mathbf{W}\left(A^{T''}, A^{T'}\right)$$

(c) Given T a λ-*Tree* on Λ and $A \subseteq \Lambda^\omega$, *II* has an obvious w.s. in the game $\mathbf{W}\left(A^T, A^{T^\xi}\right)$ which consists in copying *I*'s part of the run that belongs to T^ξ. In particular, taking $\xi = 0$ and reminding that A^{T^0} is isomorphic to A it is immediate to see that $A^T \leq_w A$. So when A is Borel, the set of all sets A^T where T ranges over all λ-*Trees* on Λ is well-founded by $<_w$. So, by Wadge's lemma, there is one minimal element up to complementation and Wadge equivalence. In fact, one can drop out the complementation condition since there may not be two such minimal non self dual elements one Wadge equivalent to the complement of the other. This holds simply because if ${}^-(A^T) \not\leq_w A^T$ and ${}^-(A^T) \equiv_w A^{T'}$ both hold, by previous remark one can easily build a λ-*Tree* T'' which "codes" both T and T' and so that both $A^T \geq_w A^{T''}$ and $A^{T'} \geq_w A^{T''}$ are satisfied.
This shows that minimal elements may be non self dual or self dual but in any cases all of them are Wadge equivalent. So this authorizes us to set:

$$\boldsymbol{A^{\sim\lambda}} = \text{a } <_w\text{-minimal element of } \{A^T : T \text{ is a } \lambda - \mathcal{T}\text{ree on } \Lambda\}.$$

We certainly don't need this operation to be made functional which can't be achieved without the use of the Axiom of Choice. We insist on the fact that $A^{\sim\lambda}$ is just a scripture commodity and none of the following proofs involving $A^{\sim\lambda}$ will require AC.

By definition of a 0-*Tree* $A^{\sim 0}$ is a set that is Wadge equivalent to A, so to simplify notations in the following, let's assume by convention that $A^{\sim 0}$ is exactly the set A. The notation $A^{\sim}$ stands for $A^{\sim 1}$.

Since for any countable λ, any λ-*Tree* T on Λ and any $A \subseteq \Lambda^\omega$, $A^T \leq_w A$ holds, one has $A^{\sim\lambda} \leq_w A$. So, obviously ${}^{\sim\lambda}$ lowers the degree of a set, this fact will be our main tool in the final proofs of this paper, allowing to apply induction hypothesis after descending the degree of the set we will deal with. But before we come to the last part of this work, we are now on going to state some properties of ${}^{\sim\lambda}$. First of all we need to verify that this pseudo operation preserves the Wadge ordering, otherwise it would really be totally useless. We also need to show that in a certain way this is the inverse operation of the conciliatory ${}^{\approx\lambda}$. Since ${}^{\approx\lambda}$ is defined on conciliatory sets (subsets of $\Lambda^{\leq\omega}$) but ${}^{\sim\lambda}$ takes care of "real" sets (subsets of Λ^ω) we need to juggle with the ^{b} operation to conciliate both points of view. The property we aim is that for any conciliatory B, $((B^{\approx\lambda})^b)^{\sim\lambda}$ is a set Wadge equivalent to B^b. But in the final proof we will only use one direction of the Wadge equivalence, namely the one that shows the sufficient lower capacity of ${}^{\sim\lambda}$: $((B^{\approx\lambda})^b)^{\sim\lambda} \leq_w B^b$. So

given any set $A \subseteq \Lambda_A^\omega$ that verifies both A and its complement reduce to $(B^{\approx\lambda})^b$, by preservation of the Wadge ordering, one will obtain that both $A^{\nsim\lambda}$ and its complement reduce to $((B^{\approx\lambda})^b)^{\nsim\lambda}$, so they both reduce to B^b. At this point we will want to apply the induction hypothesis, i.e. there exists a conciliatory set C below B in the conciliatory hierarchy such that $A^{\nsim\lambda}$ or its complement reduces to C^b. And at last we'll want to be able to go upwards, namely to get A or its complement reduces to $(C^{\approx\lambda})^b$. So this requires two different things.
First, to be able to distinguish whether or not $A^{\nsim\lambda}$ is non self dual - which we'll see happens in particular in case $A^{\nsim\xi} \geq_w A$ holds for any $\xi < \lambda$ (except for $\lambda = 1$ in which case we already now from [1] that it corresponds to A being "initializable").
Secondly, to know how, from a w.s. for player *II* in a game $\mathbf{W}\left(A^{\nsim\lambda}, C^b\right)$ we can get a w.s. for the same player in the game $\mathbf{W}\left(A, (C^{\approx\lambda})^b\right)$. So this is our schedule before coming to the end of the preliminary studies. Nevertheless, working with $^{\sim\lambda}$ instead of $^{\approx\lambda}$ will be sufficient because the result concerning the second operation will be easy to derive from those involving the first one.

Operation $^{\sim\lambda}$ preserves the conciliatory Wadge ordering. The reverse like operation $^{\nsim\lambda}$ preserves the Wadge ordering:

Proposition 26 *Let $A \subseteq \Lambda_A^\omega$, $B \subseteq \Lambda_B^\omega$ and $0 \leq \lambda \leq \delta < \omega_1$,*

$$A \leq_w B \Longrightarrow A^{\nsim\delta} \leq_w B^{\nsim\lambda}$$

$\square$ **prop.26** - First, remark that it is enough to prove the case $\delta = \lambda$, since by minimality one clearly has $A^{\nsim\delta} \leq_w A^{\nsim\lambda}$. So, the proof goes by induction on λ:

Case $\boldsymbol{\lambda = 0}$ is obvious since $A^{\nsim 0} = A$ and $B^{\nsim 0} = B$.

Case $\boldsymbol{\lambda > 0}$: let σ be some w.s. for *II* in $\mathbf{W}(A, B)$ and $B^{\nsim\lambda} = B^{T_B}$ with T_B a λ-*Tree* on Λ_B. By induction on ξ $(0 < \xi \leq \lambda)$ we define ${}^\xi T_A$ a ξ-*Tree* on Λ_A together with ${}^\xi\sigma$ a w.s. for *II* in $\mathbf{W}\left(A^{{}^\xi T_A}, B^{T_B^\xi}\right)$. By minimality this will give $A^{\nsim\lambda} \leq_w A^{{}^\lambda T_A} \leq_w B^{T_B} = B^{\nsim\lambda}$. We also want to assure that $\forall \zeta < \xi$ $\left({}^\xi T_A\right)^\zeta = {}^\zeta T_A$ holds and the "restriction" of ${}^\xi\sigma$ to ${}^\zeta T_A$ is exactly ${}^\zeta\sigma$ in the sense that if $u \in T^\xi$ and $lh(u) - 1 = \prec \theta, n \succ^\xi$ for some $\theta < \xi$ and $n < \omega$, then ${}^\xi\sigma(u) = {}^\nu\sigma(u^\nu)$ holds for any $\theta < \nu \leq \xi$.

Roughly speaking ${}^\lambda T_A$ is the pull back of T_B along the strategy σ and ${}^\lambda\sigma$ is the strategy induced by this pull back.

- $\boldsymbol{\xi = 0}$: ${}^0 T_A = \Lambda_A^{<\omega}$ and ${}^0\sigma = \sigma$.
- $\boldsymbol{\xi}$ **limit**: this is entirely inductively determined by lower degree cases. So the only interesting case is
- $\boldsymbol{\xi = \zeta + 1 > 0}$:

we just have to define $u(\prec \zeta, n \succ^\xi)$ and ${}^\xi\sigma(u)$ with $u \in {}^\xi T_A$.

Assume $lh(u) - 1 = \prec \zeta, n \succ^\xi$, ${}^\xi\sigma(u')$ has already been defined for any $u' \subsetneq u$ and

$$\left({}^\xi\sigma * (u \upharpoonright lh(u) - 1)\right)(\ /s) = v \text{ with } lh(v) = \prec \theta, n \succ^\xi$$

- if $\theta < \zeta$ then ${}^{\xi}\sigma(u) = s$ =skip: player *II* must wait and see until by inductive construction *I* (who can't skip) reach a position $u^\frown u'$ such that $lh(u^\frown u') - 1 = \prec \theta, n' \succ^{\xi}$ where *II* at that moment just apply ${}^{\theta}\sigma$.
- if $\theta = \zeta$ then next move for *II* is to answer about some set Q_v of positions in T_B^{ζ}. Let Q_u be the set of positions for *I* in ${}^{\zeta}T_A$ defined by

  $$Q_u = \{u'' \in {}^{\zeta}T_A : \; ({}^{\zeta}\sigma * u'')(\ /s) \in Q_v\}.$$

  Let $u \in {}^{\xi}T$ be the position where *I* is questioned about the set of positions Q_u, and ${}^{\xi}\sigma(u)$ is defined by:
  - if *I* answers yes (i.e. takes option $< 0 >$) then *II* also answers $< 0 >$.
  - if *I* answers no (i.e. takes option $< 1 > {}^\frown u'$) then *II* answers $< 1 > {}^\frown v'$ with $v' = ({}^{\zeta}\sigma * u')(\ /s)$

26 □

Operation ${}^{\propto\lambda}$ satisfies an elementary property very similar to arithmetical exponentiation:

Proposition 27 *Let $A \subseteq \Lambda_A^{\omega}$, $0 < \zeta, \xi < \omega_1$,*

$$A^{\propto(\zeta+\xi)} \equiv_w \left(A^{\propto\zeta}\right)^{\propto\xi}$$

□ **prop.27** - Assume $A^{\propto(\zeta+\xi)} = A^T$ so that T is some $\zeta+\xi$-*Tree* on Λ_A. Then $A^T \equiv_w \left(A^{T^{\zeta}}\right)^{T_\xi}$ where T_ξ is "the rest of T when T^{ζ} has been removed from T". So T_ξ is some ξ-*Tree* on Λ_{T^ξ} (the alphabet of $A^{T^{\zeta}}$). By minimality one obtains

$$A^{\propto(\zeta+\xi)} = A^T \equiv_w \left(A^{T^{\zeta}}\right)^{T_\xi} \geq_w \left(A^{T^{\zeta}}\right)^{\propto\xi} \geq_w \left(A^{\propto\zeta}\right)^{\propto\xi}.$$

On the other direction, assume $\left(A^{\propto\zeta}\right)^{\propto\xi} = \left(A^{T_\zeta}\right)^{T_\xi}$. Clearly from T_ζ and T_ξ one builds T a $\zeta+\xi$-*Tree* on Λ_A such that $\left(A^{T_\zeta}\right)^{T_\xi} \geq_w A^T$. And by minimality $A^T \geq_w A^{\propto(\zeta+\xi)}$.

27 □

Proposition 28 *Let $A \subseteq \Lambda_A^{\leq\omega}$, A^b Borel, and $0 < \lambda < \omega_1$,*

(a)

$$\left(A^{\sim\lambda}\right)^{b^{\propto\lambda}} \leq_w A^b$$

(b)

$$\left(A^{\approx\lambda}\right)^{b^{\propto\lambda}} \leq_w A^b$$

□ **prop.28** -

(a) It is enough to show that for each ordinal λ there exists a λ-Tree T_λ such that $((A^{\sim\lambda})^b)^{T_\lambda} \leq_w A^b$. The proof goes by induction on λ. The induction hypothesis is a little bit stronger than what is required: we will show that there exists T_λ a λ-Tree on the alphabet of $\left(A^{\sim\lambda}\right)^b$ such that for all $\alpha \in [T_\lambda]$, the sequence α "contains a positive answer" to each question of the following form:

$$\text{will } (\alpha^0)^{\circlearrowleft}(n) \text{ be the letter } a \text{ ?}$$

$$\text{for each integer } n \text{ and each letter } a \in \Lambda_A \cup \{b\} \text{ that is } \textbf{appearing in } \alpha^0,$$

$$\text{when both } lh((\alpha^0)^{\circlearrowleft}) > n \text{ and } (\alpha^0)^{\circlearrowleft}(n) = a \text{ will be true at the end of the run.}$$

By "contains an answer" we mean for all n and $a \in \Lambda_A \cup \{b\}$ appearing in α^0 there exists an integer p such that $\alpha(p)$ is the response to the question: will $(\alpha^1)^{\circlearrowleft}(n)$ be the letter a ? $< 0 >$ meaning **no** and $< 1 > {}^\frown v$ meaning **yes**. So from that position the w.s. for player *II* in the underlying game is obvious: wait (which means play b) until *I* responds **yes** to some question $(\alpha^1)^{\circlearrowleft}(0) = a_0$?, then play a_0 and wait until *I* responds **yes** to some question $(\alpha^1)^{\circlearrowleft}(1) = a_1$?, then play a_1, etc...

Assume $\lambda = 0$: this is obvious since the identity works.

Assume $\lambda = 1$: was proved in [1] (see proposition 26).

Assume $\lambda \neq \omega^\xi.n$ for all $\xi \geq 0$ and $n > 0$: then for some $\nu < \lambda$ one has $\lambda \overset{\prec}{\approx} \nu$. So there exists a w.s. σ for *II* in the game $\mathbf{W}\left((A^{\sim\lambda})^b, (A^{\sim\nu})^b\right)$ such that:

$$\forall \alpha \in (\Lambda_A \cup \{b\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \lambda\})^\omega \quad (\alpha(\ /b))^{\circlearrowleft} = ((\sigma * \alpha)(\ /b))^{\circlearrowleft}.$$

By induction hypothesis there exists a ν-Tree T_ν that fits the conditions of the induction hypothesis. Now get T_λ any λ-Tree that "contains" the pull back of T_ν along the strategy σ. Clearly T_λ works.

Assume $\lambda = \omega^\xi$ with $\xi > 0$: we describe a λ-Tree T_λ by enunciating questions directly asked by player *II* to player *I* in the Wadge game $\mathbf{W}\left(\left(\left(A^{\sim\omega^\xi}\right)^b\right)^{T_\lambda}, A^b\right)$:

First, as long as he remains waiting for the letter $(\alpha^0(\ /b))(0)$ player *II* plays b and in that case the possible questions along the Tree are totally irrelevant. This may go until the end of the run, but if *I* finally plays the letter $(\alpha^0(\ /b))(0)$ then

(i) if $(\alpha^0(\ /b))(0) \in \Lambda_A$ then everything is easy since it is the same as if *I* was simply in charge of A^b.

(ii) if $(\alpha^0(\ /b))(0) = \twoheadleftarrow_{(\theta)}$ with $\theta < \omega^\xi$ then everything goes as if *I* was in charge of $\left(A^{\sim\theta}\right)^b$ instead of $\left(A^{\sim\omega^\xi}\right)^b$. By induction hypothesis there exists a θ-Tree T_θ that works, so any λ-Tree "containing" T_θ is suitable.

(iii) if $(\alpha^0(\ /b))(0) = \twoheadleftarrow_{(\omega^\xi)}$, the sequence $\alpha^0(\ /b)(\ /\twoheadleftarrow_{(\omega^\xi)})$ is unfolded into ω sequences $(\alpha_n)_{n\in\omega}$ (remind that $\alpha^{0^{\circlearrowleft}} =< \alpha_n^{\circlearrowleft}(0)/n \in \omega >$). So during *I*'s run, player *II* imagines the sequences α_n. Of course α_n is a finite or infinite sequence on the alphabet $\Lambda_A \cup \{\twoheadleftarrow_{(\iota)} :\ 0 < \iota < \omega^\xi\}$, but continuously *II* may replace α_n by an infinite sequence α_n' that belongs to $(\Lambda_A \cup \{b\} \cup \{\twoheadleftarrow_{(\iota)} :\ 0 < \iota < \omega^\xi\})^\omega$ and that is

exactly α_n once the blanks are removed.
Now consider each n such that α_n is not the empty sequence and $\theta_n = \mathcal{D}(\alpha_n(0))$. By induction hypothesis there exists a θ_n-*Tree* S_n that gives answer to each question of the form "will $\alpha_n^{\circlearrowleft}(m)$ be a ?" for each integer m and for each letter $a \in \Lambda_A$ played by I during the run. So if $\alpha_n^{\circlearrowleft}$ is not the empty sequence, climbing along S_n we will know what the value $\alpha_n^{\circlearrowleft}(0)$ will be. But it is easy to build a θ_n+1-*Tree* T_n that verifies $T_n^{\theta} = S_n$ and T_n gives also the answer to the question "will $\alpha_n^{\circlearrowleft}$ be the empty sequence ?".
At last, provided with those trees $(T_n)_{n\in\omega}$ one may mix them to get T an ω^ξ-*Tree* such that for all integer n, T^{θ_n+1} "contains" the θ_n+1-*Tree* T_n. Therefore T will give answer to every questions

(A) will α_n be the empty sequence ?
(B) will $\alpha_n^{\circlearrowleft}$ be the empty sequence ?
(C) what will be $\alpha_n^{\circlearrowleft}(0)$? (in case T answers "no" to (A) and (B)).

And this is exactly what is sufficient to answer any questions "will $\alpha^{\circlearrowleft}(n)$ be a ?" for any a in Λ_A that is played by I during the run, provided that effectively $\alpha^{\circlearrowleft}$ will be long enough and will take the value a with argument n.

(iv) **Assume $\boldsymbol{\lambda = \omega^\xi.n}$ with $\boldsymbol{0 \le \xi}$ and $\boldsymbol{1 < n}$**: this case is relatively simple since one has:
$$A^{\sim\omega^\xi.n} \equiv_c \left(A^{\sim\omega^\xi.(n-1)}\right)^{\sim\omega^\xi}$$
and by induction hypothesis there exists both:

- a ω^ξ-*Tree* T_{ω^ξ} on the alphabet of $\left(\left(A^{\sim\omega^\xi.(n-1)}\right)^{\sim\omega^\xi}\right)^b$ that fits the conditions of the induction hypothesis and thus there is a w.s. σ for *II* in the Wadge game where I and *II* are respectively in charge of $\left(\left(\left(A^{\sim\omega^\xi.(n-1)}\right)^{\sim\omega^\xi}\right)^b\right)^{T_{\omega^\xi}}$ and $\left(A^{\sim\omega^\xi.(n-1)}\right)^b$.
- a $\omega^\xi(n-1)$-*Tree* $T_{\omega^\xi(n-1)}$ on the alphabet of $\left(A^{\sim\omega^\xi.(n-1)}\right)^b$ that also fits the conditions of the induction hypothesis and
$$\left(\left(A^{\sim\omega^\xi.(n-1)}\right)^b\right)^{T_{\omega^\xi(n-1)}} \le_w A^b$$

Let T' be a pull back of $T_{\omega^\xi(n-1)}$ by the strategy σ. Now from T_{ω^ξ} and T' one derives a $\omega^\xi.n$-*Tree* T_λ that works.

(b) Assume the C.N.F. of base ω of λ is
$$\lambda = \omega^{\lambda_k}.n_k + \ldots + \omega^{\lambda_0}.n_0$$

then

$$A^{\approx\lambda} = \left(\left(\left(A^{\sim\omega^{\lambda_0}.n_0}\right)^{\cdots}\right)^{\sim\omega^{\lambda_k}.n_k}\right)$$

and by proposition 27

$$\left(A^{\approx\lambda}\right)^{b^{\nsim\lambda}} \leq_w \left(\ldots\left(\left(A^{\approx\lambda}\right)^{b^{\nsim\omega^{\lambda_k}.n_k}}\right)^{\cdots}\right)^{\nsim\omega^{\lambda_0}.n_0}$$

and by (b)

$$\left(\ldots\left(\left(\left(\left(\left(A^{\sim\omega^{\lambda_0}.n_0}\right)^{\cdots}\right)^{\sim\omega^{\lambda_k}.n_k}\right)^{b}\right)^{\nsim\omega^{\lambda_k}.n_k}\right)^{\cdots}\right)^{\nsim\omega^{\lambda_0}.n_0} \leq_w A^b$$

28 □

We reach the last problem. The checking of the non self dual character of the set $A^{\nsim\lambda}$.

Proposition 29 *Let* $A \subseteq \Lambda_A^\omega$*, A Borel* **non self dual***,* $1 < \lambda < \omega_1$

$$(\forall\xi < \lambda \quad A \leq_w A^{\nsim\xi}) \Longrightarrow A^{\nsim\lambda} \quad non\ self\ dual$$

Note that the converse is false. E.g. a $\mathbf{\Sigma}_3^\circ$-complete set A satisfies $A^{\nsim 2}$ is non self dual (and in fact is $\mathbf{\Sigma}_1^\circ$-complete) while $\forall\xi < 2\ A \leq_w A^{\nsim\xi}$ fails because $A^{\nsim 1}$ is some $\mathbf{\Sigma}_2^\circ$-complete set which is located strictly below A in the Wadge hierarchy.
□ **prop.29** - We prove it by induction on λ.

Assume $\boldsymbol{\lambda = \nu + 1}$: the result is direct since A is non self dual and the following holds

$$A^{\nsim\nu+1} \equiv_w (A^{\nsim\nu})^{\nsim} \equiv_w A^{\nsim} \equiv_w A.$$

Assume $\boldsymbol{\lambda}$ is limit: then the result is just a consequence of DC and the following claim.

Claim 30 *Assume* $(\forall\xi < \lambda \quad A \leq_w A^{\nsim\xi})$ *and* $A^{\nsim\lambda} = A^T$*,* $u \in T$ *(*$u \in T^\zeta$ *for some* ζ *large enough (*$\zeta < \lambda$*)) and* u *satisfies*

$$\forall\xi \quad \left(\zeta \leq \xi < \lambda \implies A^{T^\xi}[u] \geq_w A^{T^\xi}\right)$$

then

(a) $A^T[u] \geq_w A^T$

(b) $\exists\ t \quad u^\frown < t > \in T,\ u^\frown < t > \in T^\nu$ *(for some* ν *large enough that verifies* $\zeta \leq \nu < \lambda$*) such that* t *satisfies*

$$\forall\mu \quad \left(\nu \leq \mu < \lambda \implies \quad A^{T^\mu}[u^\frown < t >] \geq_w A^{T^\mu}\right).$$

$\square$ **cl. 30** - (a) relies on the fact $A^T[u] \geq_w \left(A^{T^\zeta}[u]\right)^{T'}$ where T' is some θ-Tree (with $\zeta+\theta=\lambda$) which is the "rest of T when T^ζ has been removed". Thus

$$A^T[u] \geq_w \left(A^{T^\zeta}[u]\right)^{T'} \geq_w \left(A^{T^\zeta}[u]\right)^{\sim\theta} \geq_w \left(A^{T^\zeta}\right)^{\sim\theta} \geq_w \left(A^{\sim\zeta}\right)^{\sim\theta} \equiv_w A^{\sim(\zeta+\theta)} = A^{\sim\lambda} = A^T.$$

(b) Get ν the least ordinal such that every sequence of T of length exactly $lh(u)+1$ is in T^ν. One obtains

$$A \geq_w A^{T^\nu}[u] \geq_w A^{T^\nu} \geq_w A$$

so that $A^{T^\nu}[u] \equiv_w A$ showing that $A^{T^\nu}[u]$ is non self dual. Therefore there exists t such that

$$\left(A^{T^\nu}[u]\right)[< t >] \equiv_w A, \text{ thus } A^{T^\nu}[u^\frown < t >] \equiv_w A$$

and for any $\mu > \nu$ ($\mu < \lambda$)

$$A^{T^\mu}[u^\frown < t >] \geq_w \left(A^{T^\nu}[u^\frown < t >]\right)^{T''}$$

where T'' is "the rest of T^μ when T^ν has been removed; T'' is some δ-Tree with δ satisfying $\nu+\delta=\mu$. Hence

$$A^{T^\mu}[u^\frown < t >] \geq_w \left(A^{T^\nu}[u^\frown < t >]\right)^{T''} \geq_w \left(A^{T^\nu}[u^\frown < t >]\right)^{\sim\delta} \geq_w A^{\sim\delta} \geq_w A \geq_w A^{T_\mu}.$$

30$\square$

Now consider $S = \{u \in T : A^T[u] \geq_w A^T\}$. This set is non empty since the empty sequence belongs to it. And by application of the above claim, if u is in S then there exists a strict extension of u that is still in S. In other words S is a non empty pruned tree. By DC let α be an infinite branch of S, it satisfies the following

$$\forall n \in \omega \quad A^{\sim\lambda}[\alpha \restriction n] = A^T[\alpha \restriction n] \geq_w A^T = A^{\sim\lambda}.$$

29 $\square$

We come to the ascending part of the preliminary results.

Proposition 31 *Let $A \subseteq \Lambda_A^\omega$, $B \subseteq \Lambda_B^{\leq\omega}$, A, B^b Borel, and $0 \leq \lambda < \omega_1$*

$$A^{\sim\lambda} \leq_w B^b \Longrightarrow A \leq_w \left(B^{\approx\lambda}\right)^b$$

$\square$ **prop.31** - The proof is a consequence of claim 33 below dealing with the game $\mathbf{G}_\lambda(T)$:

Definition 32 *Given λ some countable ordinal and T some λ-Tree on Λ, the game $\mathbf{G}_\lambda(T)$ is a two player infinite game where*

- *I starts and plays letters in Λ*

- *II plays letters in $\Lambda_T \cup \mathcal{A}_\lambda$ (Λ_T denotes the alphabet of the Tree T and $\mathcal{A}_\lambda$ is the set of arrows of different type involved by operation $^{\approx\lambda}$)*

$$G_\lambda(T) \quad \begin{array}{lccccccccc|c} \mathrm{I} & \alpha(0) & & \alpha(1) & & \ldots\ldots\ldots\ldots & & \alpha(p) & & \ldots\ldots & \alpha \\ & & \searrow \nearrow & & \searrow \nearrow & & \nearrow & & \searrow & & \\ \mathrm{II} & & \beta(0) & & \beta(1) & \ldots\ldots\ldots\ldots & \beta(p\text{-}1) & & & \beta(p) \ldots & \beta \end{array}$$

$$\alpha \in \Lambda^\omega \quad \textit{and} \quad \beta \in (\Lambda_T \cup \mathcal{A}_\lambda)^\omega$$

The winning conditions are:

$$\textit{II wins iff } ((\beta^\odot)^0 = \alpha \textbf{ and } \beta^\odot \in [T])$$

I wins otherwise.

The goal for player *II* is just to play β such that after it is "decoded", the part of it that corresponds to the run in Λ must be exactly I's run - i.e. the sequence α. Doing this is easy but the difficulty lies on having to play inside T which means answering many questions. Therefore *II*'s duty consists also in guessing what I's next moves will be because, running along T, he has to answer questions asked at each node he reaches. But fortunately besides that extreme weakness *II* is granted with the right to use some iterated priority, for he is also provided with the set of arrows $\mathcal{A}_\lambda$. The only essential thing about that game is that player *II* can make it.

Claim 33 *Given T any λ-Tree on Λ,*

$$\textit{II} \quad \textit{has a w.s. in } \mathbf{G}_\lambda(T)$$

□ **cl. 33** - The proof goes by induction on λ. In fact we show something stronger. Namely, we prove that for any countable ordinal λ, any alphabet Λ and any λ-Tree T, *II* has a w.s.

$$^{\Lambda,T}\sigma_\lambda \;:\; \Lambda \longmapsto \Lambda_T \cup \mathcal{A}_\lambda$$

that verifies

$$\forall \xi \leq \lambda \;\; \forall \alpha \in \Lambda^\omega \quad \left(\left(^{\Lambda,T}\sigma_\lambda * \alpha \right)^\odot \right)^\xi = \left(^{\Lambda,T^\xi}\sigma_\xi * \alpha \right)^\odot$$

If $\boldsymbol{\lambda = 0}$: copying I's run is a w.s. for *II*.

If $\boldsymbol{\lambda = 1}$: This is just a priority argument. The problem for player *II* is that he has both to reproduce I's run and also to answer questions about whether or not this run will remain in closed subsets of Λ^ω; but since he is allowed to use an eraser, he may first answer "yes the run will always remain in that particular closed set you ask me about" and if at some moment of the run it appears that it is not the case he has just to use the eraser to let him come back to the move where he answered "yes" and change it into some "no" indicating also the way the run will exit that closed set. (see [1] proposition 27 (a)).

$\boldsymbol{\lambda = \omega^{\xi}}$ limit :

The w.s. for *II* is the following:
First he plays $\twoheadleftarrow_{(\lambda)}$ so that it "opens an array" in which he plays different sequences $(\beta_n)_{n\in\omega}$ so that at the end there will be $({}^{\Lambda,T}\sigma_{\lambda} * \alpha)^{\circlearrowleft} =< \beta_n^{\circlearrowleft}(0)/n \in \omega >$. Without loss of generality β_n can be taken in $\Lambda_T \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\xi_n}.p_n\}$ with $\omega^{\xi_n}.p_n$ large enough so that the condition "every sequence in T of length n is also in $T^{\omega^{\xi_n}.p_n}$" is satisfied.
Get

$${}^{\Lambda,T^{\omega^{\xi_n}.p_n}}\sigma_{\omega^{\xi_n}.p_n} \ : \Lambda \longmapsto \Lambda_{T^{\omega^{\xi_n}.p_n}} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\xi_n}.p_n\}$$

a w.s. for *II* in the game $\mathbf{G}_{T^{\omega^{\xi_n}.p_n}}\left(\omega^{\xi_n}.p_n\right)$ and since the $n+1^{\text{th}}$ level in T corresponds to some $m+1^{\text{th}}$ level in $T^{\omega^{\xi_n}.p_n}$ (for some $m \leq n$) let

$$\delta_{m,\omega^{\xi_n}.p_n} \ : \ \Lambda_{T^{\omega^{\xi_n}.p_n}} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\xi_n}.p_n\} \longmapsto \Lambda_{T^{\omega^{\xi_n}.p_n}} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \omega^{\xi_n}.p_n\}$$

be some $\omega^{\xi_n}.p_n$-remote control of a strategy δ that simply puts the $m+1^{\text{th}}$ letter in first place, for example:

$$\delta \ : \ \Lambda_{T^{\omega^{\xi_n}.p_n}} \longmapsto \Lambda_{T^{\omega^{\xi_n}.p_n}} \text{ defined by } \delta * < a_i/i \in \omega >=< a_m, a_m, \ldots, a_m, \ldots >$$

The result is simply obtained by setting $\beta_n = (\delta_{m,\omega^{\xi_n}.p_n} \ \circ \ {}^{\Lambda,T^{\omega^{\xi_n}.p_n}}\sigma_{\omega^{\xi_n}.p_n}) * \alpha$.

Clearly by construction ${}^{\Lambda,T}\sigma_{\lambda}$ satisfies the condition

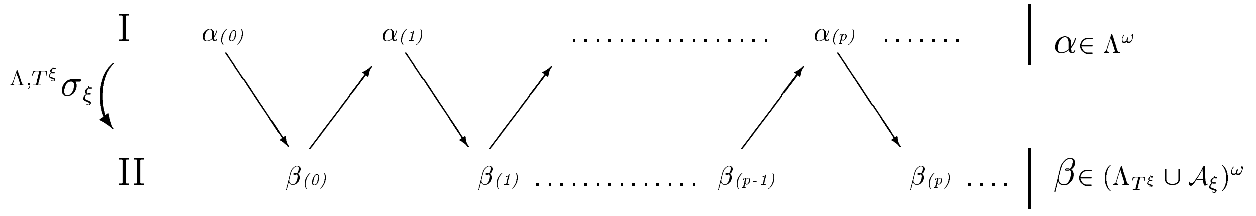
$$\forall \zeta \leq \lambda \ \ \forall \alpha \in \Lambda^{\omega} \quad \left(\left({}^{\Lambda,T}\sigma_{\lambda} * \alpha\right)^{\circlearrowleft}\right)^{\zeta} = \left({}^{\Lambda,T^{\zeta}}\sigma_{\zeta} * \alpha\right)^{\circlearrowleft}$$

which means it satisfies

$$\forall \zeta \leq \lambda \ \ \forall \alpha \in \Lambda^{\omega} \quad \left(\left({}^{\Lambda,T}\sigma_{\lambda} * \alpha\right)^{\odot}\right)^{\zeta} = \left({}^{\Lambda,T^{\zeta}}\sigma_{\zeta} * \alpha\right)^{\odot}$$

$\boldsymbol{\lambda = \omega^{\lambda_k}.n_k + \ldots + \omega^{\lambda_0}.n_0}$ with $\boldsymbol{\lambda_0 > 0}$ and $\boldsymbol{(k > 0}$ or $\boldsymbol{n_0 > 1)}$

$(\lambda_k > \ldots > \lambda_0$ and for each i $0 < n_i < \omega)$:
Set $\xi = \omega^{\lambda_k}.n_k + \ldots + \omega^{\lambda_0}.(n_0 - 1)$. assume ${}^{\Lambda,T^{\xi}}\sigma_{\xi} \ : \Lambda \longmapsto \Lambda_{T^{\xi}} \cup \mathcal{A}_{\xi}$ has been defined



and with T' denoting the "rest of T when T^{ξ} has been removed" (T' is a ω^{λ_0} -*Tree* on the alphabet of T^{ξ} - so the alphabet of T' is the same as the alphabet of T), the strategy ${}^{\Lambda_{T^{\xi}},T'}\sigma_{\omega^{\lambda_0}} \ : \Lambda_{T^{\xi}} \longmapsto \Lambda_T \cup \{\twoheadleftarrow'_{(\iota)} : 0 < \iota \leq \omega^{\lambda_0}\}$ has been defined too :

$$
{}^{\Lambda_{T^\xi},T'}\sigma_{\omega^{\lambda_0}}\Big\downarrow
\begin{array}{lcccccccc|l}
\mathrm{I} & \delta(0) & & \delta(1) & & \ldots\ldots\ldots\ldots\ldots & & \delta(p) & \ldots\ldots & \delta\in \Lambda_{T^\xi}^{\omega} \\
& \searrow & \nearrow & \searrow & \nearrow & & \nearrow & \searrow & & \\
\mathrm{II} & & \gamma(0) & & \gamma(1) & \ldots\ldots\ldots\ldots\ldots & & & \ldots\ldots & \gamma\in (\Lambda_T \cup \{\twoheadleftarrow'(\iota) : 0<\iota \leq \omega^{\lambda_0}\})^{\omega}
\end{array}
$$

Given

$$\delta^{\xi} \ : \ \Lambda_{T^\xi} \cup \mathcal{A}_{\xi} \longmapsto \Lambda_T \cup \mathcal{A}_{\lambda}$$

a strategy that "ξ-remote controls" ${}^{\Lambda_{T^\xi},T'}\sigma_{\omega^{\lambda_0}}$: in the sense that first get $\delta^{\omega^{\lambda_0}.(n_0-1)}$ a strategy that $\omega^{\lambda_0}.(n_0-1)$-remote controls ${}^{\Lambda_{T^\xi},T'}\sigma_{\omega^{\lambda_0}}$, then $\delta^{\omega^{\lambda_1}.n_1+\omega^{\lambda_0}.(n_0-1)}$ a strategy that $\omega^{\lambda_1}.n_1$-remote controls $\delta^{\omega^{\lambda_0}.(n_0-1)}$,..., and finally $\delta^{\omega^{\lambda_k}.n_k+\ldots+\omega^{\lambda_0}.(n_0-1)}$ a strategy that $\omega^{\lambda_k}.n_k$-remote controls $\delta^{\omega^{\lambda_{k-1}}.n_{k-1}+\ldots+\omega^{\lambda_1}.n_1+\omega^{\lambda_0}.(n_0-1)}$. Define

$${}^{\Lambda,T}\sigma_{\lambda} \ : \Lambda \longmapsto \Lambda_T \cup \mathcal{A}_{\lambda}$$

by

$${}^{\Lambda,T}\sigma_{\lambda} = \delta^{\xi} \circ {}^{\Lambda,T^\xi}\sigma_{\xi}$$

Clearly ${}^{\Lambda,T}\sigma_{\lambda}$ satisfies the condition

$$\forall\zeta \leq \lambda \ \ \forall\alpha \in \Lambda^{\omega} \quad \left(\left({}^{\Lambda,T}\sigma_{\lambda} * \alpha\right)^{\odot}\right)^{\zeta} = \left({}^{\Lambda,T^\zeta}\sigma_{\zeta} * \alpha\right)^{\odot}$$

because by construction

$$\forall\alpha \in \Lambda^{\omega} \quad \left(\left({}^{\Lambda,T}\sigma_{\lambda} * \alpha\right)^{\odot}\right)^{\xi} = \left({}^{\Lambda,T^\xi}\sigma_{\xi} * \alpha\right)^{\odot}$$

33□

Get a w.s. δ for *II* in the Wadge game $\mathbf{W}\left(A^T, B^b\right)$ where T is some λ-tree that satisfies $A^T \equiv_w A^{\sim\lambda}$. By above claim 33 player *II* has a w.s. ${}^{\Lambda,T}\sigma_{\lambda}$ in the game $\mathbf{G}_{\lambda}(T)$. Get δ^{λ} a strategy that "λ-remote controls" δ with same arrows as above: precisely this means that if the C.N.F of λ is $\lambda = \omega^{\lambda_k}.n_k + \ldots + \omega^{\lambda_0}.n_0$ then get $\delta^{\omega^{\lambda_0}.n_0}$ a strategy that $\omega^{\lambda_0}.n_0$-remote controls δ, then $\delta^{\omega^{\lambda_1}.n_1+\omega^{\lambda_0}.n_0}$ a strategy that $\omega^{\lambda_1}.n_1$-remote controls $\delta^{\omega^{\lambda_0}.n_0}$,..., and finally $\delta^{\omega^{\lambda_k}.n_k+\ldots+\omega^{\lambda_0}.n_0}$ a strategy that $\omega^{\lambda_k}.n_k$-remote controls $\delta^{\omega^{\lambda_{k-1}}.n_{k-1}+\ldots+\omega^{\lambda_1}.n_1+\omega^{\lambda_0}.n_0}$. Then the strategy $\delta^{\lambda} \circ {}^{\Lambda,T}\sigma_{\lambda}$ is winning for *II* in $\mathbf{W}\left(A, \left(B^{\approx\lambda}\right)^b\right)$ because when I plays α and *II* replies with $\delta^{\lambda} \circ {}^{\Lambda,T}\sigma_{\lambda} * \alpha = \beta$ one has by definition of ${}^{\Lambda,T}\sigma_{\lambda}$

$$({}^{\Lambda,T}\sigma_{\lambda} * \alpha)^{\circlearrowleft} \in [T] \text{ and } \alpha \in A \ \longleftrightarrow \ ({}^{\Lambda,T}\sigma_{\lambda} * \alpha)^{\circlearrowleft 0} \in A.$$

Thus

$$({}^{\Lambda,T}\sigma_{\lambda} * \alpha)^{\circlearrowleft 0} \in A \longleftrightarrow ((\delta^{\lambda} \circ {}^{\Lambda,T}\sigma_{\lambda} * \alpha)^{\circlearrowleft^{0}}) \in B \longleftrightarrow \beta \in B^{\approx\lambda}$$

31 □

Preliminaries are over now. We may use both operations ${}^{\approx\lambda}$ and ${}^{\sim\lambda}$ to generate Σ°_{λ}-complete sets for any non null countable ordinal λ on the basis of any complete open set.

5 Canonical Σ°_λ-complete conciliatory sets

Remind that the symbol $\ominus\!\overset{\longrightarrow}{}\!\oplus$ stands for a set of the form $\{\alpha \in \{0\}^{\leq\omega} : \exists n\ \alpha(n) = 0\}$ therefore $(\ominus\!\overset{\longrightarrow}{}\!\oplus)^b$ stands for $\{\alpha \in \{0, b\}^\omega : \exists n\ \alpha(n) = 0\}$.

Proposition 34 *Let O be some conciliatory set such that O^b is $\mathbf{\Sigma}^\circ_1$-complete (e.g. $O = \ominus\!\overset{\longrightarrow}{}\!\oplus$ or $O = \ominus \bullet 2$), and $0 < \lambda < \omega_1$,*

$$(O^{\approx\lambda})^b \text{ is } \mathbf{\Sigma}^\circ_{1+\lambda} - \text{complete}$$

$\square$ **prop.34** - First we show that for any $A \in \mathbf{\Sigma}^\circ_{1+\lambda}$:

$$A \leq_w \left(O^{\approx\lambda}\right)^b$$

This relies on the fact there clearly exists some λ-$\mathcal{T}$ree T such that $A^T \in \mathbf{\Sigma}^\circ_1$. This holds because more generally: given $(A_i)_{i\in\omega}$ a sequence of $\mathbf{\Sigma}^\circ_\xi$ sets, there exists some $\xi + 1$-$\mathcal{T}$ree T that "turns" any A_i into some $\mathbf{\Delta}^\circ_1$ set. This means every question of the form "will $(\alpha)^0$ be in A_i ?" appears in T, a player in charge of A^T must answer yes or no; so regarding T as changing the topology on Λ^ω, the effect of T is to turn any A_i into some $\mathbf{\Delta}^\circ_1$ set. This point is clear by induction on $\xi \geq 1$. Thus $A^{\sim\lambda} \in \mathbf{\Sigma}^\circ_1$ which means $A^{\sim\lambda} \leq_w O^b$.
Therefore by proposition 31:

$$A^{\sim\lambda} \leq_w O^b \Longrightarrow A \leq_w \left(O^{\approx\lambda}\right)^b.$$

We show now that $\left(O^{\approx\lambda}\right)^b \in \mathbf{\Sigma}^\circ_{1+\lambda}$. We need the following claim.

Claim 35 *Let Λ be any alphabet and T any λ-$\mathcal{T}$ree on Λ. (Remind that $[T]$ is a subset of Λ_T^ω where Λ_T is the alphabet of T).*
Let f_λ be the function from Λ^ω into Λ_T^ω defined by

> $f_\lambda(\alpha)$ *is the* **one and only** *infinite sequence belonging to $[T]$ that verifies $(f_\lambda(\alpha))^0 = \alpha$*

we claim that f_λ is a Baire class λ function.

Let Λ be the alphabet of $\left(O^{\approx\lambda}\right)^b$, by proposition 28(b)

$$\left(\left(O^{\approx\lambda}\right)^b\right)^{\sim\lambda} \leq_w O^b.$$

So that there exists a λ-$\mathcal{T}$ree T such that

$$\left(\left(O^{\approx\lambda}\right)^b\right)^T \in \mathbf{\Sigma}^\circ_1 \restriction \Lambda_T^\omega.$$

Let f_λ be as in claim above. The following is verified:

$$\forall \alpha \in \Lambda^\omega, \quad \alpha \in \left(O^{\approx\lambda}\right)^b \text{ iff } f_\lambda(\alpha) \in \left(\left(O^{\approx\lambda}\right)^b\right)^T$$

Therefore $\left(O^{\approx\lambda}\right)^b$ is the inverse image of an open set by a Baire class λ function, which means

$$\left(O^{\approx\lambda}\right)^b \in \mathbf{\Sigma}^\circ_{1+\lambda} \restriction \Lambda^\omega_T.$$

34 $\square$

$\square$ **cl. 35** - The proof is by induction on λ.

- $\lambda = 0$ is obvious.
- $\lambda = \xi + 1$. Let f_ξ from λ^ω into Λ^ω_T be defined exactly as f_λ is except that the ξ-$\mathcal{T}$ree T^ξ replaces T:

  $f_\xi(\alpha)$ is the one and only infinite sequence such that $f_\xi(\alpha) \in [T^\xi] \wedge (f_\xi(\alpha))^0 = \alpha$

  Let g from $\Lambda^\xi_{T^\xi}$ into Λ^ω_T be defined by:

  $$g(\alpha) \in [T] \wedge (g(\alpha))^\xi = \alpha.$$

  We know by induction hypothesis that f_ξ is a Baire class ξ function. We are going to show that g is a Baire class 1 function.
  Let $\mathcal{O}$ be an open subset of Λ^ω_T. $\mathcal{O} = \bigcup\limits_{i\in I} u_i{}^\frown\Lambda^\omega_T$ for some family $(u_i)_{i\in I}$ of finite sequences of elements among Λ_T. (Without loss of generality we may assume that given i and j two different elements in I, the sequences u_i and u_j are incomparable $(u_i \perp u_j)$. We need to show that $f_\lambda^{-1''}\mathcal{O} \in \mathbf{\Sigma}^\circ_{1+lambda}$.

  (a) Let $J \subseteq I$ be the subset of I such that $i \in J$ iff $u_i \in T$.

  $$\mathcal{O} = \Big(\bigcup_{i\in J} u_i{}^\frown\Lambda^\omega_T\Big) \cup \Big(\bigcup_{i\in I\smallsetminus J} u_i{}^\frown\Lambda^\omega_T\Big)$$

  For any $i \in I \smallsetminus J$ $f_\lambda^{-1''} u_i{}^\frown\Lambda^\omega_T = \emptyset$. So that

  $$f_\lambda^{-1''}\mathcal{O} = f_\lambda^{-1''}\Big(\bigcup_{i\in J} u_i{}^\frown\Lambda^\omega_T\Big) = \Big(\bigcup_{i\in J} f_\lambda^{-1''} u_i{}^\frown\Lambda^\omega_T\Big).$$

  (b) For any $i \in J$, let $v_i = \cup\{w_{i,n} : \exists n \ \ u_i(\prec \xi, n \succ^\lambda) =< 1 >{}^\frown w_{i,n}\}$. i.e. v_i is the sequence from $\Lambda^{<\omega}_{T^\xi}$ that extends u_i^ξ gathering all promises made by taking option $< 1 >$ when asked about remaining or not in a closed subset of $\Lambda^{<\omega}_{T^\xi}$.

  (c) For any $i \in J$ let F_i be the intersection of all (finitely many) closed subsets of $\Lambda^{<\omega}_{T^\xi}$ in which u_i promises to remain by taking option $< 0 >$ (at nodes of the form $u_i(\prec \xi, n \succ^\lambda$ $) =< 0 >$).

  (d) Let $C_i = F_i \cap v_i{}^\frown\Lambda^{<\omega}_{T^\xi}$, and for each integer n let J_n be the subset of J such that $i \in J_n$ iff $lh(v_i) = n$.

(e) Note that given i and j two different elements in J_n the sequences v_i and v_j are incomparable. Indeed $u_i \perp u_j$ holds; so either $u_i^\xi \perp u_j^\xi$ which implies $v_i \perp v_j$; either u_i^ξ and u_j^ξ are comparable, in which case there exists an integer n such that $u_i(\prec \xi, n \succ^\lambda) \neq u_j(\prec \xi, n \succ^\lambda)$:

- ◇ if both take option $<1>$, that means $u_i(\prec \xi, n \succ^\lambda) = <1>^\frown w_{i,n}$ and $u_j(\prec \xi, n \succ^\lambda) = <1>^\frown w_{j,n}$, with $w_{i,n} \perp w_{j,n}$. Since $w_{j,n} \subseteq v_j$ and $w_{i,n} \subseteq v_i$ holds, sequences v_i and v_j are incomparable.
- ◇ if one takes option $<0>$, necessarily the other sequence takes option $<1>$: (by symmetry)

$$u_i(\prec \xi, n \succ^\lambda) = <0> \text{ and } u_j(\prec \xi, n \succ^\lambda) = <1>^\frown w_{j,n}$$

therefore $w_{j,n} \subseteq v_j$ and $w_{j,n} \not\subseteq v_i$. Since v_i and v_j have same length v_i and v_j are incomparable.

Set $D_n = \bigcup_{i \in J_n} C_i$. By previous remark D_n is a closed subset of $\Lambda^\omega_{T^\xi}$ and

$$\bigcup_{n\in\omega} D_n = \bigcup_{n\in\omega} g^{-1''}\Big(\bigcup_{i\in J_n} u_i{}^\frown\Lambda^\omega_T\Big) = g^{-1''}\Big(\bigcup_{n\in\omega}\Big(\bigcup_{i\in J_n} u_i{}^\frown\Lambda^\omega_T\Big)\Big) = g^{-1''}\Big(\bigcup_{i\in J} u_i{}^\frown\Lambda^\omega_T\Big) = g^{-1''}\mathcal{O}.$$

Therefore $g^{-1''}\mathcal{O}$ is some $\mathbf{\Sigma}^\circ_2$ subset of $\Lambda^\omega_{T^\xi}$. And finally

$$f_\lambda^{-1''}\mathcal{O} = f_\xi^{-1''} g^{-1''}\mathcal{O} \in \mathbf{\Sigma}^\circ_{(1+\xi)+1} \restriction \Lambda^\omega$$

∘ λ is limit. Let $\mathcal{O} = \bigcup_{i\in I} u_i{}^\frown\Lambda_T$ be any open subset of Λ^ω_T. For each u_i there exists a minimum ordinal $m(u_i) = \xi_i$ such that $u_i \in T^{\xi_i}$. For each ξ less than λ set

$$I_\xi = \{i \in I : \ m(u_i) = \xi\}.$$

Since u_i belongs to T^ξ it is immediate to see that $f_\lambda^{-1''} u_i{}^\frown\Lambda_T = f_\xi^{-1''} u_i{}^\frown\Lambda^\omega_{T^\xi} =$ where f_ξ is the function from Λ^ω into $\Lambda^\omega_{T^\xi}$ defined by $f_\xi(\alpha) \in [T^\xi]$ and $(f_\xi(\alpha))^0 = \alpha$. At last

$$f_\lambda^{-1''}\mathcal{O} = f_\lambda^{-1''}\Big(\bigcup_{i\in I} u_i{}^\frown\Lambda^\omega_T\Big) = \bigcup_{\xi<\lambda}\Big(f_\xi^{-1''}\Big(\bigcup_{i\in I_\xi} u_i{}^\frown\Lambda^\omega_{T^\xi}\Big)\Big)$$

since by induction hypothesis $\Big(f_\xi^{-1''}\Big(\bigcup_{i\in I_\xi} u_i{}^\frown\Lambda^\omega_{T^\xi}\Big)\Big) \in \mathbf{\Sigma}^\circ_{1+\xi} \restriction \Lambda^\omega$, one gets $f_\lambda^{-1''}\mathcal{O} \in \mathbf{\Sigma}^\circ_{1+\lambda} \restriction \Lambda^\omega$

35□

So, we know that operations $\sim^\lambda$ are strong enough to unboundedly climb along the Wadge hierarchy on Borel sets with the help of any true open set. We have now everything to be able to generate the whole hierarchy.

6 A normal form for Borel sets

We use the Veblen hierarchy to define the following "canonical" conciliatory Borel sets.
But first we link operation $^{\sim\nu}$ to the Veblen functions by setting

Notation 36 *Let $A \subseteq \Lambda_A^{\leq\omega}$, $0 \leq \nu < \omega_1$*

$$^{\kappa}V_c^{\nu}(A) = A^{\sim\omega^{\nu}}$$

In case $\nu = 0$, $^{\kappa}V_c^{A} = A^{\sim\omega^0} = A^{\sim 1} = A^{\sim}$. It corresponds to the "quasi exponentiation" of base κ we dealed with in [1]. This notation is totally justified since the Wadge degree of the set $^{\kappa}V_c^{\nu}(A)$ is exactly the value of the Veblen function $^{\kappa}V^{\nu}$ taken at the degree of the set A (see proposition 41).

Definition 37 *Let $OT_{\mathcal{B}}$ (for order type of the Wadge hierarchy on Borel sets) designate the ordinal $\sup_{\nu<\omega_1} {}^{\kappa}V^{\nu}(2)$.*
Let $0 < \alpha < OT_{\mathcal{B}}$, α admits a unique Cantor Normal Form of base κ:

$$\alpha = \kappa^{\alpha_n}.\nu_n + \ldots + \kappa^{\alpha_0}.\nu_0$$

with

$$OT_{\mathcal{B}} > \alpha_n > \ldots > \alpha_0 \geq 0 \ et\ \forall i \leq n, \quad 0 < \nu_i < \kappa$$

Define:

$$\Omega(\alpha) = \Omega(\kappa^{\alpha_n}) \bullet \nu_n + \ldots\ldots + \Omega(\kappa^{\alpha_0}) \bullet \nu_0$$

where $\Omega(\kappa^{\beta})$ with $0 \leq \beta < OT_{\mathcal{B}}$ is defined by:

$$\begin{array}{ll} \beta = 0 : & \Omega(\kappa^{\beta}) = \ominus \\ \beta \ \textit{limit of cofinality less than } \kappa : & \Omega(\kappa^{\beta}) = \sup_{\theta\in\Theta_\beta} \Omega(\kappa^{\theta}) \\ \left.\begin{array}{l} \beta \ \textit{successor} \\ \beta \ \textit{limit of cofinality } \kappa \end{array}\right\} & \Omega(\kappa^{\beta}) = {}^{\kappa}V_c^{\nu}(\beta) = \Omega(\beta)^{\sim\omega^{\nu}} \end{array}$$

where ν is the least ordinal such that there exists $\gamma < \kappa^{\beta}$ satisfying $\kappa^{\beta} = {}^{\kappa}V^{\nu}(\gamma)$ - i.e. γ is not a fixpoint of cofinality κ of the function: $x \longmapsto {}^{\kappa}V^{\nu}(x)$.

Where Θ_β is any set of cardinality less than κ, definable in ZF and such that $sup\{\theta \in \Theta_\beta\} = \beta$. For example:

(a) if $\beta < \kappa^{\beta}$ then the Cantor Normal Form of base κ of β is

$$\beta = \kappa^{\beta_n}.\mu_n + \ldots + \kappa^{\beta_0}.\mu_0$$

$$\textit{with } \beta > \beta_n > \ldots > \beta_0 \geq 0 \ \textit{ and } \ \forall i \leq n, \ 0 < \mu_i < \kappa.$$

(i) if μ_0 is limit

$$\Theta_\beta = \{\kappa^{\beta_n}.\mu_n + \ldots + \kappa^{\beta_0}.\mu : \ \mu < \mu_0\}$$

(ii) if μ_0 is not limit

$$\Theta_\beta = \{\kappa^{\beta_n}.\mu_n + \ldots + \kappa^{\beta_0}.(\mu_0 - 1) + \kappa^\theta : \ \theta \in \Theta_{\beta_0}\}$$

(b) if $\beta = \kappa^\beta$ then consider the hierarchy $(W_\nu)_{\nu<\omega_1}$ of ordinal functions defined by:

- *W_0 is the exponentiation of base κ: $x \longmapsto \kappa^x$*
- *W_ν for $\nu > 0$ is the function that enumerates ordinals that are fixpoint of all functions W_μ for all $\mu < \nu$*

Now get the least ordinal ν such that $W_\nu(\gamma) = \beta$ and $\gamma < \beta$ and set:

$\Theta_\beta = \{W_\nu(\theta) : \ \theta \in \Theta_\gamma\}$ *when γ is limit*

- $\Theta_\beta = \{\underbrace{W_\mu \circ W_\mu \circ \ldots \circ W_\mu}_{n}(W_\nu(\delta) + 1) : \ n \in \omega, \ \mu < \mu\}$ *when $\gamma = \delta + 1$*

Those canonical sets constitute a hierarchy which describes (up to complementation) the whole Wadge hierarchy on Borel non self dual sets. Namely:

Theorem 38
Let $A \subseteq \Lambda^\omega$ be Borel and non self dual then there exists a unique ordinal $0 < \alpha < OT_\mathcal{B}$ such that

$$^{-}A \textit{ or } A \ \equiv_w \Omega(\alpha)^b$$

The proof of this theorem requires two different facts. The first one says that the sequence $(\Omega(\alpha))_{0<\alpha<OT_\mathcal{B}}$ equipped with $<_c$ is a well ordering - this is stated by next lemma 39. The second one is what we call a weak separation lemma; asserting that when a non self dual set is situated strictly below a canonical set in the Wadge hierarchy then it (or its complement) is situated at the same level (or below) the one of another canonical set but with a smaller ordinal. Therefore this lemma 40 shows that there is no Wadge degree missing in the hierarchy of canonical sets.

Lemma 39 *Let $0 < \beta < \alpha < OT_\mathcal{B}$,*

$$^{-}\Omega(\beta), \ \ \Omega(\beta) \ \leq_c \ \Omega(\alpha)$$

Remark that since any conciliatory set is non self dual, this lemma states that when β is strictly less than α then $\Omega(\beta) <_c \Omega(\alpha)$, which shows that the structure $\langle(\Omega(\alpha))_{0<\alpha<OT_\mathcal{B}}, <_c\rangle$ is a well ordering.

□ **lem. 39** - The proof goes by induction on α and for α fixed by induction on β.

The first case $\boldsymbol{\alpha = 2}$ **and** $\boldsymbol{\beta = 1}$ is obvious since it just says that $\oplus$, $\ominus \ \leq_c \ominus \bullet 2$.

Assume The C.N.F of base κ of respectively α and β are

$$\alpha = \kappa^{\alpha_n}.\nu_n + \ldots + \kappa^{\alpha_0}.\nu_0 \text{ and } \beta = \kappa^{\beta_q}.\mu_q + \ldots + \kappa^{\beta_0}.\mu_0.$$

If $\alpha \neq \kappa^{\alpha_0}$ or $\beta \neq \kappa^{\beta_0}$ then the result easily derives from induction hypothesis and from the following results stated in [1]:

$$d_w^\circ(A+B) = (d_w^\circ A) + (d_w^\circ B) \text{ and } d_w^\circ(\mathbf{sup}_{i\in I} A_i) = \sup_{i\in I}(d_w^\circ A_i)$$

So the only interesting case is $\boldsymbol{\alpha = \kappa^{\alpha_0}}$ **and** $\boldsymbol{\beta = \kappa^{\beta_0}}$:

If $\alpha_0 = 0$: then there is nothing to prove since in that case $\Omega(\beta)$ does not exist.

If α_0 is finite : then without loss of generality one may assume that $\beta_0 = \alpha_0 - 1$, $\kappa^{\beta_0} = {}^\kappa V^0(\delta - 1)$ and $\kappa^{\alpha_0} = {}^\kappa V^0(\delta)$. So that $\Omega(\alpha) = \Omega(\delta)^\sim$ and $\Omega(\beta) = \Omega(\delta-1)^\sim$. By induction hypothesis one has

$$^-\Omega(\delta-1),\ \Omega(\delta-1)\ \leq_c \Omega(\delta)$$

so by 1-remote control one gets

$$^-\Omega(\delta-1)^\sim,\ \Omega(\delta-1)^\sim\ \leq_c \Omega(\delta)^\sim$$

***If* α_0 is successor** : it satisfies $\kappa^{\alpha_0} = {}^\kappa V^0(\delta)$ and $\Omega(\alpha) = \Omega(\delta)^\sim$.

(a) if δ is limit of cofinality strictly less than κ:

we may assume without loss of generality that $\beta_0 = \delta$ and $\alpha_0 = \delta + 1$:
then $\Omega(\beta) = \sup_{\theta\in\Theta_\delta} \Omega(\kappa^\theta)$. But $\Omega(\alpha) = \Omega(\delta)^\sim$ is initializable and for any $\theta \in \Theta_\delta$ one has already proved

$$^-\Omega(\kappa^\theta),\ \Omega(\kappa^\theta) \leq_c \Omega(\kappa^{\theta+1}) \text{ and } \Omega(\kappa^{\theta+1}) \leq_c \Omega(\alpha).$$

(b) If δ is successor:

then without loss of generality we may assume that $\beta_0 = \alpha_0 - 1$ so that

$$\kappa^{\beta_0} = {}^\kappa V^0(\delta-1) \text{ and } \Omega(\beta) = \Omega(\delta-1)^\sim,\quad \Omega(\alpha) = \Omega(\delta)^\sim.$$

And since

$$^-\Omega(\delta-1),\ \Omega(\delta-1) \leq_c \Omega(\delta)$$

holds by induction hypothesis, by 1-remote control one gets

$$^-\Omega(\delta-1)^\sim,\ \Omega(\delta-1)^\sim \leq_c \Omega(\delta)^\sim.$$

There aren't any other possible cases when α_0 is successor.

If α_0 is limit of cofinality κ and $\alpha_0 < \kappa^{\alpha_0}$: then $\kappa^{\alpha_0} = {}^\kappa V^0(\alpha_0)$ and $\Omega(\alpha) = \Omega(\alpha_0)^\sim$. Clearly $\beta_0 < \alpha_0$ implies that there exists a limit ordinal of countable cofinality $\lambda < \alpha_0$ such that $\beta_0 \leq \lambda + 1$. Without loss of generality we may argue that $\beta_0 = \lambda + 1$. So $\Omega(\beta) = \Omega(\beta_0)^\sim$ and since

$$^-\Omega(\beta_0),\ \Omega(\beta_0) \leq_c \Omega(\alpha_0)$$

holds by induction hypothesis, the result follows by 1-remote control:

$$^-\Omega(\beta_0)^\sim,\ \Omega(\beta_0)^\sim \leq_c \Omega(\alpha_0)^\sim.$$

If α_0 is limit of cofinality κ and $\alpha_0 = \kappa^{\alpha_0}$ (so that $\alpha = \kappa^{\alpha}$) : in that case $\alpha = {}^{\kappa}V^{\nu}(\gamma)$ for ν the least ordinal such that $\gamma < \alpha$. First remark that for any ordinal ν if one has

$$B,\ A,\ A_i \leq_c \Omega(\gamma)^{\sim\omega^{\nu}} (\text{any } i \in I,\ \text{any } I \text{ such that } |I| < \kappa)$$

then

- $A+B \leq_c \Omega(\gamma)^{\sim\omega^{\nu}}$ simply because $\Omega(\gamma)^{\sim\omega^{\nu}}$ is initializable.
- $\sup_{i\in I} A_i \leq_c \Omega(\gamma)^{\sim\omega^{\nu}}$ also because $\Omega(\gamma)^{\sim\omega^{\nu}}$ is initializable.
- $A^{\sim\omega^{\mu}} \leq_c \Omega(\gamma)^{\sim\omega^{\nu}}$ for any $\mu < \nu$ simply because by ω^{μ}-remote control

  $$A \leq_c \Omega(\gamma)^{\sim\omega^{\nu}} \Longrightarrow A^{\sim\omega^{\mu}} \leq_c (\Omega(\gamma)^{\sim\omega^{\nu}})^{\sim\omega^{\mu}}$$

  and by remark 6 and corollary 16(a)

  $$(\Omega(\gamma)^{\sim\omega^{\nu}})^{\sim\omega^{\mu}} \leq_c \Omega(\gamma)^{\sim\omega^{\nu}+\omega^{\mu}} \leq_c \Omega(\gamma)^{\sim\omega^{\nu}}$$

  so that

  $$A \leq_c \Omega(\gamma)^{\sim\omega^{\nu}} \Longrightarrow A^{\sim\omega^{\mu}} \leq_c \Omega(\gamma)^{\sim\omega^{\nu}}$$

Now if $\gamma = 2$ then, reminding that ${}^{-}A,\ A \leq_c A+\ominus$, the result is immediate since only operations $+$, sup and ${}^{\sim\omega^{\mu}}$ for $\mu < \nu$ are involved in the definition of $\Omega(\beta)$ on the basis of $\ominus$.

If $\gamma > 2$ then since $\beta < {}^{\kappa}V^{\nu}(\gamma)$, one may assume without loss of generality that β is some fixpoint of cofinality $< \kappa$ of the function ${}^{\kappa}V^{\nu}$ strictly greater than all the ${}^{\kappa}V^{\nu}(\theta)$ for $\theta < \gamma$. This is clearly possible since such an ordinal greater than β and less than α always exists. Now by induction hypothesis

$${}^{-}\Omega(\theta),\ \Omega(\theta)\ \leq_c \Omega(\delta)$$

holds for any $\theta < \delta$; thus by ω^{ν}-remote control one obtains

$${}^{-}\Omega(\theta)^{\sim\omega^{\nu}},\ \Omega(\theta)^{\sim\omega^{\nu}}\ \leq_c \Omega(\delta)^{\sim\omega^{\nu}}.$$

But $\Omega(\beta)$ is defined only by use of such sets $\Omega(\theta)^{\sim\omega^{\nu}}$ with operations $+$, sup and ${}^{\sim\omega^{\mu}}$ for $\mu < \nu$. So here also the result is straightforward.

If α_0 is limit of cofinality less than κ : then $\Omega(\kappa^{\alpha_0}) = \mathbf{sup}_{\theta\in\Theta_{\alpha_0}}\Omega(\kappa^{\theta})$. But there exists $\theta_0 \in \Theta_{\alpha_0}$ such that $\theta_0 > \beta_0$. Therefore

$${}^{-}\Omega(\kappa^{\beta_0}),\ \Omega(\kappa^{\beta_0}) \leq_c \Omega(\kappa^{\theta_0}) \leq_c \mathbf{sup}_{\theta\in\Theta_{\alpha_0}}\Omega(\kappa^{\theta})$$

which gives the result.

39□

We come to the final lemma showing there is no gap in our construction.

Lemma 40 (weak separation lemma) *Let A be non self dual and $0 < \alpha < OT_{\mathcal{B}}$,*

$$^{\neg}A,\ A\ \leq_w \Omega(\alpha)^b \Longrightarrow \exists \beta < \alpha\ \ (^{\neg}A\ \leq_w \Omega(\beta)^b\ \ \vee\ \ A\ \leq_w \Omega(\beta)^b)$$

$\Box$ **lem. 40** - The proof goes by induction on α and for α fixed by induction on $d_w^\circ A$ - i.e. by induction on $(\alpha, d_w^\circ A)$ with lexicographic order.

If $\alpha = 1$ nothing needs to be proved since $\Omega(1) = \ominus$ and $^{\neg}A, A \leq_w \ominus^b$ is impossible.

Assume the C.N.F. of base κ of α is

$$\alpha = \kappa^{\alpha_k}.\nu_k + \ldots + \kappa^{\alpha_0}.\nu_0$$

If $k > 0$ or $\nu_0 > 1$ then the result is direct by use of proposition 14 and proposition 16 (b) in [1]. Because in that case by induction hypothesis the following holds:

$$d_c^\circ\Omega(\kappa^{\alpha_k}\bullet\nu_k+\ldots+\kappa^{\alpha_0}\bullet\nu_0) = d_c^\circ\Omega(\kappa^{\alpha_k}\bullet\nu_k)+\ldots+d_c^\circ\Omega(\kappa^{\alpha_0}\bullet\nu_0) = d_c^\circ\Omega(\kappa^{\alpha_k}).\nu_k+\ldots+d_c^\circ\Omega(\kappa^{\alpha_0}).\nu_0 = \alpha$$

So that $^{\neg}A,\ A\ \ \leq_w\ \Omega(\alpha)^b\ \Longrightarrow\ d_w^\circ A\ =\ \beta\ <\ \alpha$ and then the induction hypothesis leads to $^{\neg}A$ or $A\ \leq_w \Omega(\beta)^b$.

So **assume $k = 0$ and $\nu_0 = 1$** so that $\kappa^{\alpha_0} = \alpha$ (with $\alpha_0 > 0$). We consider different cases:

$\omega \leq cof(\alpha_0) < \kappa$

This case also derives directly from lemma 15 (b) (i) in [1] since $\Omega(\kappa^{\alpha_0}) = \sup_{\theta<\Theta_{\alpha_0}} \Omega(\kappa^\theta)$ so that by induction hypothesis

$$d_c^\circ\Omega(\kappa^{\alpha_0}) = \sup_{\theta<\Theta_{\alpha_0}} d_c^\circ\Omega(\kappa^\theta) = \sup_{\theta<\Theta_{\alpha_0}} \kappa^\theta.$$

So that the same argument as above works.

If α_0 successor or both $cof(\alpha_0) = \kappa$ and $\alpha_0 < \kappa^{\alpha_0}$

One has $\kappa^{\alpha_0} = {}^{\kappa}V^0(\delta)$ and $\Omega(\alpha) = \Omega(\delta)^{\sim}$.

(a) If A is **not initializable** then $A \equiv_w B^{\longrightarrow}\sum_{i\in I} C_i$ for some I of cardinality less than κ, each C_i non self dual, B initializable, $B <_w A$ and $C_i <_w A$ for all $i \in I$.
So by induction hypothesis there exists $\beta_B < \alpha$ and for each $i \in I$ some ordinal $\gamma_i < \alpha$ such that $B \leq_w \Omega(\beta_B)^b$ and $C_i \leq_w \Omega(\gamma_i)$ holds. Now set $\gamma = \sup_{i\in I} \gamma_i + \omega$, since $|I| < \kappa$ and $cof(\kappa^{\alpha_0}) = \kappa$ one has $\gamma < \kappa^{\alpha_0}$. But also $\gamma + \beta_B < \kappa$ holds because κ^{α_0} is closed under ordinal sum.
So clearly $B^{\longrightarrow}\sum_{i\in I} C_i \leq_w \Omega(\gamma)^b{+}\Omega(\beta_B)^b$ holds. And one has also:

$$d_w^\circ(\Omega(\gamma)^b{+}\Omega(\beta_B))^b = d_w^\circ(\Omega(\gamma)^b) + d_w^\circ(\Omega(\beta_B)^b) = \gamma + \beta_B$$

So set $\beta = \gamma + \beta_B + 1$, one still have $\beta < \alpha$ and also

$$B^{\longrightarrow}\sum_{i\in I} C_i \leq_w \Omega(\gamma)^b{+}\Omega(\beta_B)^b \leq_w \Omega(\beta)^b$$

Which proves $A \leq_w \Omega(\beta)^b$.

(b) If A is **initializable** then ${}^{-}A,\ A\ \leq_w \Omega(\delta)^{\sim}$. So by proposition 26 (a) in [1], one can derive
$$^{-}A^{\approx},\ A^{\approx} \leq_w \Omega(\delta)^{\sim b^{\approx}} \leq_w \Omega(\delta)^b.$$
And by proposition 25 (c) in [1] ${}^{-}A^{\approx}$ and $A^{\approx}$ are both non self dual. Elsewhere $\delta < \alpha$ holds so that by induction hypothesis there exists an ordinal $\gamma < \delta$ such that
$$^{-}A^{\approx} \text{ or } A^{\approx} \leq_w \Omega(\gamma)^b.$$
And finally by proposition 27 (a) in [1] one gets
$$^{-}A \text{ or } A \leq_w \Omega(\gamma)^{\sim b} = \Omega({}^{\kappa}V^0(\gamma))^b.$$
But since $\gamma < \delta$ holds and the function $x \longmapsto {}^{\kappa}V^0(x)$ is strictly increasing one has ${}^{\kappa}V^0(\gamma) < \alpha = {}^{\kappa}V^0(\delta)$ which gives the result.

If $\boldsymbol{\alpha_0 = \kappa^{\alpha_0}}$ and $\boldsymbol{cof(\alpha_0) = \kappa}$:

In that case $\Omega(\kappa^{\alpha_0}) = \Omega(\gamma)^{\sim\omega^{\nu}}$ with ν the least ordinal such that there exists γ satisfying $\alpha_0 = {}^{\kappa}V^{\nu}(\gamma)$ and $\gamma < \alpha_0$.

(a) Assume that A is **not initializable**. The same argument as above gives the result.

(b) Assume that A is **initializable** and for some $\mu < \omega^{\nu}$ one has ${}^{-}A^{\approx\mu}$ and $A^{\approx\mu} \leq_w A$. Get μ the least such ordinal. By lemma 29 one has $A^{\approx\mu}$ is non self dual. So there exists an ordinal $\delta < \alpha$ such that
$$^{-}A^{\approx\mu} \text{ or } A^{\approx\mu} \leq_w \Omega(\delta).$$
But since $\mu < \omega^{\nu}$ holds one has $\mu < \omega^{\theta}.n$ for some $\theta < \nu$ and some non zero integer n. So by lemma 31:
$$^{-}A \text{ or } A \leq_w (\Omega(\delta)^{\approx\mu})^b \leq_w (\Omega(\delta)^{\sim\omega^{\theta.n}})^b = ((\ (\Omega(\delta)^{\sim\omega^{\theta.}})^{\cdots})^{\sim\omega^{\theta}})^b$$
with operation ${}^{\sim\omega^{\theta.}}$ repeated n times. And we know that
$$(\Omega(\delta)^{\sim\omega^{\theta.n}})^b = \Omega(\underbrace{{}^{\kappa}V^{\theta} \circ \ldots \circ {}^{\kappa}V^{\theta}}_{n}(\delta))$$
But, by definition of ν, the ordinal α is a fixpoint of ${}^{\kappa}V^{\theta}$ therefore it is closed under ${}^{\kappa}V^{\theta}$; so that $\underbrace{{}^{\kappa}V^{\theta} \circ \ldots \circ {}^{\kappa}V^{\theta}}_{n}(\delta) < \alpha$ holds which gives the result.

(c) Assume now that for all $\mu < \omega^{\nu}$ the relation $A^{\approx\mu} \geq_w A$ holds. From lemma 29 we know that $A^{\approx\omega^{\nu}}$ is non self dual. And by lemma 28:
$$A^{\approx\omega^{\nu}},\ A^{\approx\omega^{\nu}}\ \leq_w ((\Omega(\gamma)^{\sim\omega^{\nu}})^b)^{\approx\omega^{\nu}} \leq_w \Omega(\gamma)^b.$$
Since $\gamma < \alpha$ holds, the induction hypothesis gives
$$^{-}A^{\approx\omega^{\nu}} \text{ or } A^{\approx\omega^{\nu}}\ \leq_w \Omega(\delta) \text{ for some } \delta < \gamma.$$

Remark that the case $\delta = 1$ leads to ${}^{\neg}A^{\sim\omega^\nu}$ or $A^{\sim\omega^\nu} \leq_w \ominus$ which means exactly that ${}^{\neg}A$ or A is the empty set and thus ${}^{\neg}A$ or $A \leq_w \Omega(1)$.

So let us consider that $\delta > 1$. Lemma 31 leads to:

$$ {}^{\neg}A \text{ or } A \leq_w \Omega(\delta)^{\sim\omega^\nu} = \Omega({}^{\kappa}V^\nu(\delta)) $$

finally, since $\delta < \gamma$ and ${}^{\kappa}V^\nu$ is a strictly increasing function one gets ${}^{\kappa}V^\nu(\gamma) < {}^{\kappa}V^\nu(\delta)$ thus the result.

40□

At last, we come to the proof of the main result stated in introduction and saying that for any non self dual Borel set A there exists a unique α such that A or ${}^{\neg}A \equiv_w \Omega(\alpha)^b$.

□ **thm. 38** - Since ${}^{\neg}A$ and A are Borel there exists some countable ordinal δ such that both ${}^{\neg}A$ and A are $\mathbf{\Sigma}^\circ_{\omega^\delta}$. So by lemma 34 ${}^{\neg}A,\ A \leq_w (\Omega(\ominus \bullet 2)^{\sim\omega^\delta})^b$. But $\Omega(\ominus \bullet 2)^{\sim\omega^\delta} \equiv_c \Omega({}^{\kappa}V^\delta(2))$. Now get the least ordinal $\alpha \leq {}^{\kappa}V^\delta(2)$ such that ${}^{\neg}A$ or $A \leq_w \Omega(\alpha)^b$. By lemma 40 and minimality of α one cannot have ${}^{\neg}A$ and $A \leq_w \Omega(\alpha)^b$; so that ${}^{\neg}A \not\leq_w \Omega(\alpha)^b$ or $A \not\leq_w \Omega(\alpha)^b$ holds. By symmetry assume that ${}^{\neg}A \not\leq_w \Omega(\alpha)^b$ holds, then by determinacy one obtains $\Omega(\alpha)^b \leq_w A$. So that both $A \leq_w \Omega(\alpha)^b$ and $\Omega(\alpha)^b \leq_w A$ hold, therefore $A \equiv_w \Omega(\alpha)^b$ is proved. Uniqueness of α is an immediate consequence of the fact $(\Omega(\alpha))_{0<\alpha<OT_{\mathcal{B}}}$ is well ordered by $<_c$ as stated in lemma 39.

38□

We may now prove theorems 1 and 2 as a corollary of theorem 38.

□ **thm. 2** - The order type of the Wadge hierarchy is $OT_{\mathcal{B}} = \sup_{\nu<\omega_1} {}^{\kappa}V^\nu(2)$ since any non self dual Borel set is Wadge equivalent to a some set $\Omega(\alpha)^b$ or ${}^{\neg}\Omega(\alpha)^b$ for some ordinal α verifying $0 < \alpha < OT_{\mathcal{B}}$. And on the other hand, any set $\Omega(\alpha)^b$ (for $0 < \alpha < OT_{\mathcal{B}}$) reduces to a set $\Omega(\beta)^b$ for some ordinal β that satisfies both $0 < \alpha < \beta < OT_{\mathcal{B}}$ and $\Omega(\beta)^b$ is $\mathbf{\Sigma}^\circ_\lambda$-complete for some countable ordinal λ; showing that every $\Omega(\alpha)^b$ is Borel.

2□

□ **thm. 1** - By theorem 38, for any Borel non self dual set $A \subseteq \Lambda_A^{\leq\omega}$, there exists an ordinal $\alpha < OT_{\mathcal{B}}$ such that $A \equiv_w \Omega(\alpha)^b$ or ${}^{\neg}A \equiv_w \Omega(\alpha)^b$. By symmetry assume that $A \equiv_w \Omega(\alpha)^b$ holds, fix a winning strategy σ for *II* in the Wadge game where *I* is in charge of A and *II* is in charge of $\Omega(\alpha)^b$ and set

$$ \forall u \in \Lambda_A^{<\omega} \quad \left(u \in S(A) \iff \sigma * (u) \in \Omega(\alpha)^b\right) $$

Now the following strategy σ' induced by σ:

$$ \begin{array}{rccl} \sigma' : & (\Lambda_A \cup \{b\})^{<\omega^*} & \longmapsto & \Lambda \cup \{s\} \\ & u^\frown < a > & \longmapsto & \sigma(u(\ /b)^\frown < a >) \text{ if } a \in \Lambda_A \\ & & & b \text{ otherwise} \end{array} $$

is winning for *II* in $\mathbf{W}\left((A \cup S(A))^b, \Omega(\alpha)^b\right)$.

On the other hand, any strategy τ that is winning for *II* in $\mathbf{W}\left(\Omega(\alpha)^b, A\right)$ is still winning for *II* in the game $\mathbf{W}\left(\Omega(\alpha)^b, (A \cup S(A))^b\right)$. Thus the result

$$(A \cup S(A))^b \equiv_w \Omega(\alpha)^b \equiv_w A.$$

1$\square$

We show that the conciliatory operation ${}^{\kappa}V_c^{\nu}$ forms the counterpart of the Veblen ordinal function ${}^{\kappa}V^{\nu}$ by referring to the definition of the "canonical" conciliatory sets $\Omega(\alpha)$.

Proposition 41 *Let $A \subseteq \Lambda_A^{\leq\omega}$, A^b Borel and $\nu < \omega_1$,*

$$d_c^{\circ}({}^{\kappa}V_c^{\nu}(A)) = {}^{\kappa}V^{\nu}(d_c^{\circ}(A))$$

$\square$ **prop.41** - Lemma 39 and theorem 38 prove that $d_c^{\circ}\Omega(\alpha) = \alpha$. Now, since A^b is some non self dual Borel set there exists a unique α such that $A^b \equiv_w \Omega(\alpha)^b$ or ${}^{-}A^b \equiv_w \Omega(\alpha)^b$. Which is the same as saying A or ${}^{-}A \equiv_c \Omega(\alpha)$. But since ${}^{\kappa}V_c^{\nu}$ preserves the conciliatory ordering:

$${}^{\kappa}V_c^{\nu}(A) \text{ or}^{-}({}^{\kappa}V_c^{\nu}(A)) \equiv_c {}^{\kappa}V_c^{\nu}(\Omega(\alpha))$$

One has ${}^{-}({}^{\kappa}V_c^{\nu}(A)) = {}^{\kappa}V_c^{\nu}({}^{-}A)$ and since by definition ${}^{\kappa}V_c^{\nu}(\Omega(\alpha)) = \Omega({}^{\kappa}V^{\nu}(\alpha))$ holds, one comes to:

$${}^{\kappa}V_c^{\nu}(A) \text{ or } {}^{\kappa}V_c^{\nu}({}^{-}A) \ \equiv_c \ \Omega({}^{\kappa}V^{\nu}(\alpha))$$

showing that

$$d_c^{\circ}({}^{\kappa}V_c^{\nu}(A)) = d_c^{\circ}(\Omega({}^{\kappa}V^{\nu}(\alpha))) = {}^{\kappa}V^{\nu}(\alpha) = {}^{\kappa}V^{\nu}(d_c^{\circ}A).$$

41 $\square$

We end up with a result about initializable sets that shows where they are located on the Wadge hierarchy; a generalization of The Hausdorff-Kuratowski theorem, and a computation of the Wadge degree of $\mathbf{\Sigma}_{1+\lambda}^{\circ}$ complete sets.

Proposition 42 *Let $A \subseteq \Lambda_A^{\omega}$, A Borel, the following are equivalent*

(a) A is initializable

(b) A n.s.d. and $d_w^{\circ}(A) = \kappa^{\theta}$ with θ not limit of cofinality less than κ

(c) there exists a conciliatory set B such that $A \equiv_w (B^{\sim})^b$

$\square$ **prop.42** - First we show (a)$\Rightarrow$(b). Remark that any self dual set is not initializable. Also, a set A is initializable iff its complement ${}^{-}A$ is initializable, and the property of being initializable is preserved under Wadge equivalence. Now since A is Borel and non self dual, there exists an ordinal α that verifies $0 < \alpha < OT_{\mathcal{B}}$ such that A or ${}^{-}A \equiv_w \Omega(\alpha)^b$. So it is enough to check what the conditions on α are. Assume α admits the following Cantor Normal Form of base κ:

$$\alpha = \kappa^{\alpha_k}.\nu_k + \ldots + \kappa^{\alpha_0}.\nu_0$$

- If $k > 0$ or $\nu_0 > 1$: clearly $\Omega(\alpha)$ is not initializable.

- If $k = 0$ and $\nu_0 = 1$: in that case $\Omega(\alpha) = \Omega(\kappa^{\alpha_0})$
  - if $\omega \leq cof(\alpha_0) < \kappa$ then $\Omega(\alpha) = \mathbf{sup}_{\theta \in \Theta_{\alpha_0}} \Omega(\kappa^\theta)$ is not initializable,
  - if $\alpha_0 = 0$ then $\Omega(\alpha) = \ominus$ which is initializable,
  - if α_0 is successor or limit of cofinality κ, then $\Omega(\kappa^{\alpha_0}) = {}^\kappa V_c^\nu(\Omega(\delta))$ where δ satisfies ${}^\kappa V^\nu(\delta) = \kappa^{\alpha_0}$ and $\delta < \kappa^{\alpha_0}$ with ν the least such ordinal. Clearly ${}^\kappa V_c^\nu(\Omega(\delta)) = \Omega(\delta)^{\sim\omega^\nu}$ is an initializable set.

We show that (b)$\Rightarrow$(c). Since A is non self dual $A \equiv_w \Omega(\kappa^\theta)$ or ${}^-\Omega(\kappa^\theta)$. If $\theta = 0$ then clearly $A \equiv_w \ominus^b \equiv_w (\ominus^\sim)^b$ or $A \equiv_w \oplus^b \equiv_w (\oplus^\sim)^b$. Assume that $\theta > 0$: due to the fact θ is not limit of cofinality less than κ, one has A or ${}^-A \equiv_w {}^\kappa V_c^\nu(\Omega(\delta))$ for some ν and δ. If $\nu = 0$ then we are done because by definition ${}^{\sim 1}$ is the same operation as ${}^\sim$. If $\nu > 0$ then the following holds which gives the result:

$$^\kappa V_c^\nu(\Omega(\delta)) = \Omega(\delta)^{\sim\omega^\nu} \equiv_w \left(\Omega(\delta)^{\sim\omega^\nu}\right)^\sim.$$

The remaining (c)$\Rightarrow$(a) is obvious.

42 □

Proposition 43 *Assume $\lambda < \omega_1$ admits the following Cantor Normal Form of base κ:*

$$\lambda = \omega^{\lambda_k}.n_k + \ldots + \omega^{\lambda_0}.n_0$$

(a) then the Wadge degree of any $\mathbf{\Sigma}^\circ_{1+\lambda}$-complete set (or $\mathbf{\Pi}^\circ_{1+\lambda}$-complete) is

$$\underbrace{^\kappa V^{\lambda_k} \circ \ldots \circ {}^\kappa V^{\lambda_k}}_{n_k} \circ \ldots\ldots \circ \underbrace{^\kappa V^{\lambda_0} \circ \ldots \circ {}^\kappa V^{\lambda_0}}_{n_0}(2)$$

(b) generalization of the Hausdorff-Kuratowski theorem on differences.

$$\mathbf{\Delta}^\circ_{1+\lambda+1} = \bigcup_{0<\nu<\kappa} D_\nu(\mathbf{\Sigma}^\circ_{1+\lambda})$$

□ **prop.43** -

(a) is an immediate consequence of proposition 34 since $(O^{\approx\lambda})$ is $\mathbf{\Sigma}^\circ_{1+\lambda}$-complete and by definition, considering $O = \ominus \bullet 2$:

$$(O^{\approx\lambda}) = \underbrace{^\kappa V_c^{\lambda_k} \circ \ldots \circ {}^\kappa V_c^{\lambda_k}}_{n_k} \circ \ldots\ldots \circ \underbrace{^\kappa V_c^{\lambda_0} \circ \ldots \circ {}^\kappa V_c^{\lambda_0}}_{n_0}(\Omega(2))$$

which degree is exactly

$$\underbrace{^\kappa V^{\lambda_k} \circ \ldots \circ {}^\kappa V^{\lambda_k}}_{n_k} \circ \ldots\ldots \circ \underbrace{^\kappa V^{\lambda_0} \circ \ldots \circ {}^\kappa V^{\lambda_0}}_{n_0}(2)$$

(b) Remark that for any non null ordinal $= \nu$ less than κ, the set $\ominus \bullet (1+\nu)$ is equivalent to the following set:

$$A_\nu = \{\alpha \in (\nu)^{\leq \omega^*} : lst(\hat{\alpha}) = \text{the least ordinal occuring in } \hat{\alpha}, \text{ has not the same parity as } \nu\}$$

where $\hat{\alpha}$ stands for the longest initial segment of α that is a decreasing (not necessarily strictly) sequence of ordinals.

Now for each non null ordinal $\mu < \nu$, set $B_\mu = \{\alpha \in (\nu)^{\leq \omega^*} : lst(\hat{\alpha}) < \mu\}$. The set B^b_μ is $\mathbf{\Sigma}^\circ_1$-complete and the sequence $(B^b_\mu)_{\mu<\nu}$ is strictly increasing for inclusion. Moreover, A^b_ν is the difference of the sets $(B^b_\mu)_{\mu<\nu}$. Therefore $A^b_\nu \in D_\nu(\mathbf{\Sigma}^\circ_1)$. And the fact it is $D_\nu(\mathbf{\Sigma}^\circ_1)$-complete is immediate playing the reduction game.

Showing that $(A_\nu{}^{\approx\lambda})^b$ belongs to $D_\nu(\mathbf{\Sigma}^\circ_{1+\lambda})$ relies on the $\mathbf{\Sigma}^\circ_{1+\lambda}$-completeness of each set $(B_\mu{}^{\approx\lambda})^b$, the fact $((B_\mu{}^{\approx\lambda})^b)_{\mu<\nu}$ is an increasing (for inclusion) sequence of sets, and the following remark:

$$\begin{aligned}
\alpha \in (A_\nu{}^{\approx\lambda})^b &\iff \alpha(\ /b)^\odot \in A_\nu \\
&\iff \exists\mu \ \Big(parity(\mu) \neq parity(\nu) \ \wedge \ \alpha(\ /b)^\odot \in \Big(B_\mu \smallsetminus \bigcup_{\theta<\mu} B_\theta\Big)\Big) \\
&\iff \exists\mu \ \Big(parity(\mu) \neq parity(\nu) \ \wedge \ \alpha \in \Big((B_\mu{}^{\approx\lambda})^b \smallsetminus \bigcup_{\theta<\mu} (B_\theta{}^{\approx\lambda})^b\Big)\Big)
\end{aligned}$$

At last, a set A belongs to $\mathbf{\Delta}^\circ_{1+\lambda+1}$ iff its Wadge degree is less than the one of any $\mathbf{\Sigma}^\circ_{\lambda+1}$ -complete set. But we now that this ordinal is the one of $(\ominus \bullet 2)^{\approx\lambda+1}$. That is

$$\underbrace{{}^\kappa V^{\lambda_k} \circ \ldots \circ {}^\kappa V^{\lambda_k}}_{n_k} \circ \ldots\ldots \circ \underbrace{{}^\kappa V^{\lambda_0} \circ \ldots \circ {}^\kappa V^{\lambda_0}}_{n_0+1}(2).$$

Therefore for some ordinal $0 < \nu < \kappa$, the Wadge degree of A is less than

$$\underbrace{{}^\kappa V^{\lambda_k} \circ \ldots \circ {}^\kappa V^{\lambda_k}}_{n_k} \circ \ldots\ldots \circ \underbrace{{}^\kappa V^{\lambda_0} \circ \ldots \circ {}^\kappa V^{\lambda_0}}_{n_0}(1+\nu)$$

which is the Wadge degree of the set

$$\underbrace{{}^\kappa V_c^{\lambda_k} \circ \ldots \circ {}^\kappa V_c^{\lambda_k}}_{n_k} \circ \ldots\ldots \circ \underbrace{{}^\kappa V_c^{\lambda_0} \circ \ldots \circ {}^\kappa V_c^{\lambda_0}}_{n_0}(\Omega(1+\nu)) = \Omega(1+\nu)^{\approx\lambda}$$

Since $\Omega(1+\nu) = \ominus \bullet (1+\nu) \equiv_w A_\nu$ this shows that A is in the class $D_\nu(\mathbf{\Sigma}^\circ_{1+\lambda})$.

43 □

Besides, one can show that the set

$$(\ominus \bullet \nu)^{\approx\lambda} = \underbrace{{}^\kappa V_c^{\lambda_k} \circ \ldots \circ {}^\kappa V_c^{\lambda_k}}_{n_k} \circ \ldots\ldots \circ \underbrace{{}^\kappa V_c^{\lambda_0} \circ \ldots \circ {}^\kappa V_c^{\lambda_0}}_{n_0}(\Omega(1+\nu))$$

is $D_\nu(\mathbf{\Sigma}^\circ_{1+\lambda})$-complete. Which immediately gives the Wadge degree of any $D_\nu(\mathbf{\Sigma}^\circ_{1+\lambda})$ -complete set by replacing the conciliatory Veblen operations by the ordinal ones:

$$\underbrace{{}^\kappa V^{\lambda_k} \circ \ldots \circ {}^\kappa V^{\lambda_k}}_{n_k} \circ \ldots\ldots \circ \underbrace{{}^\kappa V^{\lambda_0} \circ \ldots \circ {}^\kappa V^{\lambda_0}}_{n_0}(1+\nu)$$

To conclude, note that in case $\kappa = \omega_1$, i.e. in case we are interested in countable alphabets, the ordinal $\sup_{\nu<\omega_1} {}^{\omega}V^{\nu}(2)$ which is the order type of the Wadge hierarchy is also the ordinal ${}^{\kappa}V^{\omega_1}(2) = {}^{\omega}V^{\omega_1}(2)$: the first fixpoint of cofinality ω_1 of each function ${}^{\omega}V^{\nu}$. But this property fails as soon as κ is greater than ω_1; and in that case, there is a difference between ${}^{\kappa}V^{\omega_1}(2)$ and $\sup_{\nu<\omega_1} {}^{\omega}V^{\nu}(2)$. Moreover, with $\kappa{=}\omega_1$ both ordinals $\sup_{\nu<\kappa} {}^{\kappa}V^{\nu}(2)$ and $\sup_{\nu<\omega_1} {}^{\kappa}V^{\nu}(2)$ are the same. But as soon as κ is not ω_1 holds the greater κ is, the larger the gap between $\sup_{\nu<\omega_1} {}^{\kappa}V^{\nu}(2)$ and $\sup_{\nu<\kappa} {}^{\kappa}V^{\nu}(2) = {}^{\kappa}V^{\kappa}(2)$ gets. Thus a question arise. When considering spaces of the form Λ^{ω} with alphabet Λ of cardinality less than some regular cardinal $\kappa > \omega_1$, is there a hierarchy of sets, extending the Wadge hierarchy of Borel sets, of order type exactly $\sup_{\nu<\kappa} {}^{\kappa}V^{\nu}(2)$?

Moreover, operations ${}^{\kappa}V_c^{\nu}$ can be defined not only for $\nu < \omega_1$ but for $\nu < \kappa$, simply because $^{\circlearrowleft}$ can be defined exactly the same way we did but with a set of arrows of cardinality κ instead of ω_1. Therefore, we may define sets $\Omega(\alpha)$ for any non null ordinal α that is less than $\sup_{\nu<\kappa} {}^{\kappa}V^{\nu}(2)$ and show that the hierarchy $(\Omega(\alpha))_{0<\alpha<\sup_{\nu<\kappa} {}^{\kappa}V^{\nu}(2)}$ is well ordered by $<_c$. The previous question may be reformulated into another one related to this hierarchy: is there a class of subsets of spaces of the form Λ^{ω} with $|\Lambda| < \kappa$ for which Wadge determinacy holds and such that every non self dual set is Wadge equivalent (up to complementation) to one of these sets $\Omega(\alpha)^b$ for $0 < \alpha < \sup_{\nu<\kappa} {}^{\kappa}V^{\nu}(2)$?
The answer is positive. It simply corresponds to the Wadge hierarchy of **quasi-Borel** sets, namely the notion of $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$ sets extended to uncountable alphabets:

$$A \in \mathbf{\Sigma}^1_1 \restriction \Lambda^{\omega} \iff \exists B \in \mathbf{\Pi}^{\circ}_1 \restriction (\Lambda^{\omega} \times \omega^{\omega}) \quad (\alpha \in A \longleftrightarrow \exists \beta \quad (\alpha, \beta) \in B).$$

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