The Steel Hierarchy of Ordinal Valued Borel Mappings

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Abstract

Given well ordered countable sets of the form Λ_ϕ, we consider Borel mappings from ${\Lambda_\phi}^\omega$ with countable image inside the ordinals. The ordinals and Λ_ϕ^ω are respectively equipped with the discrete topology and the product of the discrete topology on Λ_ϕ. The Steel well-ordering on such mappings is defined by $\phi \leq \psi$ iff there exists a continuous function f such that $\phi(x) \leq \psi \circ \psi(x)$ holds for any $x \in \Lambda_\phi^\omega$. It induces a hierachy of mappings which we give a complete description of. We provide, for each ordinal α, a mapping $\mathbb{T}(\alpha)$ whose rank is precisely α in this hierarchy and we also compute the height of the hierarchy restricted to mappings with image bounded by α. These mappings being viewed as partitions of the reals, there is, in most cases, a unique distinguished element of the partition. We analyze the relation between its topological complexity and the rank of the mapping in this hierarchy.

1 Introduction

We deal with mappings from ω-sequences of elements from countable well-ordered alphabets, into the ordinals. Given the discrete topology on ordinals and the usual topology on sets of the form Λ^ω (the product topology of the discrete topology on Λ), we are only concerned with **Borel** mappings with countable image for reasons of using determinacy without large cardinal hypothesis. The Steel ordering between two such mappings $\phi :\ {\Lambda_\phi}^\omega \overset{\mathbf{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$ and $\psi :\ {\Lambda_\psi}^\omega \overset{\mathbf{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$, denoted $\phi \leq \psi$ is based on the existence of a continuous function f from ${\Lambda_\phi}^\omega$ to ${\Lambda_\psi}^\omega$ such that $\phi(x) \leq \psi \circ f(x)$ holds for any $x \in \Lambda_\phi^\omega$. We write $\phi \equiv \psi$ when both relations $\phi \leq \psi$ and $\psi \leq \phi$ hold; and $\phi < \psi$ when $\phi \leq \psi$ is satisfied but not $\psi \leq \phi$. Steel was the first to remark that this partial ordering becomes a well-ordering under determinacy.
The aim of this paper is to give a complete description of this well-ordering by analyzing the hierarchy of mappings that it induces.
This work takes place in the larger domain of the theory of BQO (*Better Quasi Ordering*) which is a strengthening of the notion of WQO. (*well quasi ordering*): the partial orders with no infinite antichain and no infinite descending chain. More precisely, by replacing in the definition of the Steel ordering the ordering on ordinals by any other BQO, one defines in the same way a partial ordering on the functions from the reals into this BQO. And Louveau and St-Raymond showed that this partial ordering turns out to be itself a BQO (see [7]). Hence the issue is then to give a description of the BQO of functions from the very description of the original BQO. From this point of view, the Steel hierarchy stands as the simple case of a BQO without antichain, and the Wadge

hierarchy as the most simple case of a BQO with an antichain: it is equality on $\{0,1\}$, with only two elements, none being compatible with the other.
We recall the proof of the well-orderingness of the Steel ordering.

Theorem 1 (Steel)

(a) *Given any* $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$, $\psi : \Lambda_\psi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$, *either* $\phi \leq \psi$ *or* $\psi \leq \phi$ *holds.*

(b) *There is no sequence* $(\phi_n)_{n\in\mathbb{N}}$ *that satisies* $\phi_i > \phi_{i+1}$ *for all integer* i.

The proof relies on a suitable two player infinite game:

Definition 2 *Given* $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$ *and* $\psi : \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$, *the game* $\mathbb{G}_{\leq}(\phi,\psi)$ *is a two player infinite game with full information where both players take turns, I plays letters in* Λ_ϕ *and II plays letters in* Λ_ψ. *Player I starts first and is not allowed to skip while II is allowed to skip provided he plays infinitely many times. After* ω *moves, I has produced* $x \in \Lambda_\phi^\omega$ *and II has played* $y \in \Lambda_\phi^\omega$. *We say Player II wins that run if* $\phi(x) \leq \psi(y)$; *I wins otherwise.*
We also define variants of this game:

- *The* **strict** *game* $\mathbb{G}_{<}(\phi,\psi)$ *is exactly the same as the game* $\mathbb{G}_{\leq}(\phi,\psi)$ *except that the winning conditions are strict: II wins if* $\phi(x) < \psi(y)$; *I with otherwise.*
- *And for some ordinal* ξ, $\mathbb{G}_{<,\ \xi}(\phi,\psi)$, *the* **strict game with restriction $\boldsymbol{\xi}$**, *is as* $\mathbb{G}_{<}(\phi,\psi)$ *except that II wins iff* $(\phi(x) < \phi(y) \ \vee \ \xi = \phi(x) = \psi(y))$.

Remarks 3 These games are designed so that player *II* has a winning strategy iff there exists a continuous function $f : \Lambda_\phi^\omega \longmapsto \Lambda_\psi{}^\omega$ such that $\forall x \in \Lambda_\phi^\omega \ \ \phi(x) \leq \psi \circ f(x)$ in the case of game $\mathbb{G}_{\leq}(\phi,\psi)$; and $\forall x \in \Lambda_\phi^\omega \ \ \phi(x) < \psi \circ f(x)$ in the case of game $\mathbb{G}_{<}(\phi,\psi)$.
In other words, *II* has a winning strategy in $\mathbb{G}_{\leq}(\phi,\psi)$ iff $\phi \leq \psi$. But the reader should notice that the fact *II* has a w.s. in the strict game $\mathbb{G}_{<}(\phi,\psi)$ does not mean that $\phi < \psi$.
If σ is a strategy for a player in one of the games described above, we write $\sigma\star$ for the function defined on $\Lambda_\phi{}^{\leq\omega}$ by $\sigma \star x = \langle \sigma(x \restriction i)/i < lh(x)\rangle$.

□ **thm. 1 -**

(a) Assume $\phi \not\leq \psi$. By determinacy, this means that player I has a w.s. in the game $\mathbb{G}_{\leq}(\phi,\psi)$; from which one easily derives a w.s. for *II* in $\mathbb{G}_{\leq}(\psi,\phi)$. Thus $\psi \leq \phi$ holds.

(b) Towards a contradiction, we assume that for each integer i there is a w.s. σ_i for I in the game $\mathbb{G}_{\leq}(\phi_i,\phi_{i+1})$

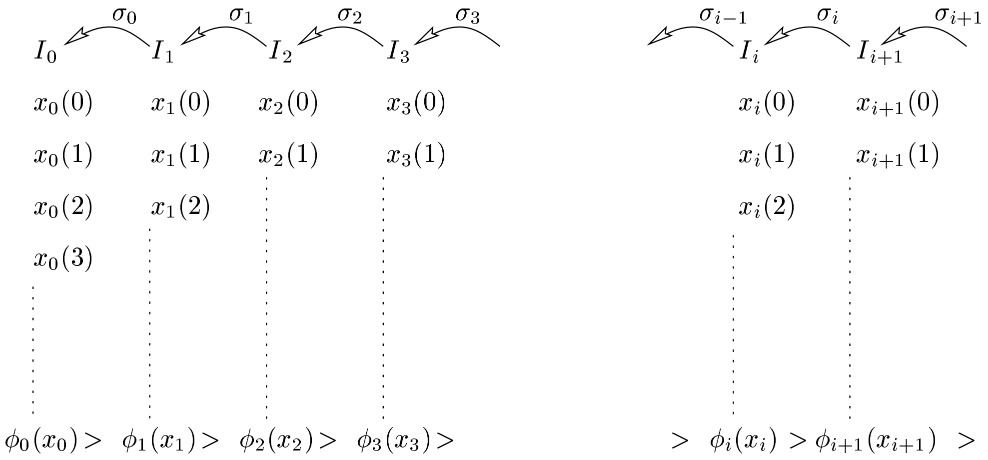
We play the following combination of games where I_i applies σ_i to I_{i+1}'s run - see figure 1. Let x_i be I_i's run. Since σ_i is winning for I, the sequence $\langle \phi_i(x_i)/i \in \omega\rangle$ is strictly decreasing; a contradiction.

1□

Activity in this domain was renewed by an intriguing result due to Chalons [1]:

Theorem 4 (Chalons) *Let* $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$ *be Borel, then*

Figure 1:



(a) *Either II has a winning strategy in the strict game* $\mathbb{G}_{<}(\phi,\phi)$*).*

(b) *Or (a) is false but there exists a unique ordinal* ξ *such that II has a winning strategy in* $\mathbb{G}_{<,\ \xi}(\phi,\phi)$*.*
This unique ordinal ξ *is called the* **target** *of* ϕ *and denoted* $\xi = tg\,(\phi)$*.*

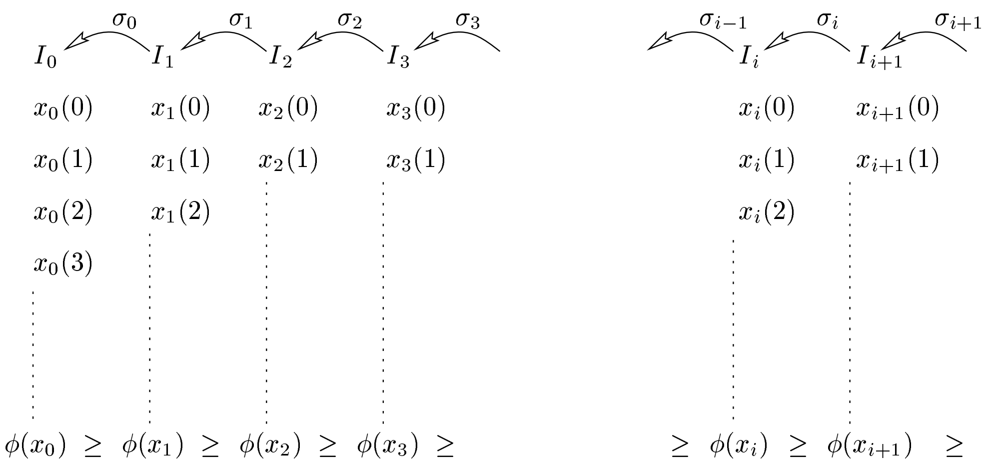
For instance the function $\phi : \ \omega^\omega \longmapsto \mathbb{N}$ defined by $\phi(x) = x(0)$ satisfies condition (a): it is target-free. On the contrary, any constant function satisfies condition (b).
So there are two radically different kinds of Borel functions: the ones that satisfy conditions (a) and don't admit a target, and the ones that verify condition (b) and have a target. Chalons also proved that when compact spaces are involved (that is Λ_ϕ is finite) every function admits a target.
$\square$ **thm. 4** - Towards a contradiction, assume (a) is false and there is no ordinal ξ such that *II* has a winning strategy in $\mathbb{G}_{<,\ \xi}(\phi,\phi)$.
Let $\langle \xi_i / i \in \omega \rangle$ be an enumeration of the countable image of ϕ with infinite repetition of each ordinal. By determinacy, let $\boldsymbol{\sigma}_i$ be a w.s. for I in the game $\mathbb{G}_{<,\ \xi_i}(\phi,\phi)$. Play the following combination of reduction games where I_i applies $\boldsymbol{\sigma}_i$ to I_{i+1}'s run - see figure 2. Let x_i be I_i's run. Since $\boldsymbol{\sigma}_i$ is winning for I, the sequence $\langle \phi(x_i)/i \in \omega\rangle$ is decreasing hence stationary. This means there exists some $n \in \mathbb{N}$ and $\xi_i \in \mathcal{I}m(\phi)$ such that $\forall p > n \ \ \phi(x_p) = \xi_i$. Take some $q > n$ such that $\xi_i = \xi_q$, then $\xi_q = \boldsymbol{\sigma}_q \star \xi_{q+1} = \boldsymbol{\sigma}_q \star \xi_q$ a contradiction.

For unicity, we assume there exists $\xi \neq \zeta$ such that player *II* has winning strategies in both games $\mathbb{G}_{<,\ \xi}(\phi,\phi)$ and $\mathbb{G}_{<,\ \zeta}(\phi,\phi)$. By composing strategies, we obtain a w.s. for *II* in $\mathbb{G}_{<}(\phi,\phi)$; a contradiction.

4$\square$

Remarks 5 Let $\phi :\ {\Lambda_\phi}^\omega \overset{\boldsymbol{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$, $\psi :\ {\Lambda_\psi}^\omega \overset{\boldsymbol{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$,

Figure 2:



(a) if $\phi \equiv \psi$ then (ϕ has a target $\iff$ ψ has a target). Also when both ϕ and ψ have a target it is the same ordinal.

(b) Assume $tg\,(\phi) = \xi$; then there exists a strategy σ that is winning for *II* in the game $\mathbb{G}_{<,\ \xi}\,(\phi, \phi)$ such that $\forall x \in \Lambda^\omega_\phi\ \ \phi(\sigma \star x) \geq \xi$.

(c) Assume ψ has a target, then

$$\textit{II} \text{ has a winning strategy in } \mathbb{G}_<\,(\psi, \phi) \iff \psi < \phi.$$

□ **rem.5** -

(a) This holds because, assuming $\phi \equiv \psi$, if player *II* has a w.s. in $\mathbb{G}_<\,(\phi, \phi)$ then *II* also has a w.s. in both $\mathbb{G}_<\,(\psi, \phi)$ (since $\psi \leq \phi$ holds) and $\mathbb{G}_<\,(\psi, \psi)$ (since $\phi \leq \psi$ holds). On the contrary, if *I* has a w.s. in $\mathbb{G}_<\,(\phi, \phi)$ then player *I* has also a w.s. in $\mathbb{G}_<\,(\psi, \psi)$ (otherwise *II* would have a w.s. in that game, hence in $\mathbb{G}_<\,(\phi, \phi)$, a contradiction).
And in case $tg\,(\phi) = \xi$, this means *II* has a w.s. in $\mathbb{G}_{<,\ \xi}\,(\phi, \phi)$ hence *II* also has a w.s. in $\mathbb{G}_{<,\ \xi}\,(\psi, \phi)$ (because $\psi \leq \phi$ holds) and also in $\mathbb{G}_{<,\ \xi}\,(\psi, \psi)$ (because $\phi \leq \psi$ holds).

(b) Let ζ be the least ordinal of the image of ϕ. If $\zeta = tg\,(\phi) = \xi$ the result is obvious. So we assume $\zeta < \xi$. We define the mapping $\phi' :\ \Lambda^\omega_\phi \longmapsto \boldsymbol{On}$ by

$$\begin{array}{rcll} \phi'(x) & = & \zeta & \text{if } \phi(x) < \xi \\ & = & \phi(x) & \text{otherwise} \end{array}$$

Clearly $\phi' \leq \phi$ holds, ϕ' has a target and $tg\,(\phi') = tg\,(\phi) = \xi$. Moreover $\phi \leq \phi'$ holds because otherwise *I* would have a winning strategy τ in $\mathbb{G}_\leq\,(\phi, \phi')$. That same strategy would be winning for *II* in the game $\mathbb{G}_{<,\ \zeta}(\phi', \phi')$, contradicting the fact $tg(\phi') = \xi$. So we have $\phi \equiv \phi'$. Get any strategy σ that is winning for *II* in $\mathbb{G}_{<,\ \xi}(\phi', \phi')$. This strategy σ is also winning for *II* in $\mathbb{G}_{<,\ \xi}(\phi, \phi)$ and satisfies $\forall x \in \Lambda_\phi^\omega \quad \phi(\sigma \star x) \geq \xi$.

(c) **($\Longrightarrow$)** Assuming $\phi \leq \psi$ would give a w.s. for *II* in $\mathbb{G}_<(\psi, \psi)$, contradicting the fact ψ has a target.

($\Longleftarrow$) *I* has a w.s. in $\mathbb{G}_\leq(\phi, \psi)$ which is also a w.s. for *II* in $\mathbb{G}_<(\psi, \phi)$.

5 □

Since $<$ well orders Borel mappings, it induces a notion of rank that we call degree. There will be three different notions of degrees. Given any $\xi < \boldsymbol{On}$ we write C_ξ for the constant function ξ. It is obvious that $C_0 < \phi$ holds for any mapping ϕ, except C_0 itself.

Definition 6 *We define the degree of a Borel mapping ϕ by*

- $d^\circ_m(\phi) = 0 \iff \phi = C_0$,
- $d^\circ_m(\phi) = \sup\{d^\circ_m(\psi) + 1 :\ \psi < \phi\}$ *(for any $\phi > C_0$).*

Clearly, if $\xi < \zeta \in \boldsymbol{On}$ then $C_\xi < C_\zeta$ holds since any strategy is winning for I in the game $\mathbb{G}_<(C_\zeta, C_\xi)$. Therefore there are as many degrees as ordinals. Nevertheless we may look at the height of sub-hierarchies of mappings, for instance the one of integer valued mappings. Therefore, for any non zero ordinal η, we write $OT_{\mathcal{B}_\eta}$ for the order type of the Steel Hierarchy of Borel mappings with countable image **inside** η. In other words, $OT_{\mathcal{B}_\eta} = \sup\{d^\circ_m(\phi) :\ \phi \text{ Borel } \wedge\ |\mathcal{I}m(\phi)| \leq \aleph_0\ \wedge\ \mathcal{I}m(\phi) \subseteq \eta\}$. But obviously, the constant function C_η is, up to $\equiv$, the $<$-least mapping above all the ones with image inside the ordinal η. So computing the height of the hierarchy restricted to mappings with image in η is exactly the same as computing $d^\circ_m(C_\eta)$.
As a consequence of the whole study, we will be able at the end of this paper to prove:

Theorem 7 *Let $0 < \eta = \omega \cdot \delta + n$ with $n \in \mathbb{N}$. Then*

$$\begin{array}{rcll} d^\circ_m(C_\eta) = OT_{\mathcal{B}_\eta} & = & V_{\omega_1}(1+\delta) + n & \textit{if } \delta > 0 \\ & = & n & \textit{otherwise,} \end{array}$$

where the function V_{ω_1} is the $\omega_1{}^{th}$ Veblen function of base ω_1 given in the following definition.

Definition 8 *The Veblen hierarchy of base ω_1 consists of the sequence of functions $(V_\xi)_{\xi \in \boldsymbol{On}}$, where $V_\xi : \boldsymbol{On} \smallsetminus \{0\} \longmapsto \boldsymbol{On}$ defined by:*
V_ξ is the the function that enumerates the ordinals that are closed under ordinal sum, supremum of cofinality ω and all functions V_ζ for $\zeta < \xi$.

In other words,

- V_0 is the function that enumerates the ordinals of cofinality $\neq \omega$ that are closed under addition. It is almost the exponentiation of base ω_1: $\alpha \longmapsto \omega_1{}^{(-1+\alpha)}$; except that there is a shift to avoid ω_1^α when α is limit of cofinality strictly less than ω_1:
  - ◇ $V_0(1) = 1$

◇ $V_0(\alpha+1) = V_0(\alpha)\cdot\omega_1$

◇ if $cof(\alpha) > \omega$ then $V_0(\alpha) = \omega_1^{\alpha}$

◇ if $cof(\alpha) = \omega$ then $V_0(\alpha) = \omega_1^{\alpha+1}$.

∘ For $\mu > 0$, V_μ is the function that enumerates ordinals with cofinality $\neq \omega$ that are fixpoints of all Veblen functions V_ξ for $\xi < \mu$. That is, $V_\mu(\nu)$ (for $0 < \nu$) is the ν^{th} such one. (The only one with finite cofinality is $V_\mu(1) = 1$, all other such fixpoints have infinite cofinalities).

2 Functions with no target

In this section we characterize mappings that don't admit a target by linking them to functions with target. This will allow us to get rid of them and later on work in a world made only of targeted mappings. Given $\phi:\ {\Lambda_\phi}^\omega \xmapsto{\mathbf{\Delta}^1_1} \boldsymbol{On}$ and $u \in {\Lambda_\phi}^{<\omega}$ we write $\phi[u]$ for the function $\psi:\ {\Lambda_\phi}^\omega \xmapsto{\mathbf{\Delta}^1_1} \boldsymbol{On}$ defined by $\psi(x) = \phi(u^\frown x)$.

Definition 9 *Let* $\phi:\ {\Lambda_\phi}^\omega \xmapsto{\mathbf{\Delta}^1_1} \boldsymbol{On}$*,*
The tree $\mathcal{I}nit_\phi$ *is defined by*

$$\mathcal{I}nit_\phi = \{u \in {\Lambda_\phi}^{<\omega}:\ \phi[u] \equiv \phi\}.$$

We may think of $\mathcal{I}nit_\phi$ as the set of all positions which are equivalent to the original one for a player in charge of ϕ in a reduction game. It is closed under subsequences, hence a tree. It is also non empty for it contains the empty sequence. The wellfoundedness of this tree is a criterion for the existence of a target.

Proposition 10 *Let* $\phi:\ {\Lambda_\phi}^\omega \xmapsto{\mathbf{\Delta}^1_1} \boldsymbol{On}$*,*

$$\phi \text{ has a target} \quad\Longleftrightarrow\quad \mathcal{I}nit_\phi \text{ is illfounded.}$$

☐ **prop.10** -

($\Longrightarrow$) Let $x \in \Lambda_\phi^\omega$ be I's run in $\mathbb{G}_<(\phi,\phi)$ when II always skips and I applies a winning strategy. For each integer n, $\phi \leq \phi[x \restriction n]$ holds which proves $\mathcal{I}nit_\phi$ is illfounded.

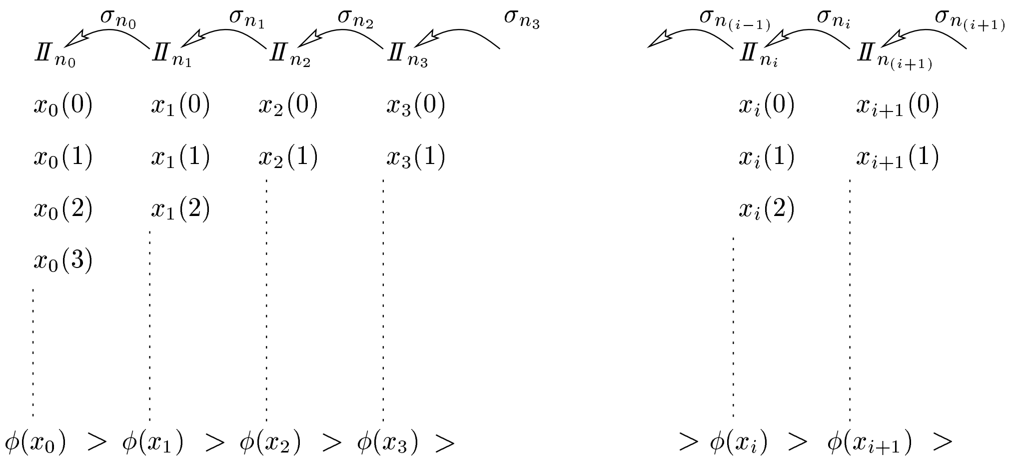
($\Longleftarrow$) Towards a contradiction, assume that II does have a w.s. in $\mathbb{G}_<(\phi,\phi)$. Get $x \in [\mathcal{I}nit_\phi]$, and for each integer k, a strategy σ_k that is winning for II in the II-imposed game $\mathbb{G}_<(\phi,\phi)$ where II must play $x \restriction k$ as first k^{th}moves - see figure 3.

Play the following combination of reduction games: II_i applies σ_{n_i} to II_{i+1}'s run where n_i is the least integer strictly greater than $n_{(i-1)}$ such that if player II_i has played $x \restriction n_i$ then player II_0 has already played a sequence of length at least i. In other words $\sigma_{n_0}\star\circ\sigma_{n_1}\star\circ\ldots\circ\sigma_{n_i}\star(x \restriction n_i)$ has length $\geq i$.
Let x_i be II_i's run in that general game. For each integer i, $\phi(x_i) > \phi(x_{(i+1)})$ holds; a contradiction.

10 ☐

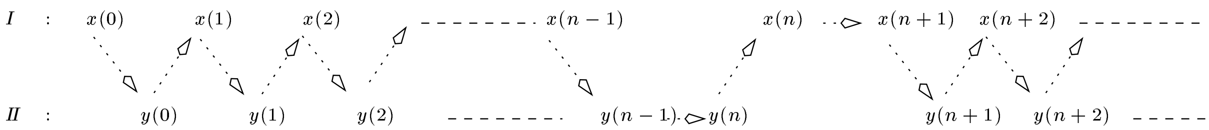
The following 1−Lipschitz reduction game was designed by Chalons in the search for some uniqueness property for the target-free functions (he already knew of the uniqueness of the target).

Figure 3:



Definition 11 (Chalons) *Given* $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$, $\psi : \Lambda_\psi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$ *and an integer* n*, the game with* **breakstep** n*,* $\mathbb{B}_n(\phi,\psi)$*, is the following two player infinite game where I plays letters in* Λ_ϕ *and II plays letters in* Λ_ψ*; none of them being allowed to skip. Player I starts and both players take turn all along the run except at move* n *where player one is allowed to skip only one time. So the run is as in figure 4.*

Figure 4:



Player II wins this game if $\phi(x) \leq \phi(y)$*; player I wins otherwise.*

Theorem 12 (Chalons) *Let* $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$*. If* ϕ *does not have a target (i.e. II does not have a w.s. in* $\mathbb{G}_<(\phi,\phi)$*) then there exists a unique integer* n *such that I has a winning strategy in the game* $\mathbb{B}_n(\phi,\phi)$*.*

We relate this notion of breakstep to the wellfoundedness of $\mathcal{I}nit_\phi$.

Proposition 13 *Assume* $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$ *and* $n \in \mathbb{N}$*, the following are equivalent:*

(a) I has a w.s. in $\mathbb{B}_n(\phi, \phi)$

(b) $\mathcal{I}nit_\phi$ *is wellfounded and its height is* $\omega \cdot \nu + n$ *(for some* $\nu < \omega_1$*).*

□ **prop.13** -

(b)$\Longrightarrow$(a) We assume $\omega \cdot \nu + n$ is the height of $\mathcal{I}nit_\phi$. We show that I has a w.s. in $\mathbb{B}_n(\phi, \phi)$.

- If $\nu = 0$ then any position u of length $n+1$ satisfies $\phi[u] < \phi$. Hence, given any $v \in \mathcal{I}nit_\phi$ of length n, playing the first moves according to v will guarantee I to be in a winning position after player $I\!I$ replies to his skip.
- If $\nu > 0$, then a winning strategy for I consists in playing as first n moves a sequence u that satisfies the height of the subtree $T_u = \{v \in {\Lambda_\phi}^{<\omega} : u^\frown v \in T\}$ is $\omega \cdot \nu$, and then in skipping. At that point player $I\!I$ will be in a position v. Either that position v is not in T in which case I obviously wins. Or that position v is still in T, but in that case, the height of the subtree T_v is strictly below $\omega \cdot \nu$. Player I, then, keeps on extending the position u into other positions u' that remain in T as long as $I\!I$'s corresponding positions v' also remain in T with furthermore I ensuring that the height of the subtree $T_{u'}$ remains strictly above the one of the corresponding $T_{v'}$. This way, $I\!I$ will be the first player to "leave" the tree T, hence will lose the game.

(a)$\Longrightarrow$(b) First we show that $\mathcal{I}nit_\phi$ is wellfounded. Towards a contradiction we assume that it is illfounded. By proposition 10, ϕ has a target which means I has a w.s. in $\mathbb{G}_<(\phi, \phi)$. Since $\mathcal{I}nit_\phi$ is illfounded, get any $u \in \mathcal{I}nit_\phi$ of length greater than n, player I still has a w.s. σ in $\mathbb{G}_<(\phi[u], \phi)$. So clearly $I\!I$ has a w.s. in $\mathbb{B}_n(\phi, \phi)$ that consists in playing u and then applying σ to I's run, a contradiction.
So assume $\omega \cdot \nu + m$ is the height of $\mathcal{I}nit_\phi$. By the previous case we know that I has a w.s. in $\mathbb{B}_m(\phi, \phi)$. Since the breakstep of ϕ (=the integer n such that I has a w.s. in $\mathbb{B}_n(\phi, \phi)$) is unique, this leads to $n = m$, that is the height of $\mathcal{I}nit_\phi$ is $\omega \cdot \nu + n$.

13 □

We introduce a very simple branching of mappings and use that operation to characterize target-free functions on the basis of the targeted ones.

Definition 14 *Assume* $\mathcal{U} \subseteq \Lambda^{<\omega}$ *is such that* $\{u^\frown \Lambda^{<\omega} : u \in \mathcal{U}\}$ *is a partition of* $\Lambda^{<\omega}$, $\phi_u : {\Lambda_\phi}^\omega \xmapsto{\Delta^1_1} \boldsymbol{On}$ *(all* $u \in \mathcal{U}$*), define* $\sum_{u \in \mathcal{U}} \phi_u : \Lambda^\omega \longmapsto \boldsymbol{On}$ *by*

$$\left(\sum_{u \in \mathcal{U}} \phi_u\right)(x) = \phi_u(y) \quad \text{for the unique } u \text{ and } y \text{ such that } u \in \mathcal{U} \text{ and } u^\frown y = x.$$

Proposition 15 *Let* $\phi : {\Lambda_\phi}^\omega \xmapsto{\Delta^1_1} \boldsymbol{On}$ *be a Borel mapping with no target, then there exists*

- $\mathcal{U} \subseteq \Lambda^{<\omega}$ *such that* $\{u^\frown \Lambda^{<\omega} : u \in \mathcal{U}\}$ *is a partition of* $\Lambda^{<\omega}$,
- *and, for each* $u \in \mathcal{U}$*, a mapping* **with target** $\phi_u : {\Lambda_\phi}^\omega \xmapsto{\Delta^1_1} \boldsymbol{On}$ *that verifies* $\phi_u < \phi$

such that:

$$\phi = \sum_{u \in \mathcal{U}} \phi_u$$

(Necessarily $\{\phi_u : u \in \mathcal{U}\}$ does not contain its $\leq$ -upper bound).

$\square$ **prop.15** - The proof goes by induction on the degree of ϕ.
By proposition 13 $\mathcal{I}nit_\phi$ is wellfounded. We set $I = \{i \in {\Lambda_\phi}^{<\omega} : i \notin \mathcal{I}nit_\phi \wedge i \restriction (lh(i)-1) \in \mathcal{I}nit_\phi\}$ and $\phi_i : {\Lambda_\phi}^\omega \xmapsto{\Delta^1_1} \boldsymbol{On}$ by $\phi_i(x) = \phi(i^\frown x)$ (for all $i \in I$). Two different cases appear:

- If ϕ_i has a target, set $\mathcal{U}_i = \{\langle\rangle\}$ and $\phi'_i = \phi_i$
- If ϕ_i has no target, by induction hypothesis there exists $\mathcal{U}_i \subseteq {\Lambda_\phi}^{<\omega}$ and for all $u \in \mathcal{U}_i$ there exists $\phi_u : {\Lambda_\phi}^\omega \xmapsto{\Delta^1_1} \boldsymbol{On}$ that has a target and satisfies $\phi_u < \phi_i$ such that

  $$\phi_i = \sum_{u \in \mathcal{U}_i} \phi_u$$

  we set $\phi'_{i^\frown u} = \phi_u$ for each $u \in \mathcal{U}_i$.

By setting $\mathcal{U} = \{i^\frown u : i \in I \wedge u \in \mathcal{U}_i\}$, we obtain $\phi = \sum_{v \in \mathcal{U}} \phi'_v$. 15 $\square$
This characterization of target-free mappings directly leads to the ordinal property of their degrees:

Proposition 16 *Let $\phi : {\Lambda_\phi}^\omega \xmapsto{\Delta^1_1} \boldsymbol{On}$ be Borel,*

$$\phi \textit{ has no target} \iff d^\circ_m(\phi) \textit{ is limit of countable cofinality.}$$

$\square$ **prop.16** -

($\Longrightarrow$) If ϕ has no target then by the previous proposition $\phi = \sum_{u \in \mathcal{U}} \phi_u$ with no $\leq$ -maximal element in $\{\phi_u : u \in \mathcal{U}\}$. Hence $d^\circ_m(\phi) \geq \sup\{d^\circ_m(\phi_u) : u \in \mathcal{U}\}$ which is a limit ordinal of countable cofinality.

In order to get $d^\circ_m(\phi) \leq \sup\{d^\circ_m(\phi_u) : u \in \mathcal{U}\}$ we make the following

Claim 17

$$\psi < \sum_{u \in \mathcal{U}} \phi_u \implies \exists u \;\; \psi \leq \phi_u.$$

$\square$ **cl. 17** - Since player *II* does not have a w.s. in the game $\mathbb{G}_\leq\left(\sum_{u \in \mathcal{U}} \phi_u, \psi\right)$, by determinacy player I has a w.s. in that game. In particular, this strategy works when *II* skips as long as I does not "choose" any particular ϕ_u (i.e. has not reached any position $u \in \mathcal{U}$). Since I can't skip, this leads to I reaching a position $u \in \mathcal{U}$ while *II* hasn't played anything at that time, showing I has a winning strategy in $\mathbb{G}_\leq(\phi_u, \psi)$. In other words $\psi < \phi_u$. 17$\square$

As a consequence,

$$d^\circ_m(\phi) = \sup\{d^\circ_m(\psi)+1 : \psi < \phi\} \leq \sup\{d^\circ_m(\psi)+1 : \exists u \;\; \psi \leq \phi_u\} \leq \sup\{d^\circ_m(\phi_u) : u \in \mathcal{U}\}$$

($\Longleftarrow$) Assume $d^\circ_m(\phi) = \sup_{n<\omega} \alpha_n$. Get $\phi_n : \Lambda_{\phi_n}{}^\omega \overset{\mathbf{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$ such that $d^\circ_m(\phi_n) = \alpha_n$. Set $\Lambda_\psi = \left(\bigcup_{n<\omega} \Lambda_{\phi_n}\right) \cup \mathbb{N}$, define $\psi : \Lambda_\psi{}^\omega \overset{\mathbf{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$ by

$$\begin{aligned} \psi(\langle i\rangle^\frown x) &= \phi_i(x) \quad \text{if } i \in \mathbb{N} \ \wedge\ x \in \Lambda^\omega_{\phi_i} \\ &= 0 \qquad\quad \text{otherwise} \end{aligned}$$

Clearly $d^\circ_m(\psi) = \sup_{n\in\mathbb{N}} \alpha_n$; i.e. $\psi \equiv \phi$. But *II* has an obvious w.s. in the strict game $\mathbb{G}_<(\psi, \psi)$. Hence neither of ψ and ϕ have a target.

16 □

3 Conciliatory mappings

We want to forget about the target-free mappings. We just saw that they were easily defined by mean of mappings with targets and that their localisation in the whole hierarchy corresponds exactly to the limit ordinal of countable cofinality.
As a preliminary we first define another notion of degree that enumerates the level of the Steel hierarchy skipping the ones allocated to target-free mappings.

Definition 18 *We define the degree of Borel mappings restricted to functions with targets by:*

- $d^\circ_t(\phi) = 0 \iff \phi = C_0$,
- $d^\circ_t(\phi) = \sup\{d^\circ_t(\psi) + 1 : \ \psi < \phi$ *and* ψ *admits a target*$\}$ *(for any* $\phi > C_0$ *that admits a target).*

There is a simple relation between d°_t and d°_m: if ϕ admits a target, then $d^\circ_m(\phi) = d^\circ_t(\phi) + \varepsilon$ with $\varepsilon = 1$ if the cofinality of the least limit ordinal less or equal than $d^\circ_t(\phi)$ is ω; and $\varepsilon = 0$ otherwise.
By ϕ bounded we mean that there exists some $x \in \Lambda^\omega_\phi$ such that $\phi(y) \le \phi(x)$ holds for any $y \in \Lambda^\omega_\phi$.

Definition 19 *We call conciliatory any mapping from* $\Lambda^{\le\omega}$ *into ordinals.*
Given two such conciliatory mappings $\phi : \ \Lambda_\phi{}^{\le\omega} \longmapsto \boldsymbol{On}$ *and* $\psi : \ \Lambda_\psi{}^{\le\omega} \longmapsto \boldsymbol{On}$ *we define the conciliatory game* $\mathbb{G}^c_\le(\psi, \phi)$ *as exactly the same as the game* $\mathbb{G}_\le(\ ,\ )$ *except that* **both** *players are allowed to skip, even indefinitely if they want to. Thus after* ω *moves, I and II will have respectively produced* $x \in \Lambda_\psi{}^{\le\omega}$ *and* $y \in \Lambda_\phi{}^{\le\omega}$. *We say II wins this game iff* $\psi(x) \le \phi(y)$. *We write* $\phi \le \psi$ *when II has a winning strategy in that game.*
We also define the game $\mathbb{G}^c_<(\psi, \phi)$ *as the game* $\mathbb{G}^c_\le(\psi, \phi)$ *except for the winning conditions which become II wins iff* $\psi(x) < \phi(y)$.

Remark 20 Given $\phi : \ \Lambda_\phi{}^{\le\omega} \longmapsto \boldsymbol{On}$ we go back to mappings from $\Lambda^\omega_\phi \longmapsto \boldsymbol{On}$ in the following way

$$\phi^* : \ (\Lambda_\phi \cup \{*\})^\omega \longmapsto \boldsymbol{On} \text{ defined by } \forall x \in (\Lambda_\phi \cup \{*\})^\omega \quad \phi^*(x) = \phi(x(\ /*))$$

where $*$ is a letter not in Λ_ϕ and $x(\ /*)$ stands for what remains from x after all occurences of the letter $*$ has been erased.
By abuse of language we say that the mapping $\phi : \ \Lambda_\phi{}^{\le\omega} \longmapsto \boldsymbol{On}$ is Borel when the mapping $\phi^* : \ (\Lambda_\phi \cup \{*\})^\omega \longmapsto \boldsymbol{On}$ is Borel. We also write it $\phi : \ \Lambda_\phi{}^{\le\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$.

Now, for the purpose of considering only Borel mappings which admit a target, it is enough to replace mappings from spaces of the form Λ^ω into $\boldsymbol{On}$ by mappings from $\Lambda^{\leq\omega}$ into $\boldsymbol{On}$. This result is based on three facts:

(a) Any Borel mapping $\phi: \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$ admits a target - precisely ϕ^* has a target since $\underbrace{\langle *, *, *, \ldots \rangle}_{\omega}$ belongs to $[\mathcal{I}nit_{\phi^*}]$.

(b) Given ϕ and ψ two conciliatory mappings, a player wins $\mathbb{G}^c_\leq(\phi,\psi)$ iff the same player wins $\mathbb{G}_\leq(\phi^*,\psi^*)$. Hence

$$\phi \leq \psi \iff \phi^* \leq \psi^*$$

There is a trivial isomorphism between strategies σ related to $\mathbb{G}^c_\leq(\phi,\psi)$ and strategies σ^* related to $\mathbb{G}_\leq(\phi^*,\psi^*)$: σ skips iff σ^* chooses the letter $*$.

From now on we omit the superscript ^{c} we simply write $\mathbb{G}_\leq(,)$ instead of $\mathbb{G}^c_\leq(,)$. Knowing which kind of game we will be dealing with will always be clear when looking at the functions involved.

(c) **Proposition 21** *Let $\phi: \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$ be a Borel mapping that admits a target, and the function $\phi_0: \Lambda^\omega_\phi \longmapsto \boldsymbol{On}$ be defined by $\phi_0(x) = \phi(x)$ for any $x \in \Lambda^\omega_\phi$ and $\phi_0(u) = 0$ for any $u \in \Lambda_\phi{}^{<\omega}$, then*

$$\phi \equiv {\phi_0}^*.$$

☐ **prop.21** - The fact $\phi \leq {\phi_0}^*$ is obvious. It is enough to show that ${\phi_0}^* \leq \phi$. Towards a contradiction, assume that ${\phi_0}^* \not\leq \phi$ holds. Then, by determinacy, I has a w.s. in the game $\mathbb{G}_\leq({\phi_0}^*,\phi)$. But to be winning this strategy needs also to ensure that I never plays some $x \in (\Lambda_\phi \cup \{*\})^\omega$ such that ${\phi_0}^*(x) = 0$. Hence I always plays infinitely many letters different from $*$. Therefore by simply turning the $*$ letter into a skip, this winning strategy for I in $\mathbb{G}_\leq({\phi_0}^*,\phi)$ becomes a winning strategy for $I\!I$ in $\mathbb{G}_\leq(\phi,\phi)$ where for any x played by I the answer y played by $I\!I$ satisfies $\phi(x) < \phi(y)$. This proves that ϕ does not have a target which contradicts the hypothesis. 21 ☐

This shows that in order to study Borel mappings with targets we can simply concentrate on conciliatory ones.

4 The successor levels of the conciliatory Steel hierarchy

This section is dedicated to the study of the operation $\mathcal{S}_{hift}$ defined below. That stands as the conciliatory counterpart of the ordinal successor.

Definition 22 *Given any $\phi: \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$ and $n \in \mathbb{Z}$, we define $\mathcal{S}_{hift}{}^n$ by,*
$\mathcal{S}_{hift}{}^n \circ \phi(x) = \phi(x) \dot{+} n$, where $\forall p \in \mathbb{N}$ $(\omega \cdot \alpha + p) \dot{+} n$ is defined by

$$\begin{aligned}(\omega\cdot\alpha+p)\dot{+}n &= \omega\cdot\alpha + (p+n) && \text{if } p+n \in \mathbb{N}\\ &= \omega\cdot\alpha && \text{otherwise.}\end{aligned}$$

We write $\mathcal{S}_{hift} \circ \phi$ instead of $\mathcal{S}_{hift}{}^1 \circ \phi$. We state a few properties of this operation that will be usefull when it comes to describe the successor levels of the hierarchy.

Proposition 23 *Let $\phi : \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$ and $\psi : \Lambda_\psi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$ be both Borel,*

(a) $\phi < \psi \Longrightarrow \mathcal{S}_{hift} \circ \phi \leq \psi$*; hence* $d_t^\circ\left(\mathcal{S}_{hift} \circ \phi\right) = d_t^\circ(\phi) + 1$.

(b) $\psi < \mathcal{S}_{hift} \circ \phi \Longrightarrow \psi \leq \phi$

(c) If $tg(\phi)$ *is successor then* $\mathcal{S}_{hift}{}^{-1} \circ \phi < \phi \equiv \mathcal{S}_{hift} \circ \mathcal{S}_{hift}{}^{-1} \circ \phi$

(d) $tg\left(\mathcal{S}_{hift} \circ \phi\right) = tg(\phi) + 1$

(e) $tg\left(\mathcal{S}_{hift}{}^{-1} \circ \phi\right) = tg(\phi) \dot{+} (-1)$.

□ **prop.23** -

(a) By hypothesis, assume σ is a winning strategy for I in $\mathbb{G}_{\leq}(\psi, \phi)$. This strategy satisfies $\phi(x) < \psi \circ \sigma\star(x)$, hence $\mathcal{S}_{hift} \circ \phi(x) \leq \psi \circ \sigma\star(x)$, holds for any $x \in \Lambda_\phi{}^{\leq\omega}$. Therefore σ is winning for $I\!I$ in $\mathbb{G}_{\leq}\left(\mathcal{S}_{hift} \circ \phi, \psi\right)$. This induces that $\mathcal{S}_{hift} \circ \phi$ is the $<$-least above ϕ hence $d_t^\circ\left(\mathcal{S}_{hift} \circ \phi\right) = d_t^\circ(\phi) + 1$.

(b) By hypothesis, player I has a w.s. in $\mathbb{G}_{\leq}\left(\mathcal{S}_{hift} \circ \phi, \psi\right)$ which is also a w.s. for $I\!I$ in $\mathbb{G}_{\leq}(\psi, \phi)$.

(c) Let ξ be the target of ϕ. By remark 5(b) there exists a strategy σ that is winning for $I\!I$ in the game $\mathbb{G}_{<,\ \xi}(\phi, \phi)$ such that $\forall x \in \Lambda_\phi^\omega \quad \phi(\sigma \star x) \geq \xi$. This same strategy is winning for I in $\mathbb{G}_{\leq}\left(\phi, \mathcal{S}_{hift}{}^{-1} \circ \phi\right)$; showing that $\mathcal{S}_{hift}{}^{-1} \circ \phi < \phi$. Therefore by (a) we obtain $\mathcal{S}_{hift} \circ \mathcal{S}_{hift}{}^{-1} \circ \phi \leq \phi$. Finally, we remark that the identity strategy is a winning strategy for $I\!I$ in $\mathbb{G}_{\leq}\left(\phi, \mathcal{S}_{hift} \circ \mathcal{S}_{hift}{}^{-1} \circ \phi\right)$, which gives the result.

(d) A winning strategy for $I\!I$ in $\mathbb{G}_{<,\ \xi}(\phi, \phi)$ is also a w.s. for $I\!I$ in $\mathbb{G}_{<,\ \xi+1}\left(\mathcal{S}_{hift} \circ \phi, \mathcal{S}_{hift} \circ \phi\right)$.

(e) If $tg(\phi)$ is successor then the result comes from (c) and (c). If $tg(\phi) = \lambda$ is limit, then there exists a w.s. σ for $I\!I$ in $\mathbb{G}_{<,\ \lambda}(\phi, \phi)$. Let τ be σ applied twice ($\tau\star = \sigma \star \circ \sigma\star$), it is a w.s. for $I\!I$ in $\mathbb{G}_{<,\ \lambda}\left(\mathcal{S}_{hift}{}^{-1} \circ \phi, \mathcal{S}_{hift}{}^{-1} \circ \phi\right)$; which proves that $\lambda = \lambda \dot{+} (-1)$ is still the target of $\mathcal{S}_{hift}{}^{-1} \circ \phi$.

23 □

The properties of $\mathcal{S}_{hift}$ allow us to relate the successor degrees to the successor targets; the integer part of both ordinals being the same.

Proposition 24 *Let $\phi : \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$ and $n \in \mathbb{N}$,*

$$\exists \alpha \quad d_t^\circ(\phi) = \omega \cdot \alpha + n \iff \exists \lambda \text{ limit} \quad tg(\phi) = \lambda + n.$$

Corollary 25 Given any $\phi : \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$,

there exists $\varphi : \Lambda_\varphi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$ and $n \in \mathbb{N}$ such that both $tg(\varphi) = 0$ or limit and $\phi \equiv \mathcal{S}_{hift}{}^n \circ \varphi$.

□ **cor. 25** - Let $\omega \cdot \alpha + n = d_t^\circ(\phi)$, any φ that satisfies $d_t^\circ(\varphi) = \omega \cdot \alpha$ works. 25□

□ **prop.24** -

($\Longrightarrow$) The proof goes by induction on $d_t^\circ(\phi)$.

$\boldsymbol{d_t^\circ(\phi) = 0}$**:** in that case $\phi = C_0$ and $tg(\phi) = 0$.

$\boldsymbol{d_t^\circ(\phi)}$ **is successor:** assume $d_t^\circ(\phi) = \omega \cdot \alpha + n + 1$, then take any ψ such that $d_t^\circ(\psi) = \omega \cdot \alpha + n$ holds. We get $\mathcal{S}_{hift} \circ \psi \equiv \phi$ which leads to $tg(\phi) = tg(\mathcal{S}_{hift} \circ \psi) = tg(\psi) + 1 = \lambda + n + 1$.

$\boldsymbol{d_t^\circ(\phi)}$ **is limit:** towards a contradiction assume that $tg(\phi)$ is successor. Then the following holds:

$$\mathcal{S}_{hift}{}^{-1} \circ \phi < \phi \equiv \mathcal{S}_{hift} \circ \mathcal{S}_{hift}{}^{-1} \circ \phi.$$

Hence $d_t^\circ(\phi) = d_t^\circ\left(\mathcal{S}_{hift}{}^{-1} \circ \phi\right) + 1$ which is not a limit ordinal, a contradiction.

($\Longleftarrow$) The proof goes by induction on n.

$\boldsymbol{tg(\phi) = \lambda}$**:** then $d_t^\circ(\phi)$ is limit because otherwise any ψ such that $d_t^\circ(\psi) + 1 = d_t^\circ(\phi)$ would verify $\mathcal{S}_{hift} \circ \psi \equiv \phi$; hence $tg(\mathcal{S}_{hift} \circ \psi) = tg(\phi)$. A contradiction since $tg(\mathcal{S}_{hift} \circ \psi)$ is successor.

$\boldsymbol{tg(\phi) = \lambda + n + 1}$**:** in that case $\mathcal{S}_{hift}{}^{-1} \circ \phi < \phi \equiv \mathcal{S}_{hift} \circ \mathcal{S}_{hift}{}^{-1} \circ \phi$ holds. By the other direction of this proof, $tg\left(\mathcal{S}_{hift}{}^{-1} \circ \phi\right) = \lambda + n$, so by induction hypothesis there exists an ordinal α such that $d_t^\circ\left(\mathcal{S}_{hift}{}^{-1} \circ \phi\right) = \omega \cdot \alpha + n$. Hence

$$d_t^\circ(\phi) = d_t^\circ\left(\mathcal{S}_{hift} \circ \mathcal{S}_{hift}{}^{-1} \circ \phi\right) = d_t^\circ\left(\mathcal{S}_{hift}{}^{-1} \circ \phi\right) + 1 = \omega \cdot \alpha + n + 1.$$

24 $\square$

Since we actually know of a simple relation between mappings at successor levels of the Steel Hierarchy and mappings at limit levels, we concentrate on the only mappings that occur at limit levels of the hierarchy. For this reason we set a third version of the notion of degree.

Definition 26 *Let* $\phi : \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta_1^1} \boldsymbol{On}$, $tg(\phi) = 0$ *or limit,*

- $d_l^\circ(\phi) = 1 \iff \phi = C_0$
- $d_l^\circ(\phi) = \sup\{d_l^\circ(\psi) + 1 : \ tg(\psi) = 0$ *or limit* $\wedge \ \psi < \phi\}$ *(if* $\phi > C_0$*).*

We remark that for any Borel function ϕ with target 0 or limit, $d_t^\circ(\phi) = \omega \cdot (-1 + d_l^\circ(\phi))$. The reason why the enumeration starts at 1 instead of 0 is because it has to fit with the operation on functions that corresponds to the ordinal sum - see §7.

5 Initializability

Before describing the conciliatory counterparts of arithmetical operations that we will use to build the whole hierarchy on the basis of constant functions, we look at a particular property that leads to a main dichotomy among mappings. This notion of initializability is contained in the very notion of game: what about if a player might start again from scratch at any time? This game-theory related idea is more important to our study than it first seems. Its futility hides the core of powerful tools. As we will see, it is very easy to produce non initializable mappings of any corresponding degrees affording initializable ones. But it is much more difficult to generate non initializable ones. We first start by defining the arrow operation.

Definition 27 *Given $\Lambda_\phi \subsetneq \Lambda_\psi$, $\phi:\ \Lambda_\phi^{\leq\omega} \longmapsto$* ***On*** *and $\psi:\ {\Lambda_\psi}^{\leq\omega} \longmapsto$* ***On****, we define the mapping $\phi^{\longrightarrow}\psi:\ {\Lambda_\psi}^{\leq\omega} \longmapsto$* ***On*** *by*

$$\begin{array}{lrcl} \forall x \in \Lambda_\phi^{\leq\omega} & \phi^{\longrightarrow}\psi(x) & = & \phi(x) \\ \forall u \in \Lambda_\phi^{<\omega}\ \forall a \in \Lambda_\psi \smallsetminus \Lambda_\phi\ \forall x \in {\Lambda_\psi}^{\leq\omega} & \phi^{\longrightarrow}\psi(u^\frown\langle a\rangle^\frown x) & = & \psi(x). \end{array}$$

Remarks 28

- The operation $^{\longrightarrow}$ preserves both $\leq$ and $<$; that is when ϕ, ϕ', ψ, ψ' are all Borel:

$$\phi \leq \phi' \ \wedge \psi \leq \psi' \implies \phi^{\longrightarrow}\psi \leq \phi'^{\longrightarrow}\psi'$$

$$\phi < \phi' \ \wedge \psi < \psi' \implies \phi^{\longrightarrow}\psi < \phi'^{\longrightarrow}\psi'$$

- We say that a player in charge of a set of the form $\phi^{\longrightarrow}\psi$ at one moment of the run "crosses the arrow" when he plays for the first time a letter not in Λ_ϕ. In other words, when that player reaches a position $u^\frown\langle a\rangle$ with $u \in \Lambda_\phi^{<\omega}$ and $a \in \Lambda_\psi \smallsetminus \Lambda_\phi$. We say so because, from such a position, everything goes as if the player was starting the run all over again; this time being in charge of ψ. The condition $\Lambda_\phi \subsetneq \Lambda_\psi$ is mandatory in the definition, of $\phi^{\longrightarrow}\psi$. But given any alphabet Λ_ϕ, any $\phi:\ {\Lambda_\phi}^{\leq\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto}$ ***On*** and any countable set Λ, there is a canonical mapping $\psi:\ (\Lambda_\phi \cup \Lambda)^{\leq\omega} \longmapsto$ ***On*** such that $\psi \equiv \phi$: just set $\psi(x) = \phi(x)$ if $x \in {\Lambda_\phi}^{\leq\omega}$ and $\phi(x) = 0$ otherwise. In other words, alphabets may be enriched as desired. For instance we allow us to write $\phi^{\longrightarrow}\phi$ which means that we consider an alphabet $\Lambda_\psi \supsetneq \Lambda_\phi$, a mapping $\phi:\ {\Lambda_\phi}^{\leq\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto}$ ***On*** such that $phi \equiv \psi$, and strictly speaking $\phi^{\longrightarrow}\phi$ **is in fact** $phi^{\longrightarrow}\psi$.

Definition 29 *Let $\phi:\ {\Lambda_\phi}^{\leq\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto}$* ***On****,*

$$\phi \textit{ is } \textbf{initializable} \textit{ iff } \phi^{\longrightarrow}\phi \equiv \phi.$$

From the point of view of a player involved in a reduction game, being in charge of an initializable set means being able to initialize the run as many times as desired. We recall that $\mathcal{I}nit_\phi$ is the tree of all positions u in ${\Lambda_\phi}^{<\omega}$ that are equivalent to the original one ($\phi[u] \equiv \phi$). In order to later be able to deal with initializable sets we introduce new notions that help to describe any conciliatory mapping.

Notation 30 Given $\phi:\ {\Lambda_\phi}^{\leq\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto}$ ***On***, we define $\phi_{Init}:\ {\Lambda_{\phi_{Init}}}^{\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto}$ ***On*** and $\phi_{Exit}:\ {\Lambda_{\phi_{Exit}}}^{\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto}$ ***On*** by

$$\begin{array}{rcccl} \phi_{Init}(x) & = & \phi(x) & \text{if} & x \in \mathcal{I}nit_\phi \cup [\mathcal{I}nit_\phi] \\ & = & 0 & \text{otherwise} & \end{array}$$

$$\begin{array}{rcccl} \phi_{Exit}(x) & = & 0 & \text{if} & x \in \mathcal{I}nit_\phi \cup [\mathcal{I}nit_\phi] \\ & = & \phi(y) & \text{if} & x = u^\frown y \text{ with } u \notin \mathcal{I}nit_\phi \wedge u \restriction (lh(u)-1) \in \mathcal{I}nit_\phi. \end{array}$$

These mappings ϕ_{Init} and ϕ_{Exit} induced by ϕ behave as a kind of canonical "deconstruction" of ϕ. This way ϕ_{Init} can be regarded as the initializable part of ϕ, and ϕ_{Exit} as what remains from ϕ after ϕ_{Init} has been thrown away. Precisely we have the following relations between ϕ_{Init}, ϕ_{Exit} and ϕ.

Proposition 31 *Let* $\phi:\ \Lambda_\phi{}^{\leq\omega} \overset{\Delta^1_1}{\longmapsto} \boldsymbol{On}$,

(a) $\phi \equiv \phi_{Init}{}^{\frown}\phi_{Exit}$.

(b) $\phi_{Init} \equiv \phi_{Init}{}^{\frown}\phi_{Init}$. *(i.e.* ϕ_{Init} *is initializable).*

(c) $\phi > C_0 \implies \phi_{Exit} < \phi^{\frown}\phi$.

(d) $\phi^{\frown}\phi \equiv \phi \iff \phi \equiv \phi_{Init}$.

(e) $\phi^{\frown}\phi > \phi \iff (\phi_{Exit} > \phi_{Init} \ \wedge\ \phi > \phi_{Init} \ \wedge\ \phi > \phi_{Exit})$.

□ **prop.31** -

(a) Straightforward and left to the reader.

(b) The direction $\phi_{Init} \leq \phi_{Init}{}^{\frown}\phi_{Init}$ is obvious, so we just have to show $\phi_{Init}{}^{\frown}\phi_{Init} \geq \phi_{Init}$. The following strategy is winning for *II* in the game $\mathbb{G}_{\leq}(\phi_{Init}{}^{\frown}\phi_{Init}, \phi_{Init})$: player *II* copies *I*'s run as long as it remains inside $\mathcal{I}nit_\phi$. If *I* exits $\mathcal{I}nit_\phi$ and reaches the position v, then at that moment player *II* is still in a position u inside $\mathcal{I}nit_\phi$. Therefore $\phi[u] \equiv \phi$ holds. From that point the strategy for *II* consists in skipping until *I* "crosses the arrow" (which is winning in case *I* never makes it) and then applying any winning strategy for *II* in $\mathbb{G}_{\leq}(\phi_{Init}, \phi_{Init}[u])$.

(c) We describe a w.s. for *II* in the game $\mathbb{G}_{<}(\phi_{Exit}, \phi^{\frown}\phi)$. As long as *I* remains inside $\mathcal{I}nit_\phi$, *II* plays any $x \in \Lambda_\phi{}^{\leq\omega}$ that verifies $\phi(x) > C_0$. If *I* reaches a position $u \notin \mathcal{I}nit_\phi$ then *II* "crosses the arrow" and applies a w.s. for *II* in $\mathbb{G}_{<}(\phi[u], \phi)$.

(d) **(⟹)** If $\phi = C_0$ then $\phi \equiv \phi_{Init}$. If $\phi > C_0$ then by (c) $\phi_{Exit} < \phi^{\frown}\phi \equiv \phi$ holds. Assuming $\phi_{Init} < \phi$ leads to $\phi_{Init}{}^{\frown}\phi_{Init} < \phi^{\frown}\phi$ which contradicts (a). Hence $\phi_{Init} \geq \phi$ holds.

(⟹) Immediate from (b).

(e) A direct consequence of all other cases.

31 □

The preliminary work is now over. Next sections are dedicated to the "arithmetical-like" operations on mappings.

6 Conciliatory supremum of countable cofinality

This first operation consists in a simple classical way of coding countably many mappings into a single one. In that sense it is very close to the operation $\sum$ on natural mappings defined in §2.

Definition 32 *Let* $I \subseteq \Lambda$, *for all* $i \in I$ $\phi_i:\ \Lambda^{\leq\omega} \longmapsto \boldsymbol{On}$, $\mathbf{sup}_{i\in I}\phi_i:\ \Lambda^{\leq\omega} \longmapsto \boldsymbol{On}$ *is defined by*

$$\begin{array}{rcll}(\mathbf{sup}_{i\in I}\phi_i(x)) & = & \phi_i(y) & \textit{if} \quad x = \langle i\rangle^{\frown}y \textit{ with } i \in I\\ & = & 0 & \textit{otherwise.}\end{array}$$

Proposition 33 *Assume* $I \subseteq \Lambda$, *for all* $i \in I$ $\phi_i:\ \Lambda^{\omega} \longmapsto \boldsymbol{On}$ *satisfies* $tg(\phi_i) = 0$ *or limit and* $\{\phi_i:\ i \in I\}$ *does not contain a* $\leq$ *-maximal element. Then:*

(a) $tg\,(\mathbf{sup}_{i\in I}\phi_i) = 0$.

(b) $d_l^\circ\,(\mathbf{sup}_{i\in I}\phi_i) = \sup\{d_l^\circ\,(\phi_i) :\ i \in I\}$.

(c) $\psi < \mathbf{sup}_{i\in I}\phi_i \implies \psi \leq \sum_{\langle a\rangle\in\Lambda^1} {\phi_{\langle a\rangle}}^*$ *where* ($\phi_{\langle a\rangle} = \phi_a$ *if* $a \in I$, *and* $\phi_{\langle a\rangle} = C_0$ *otherwise). Hence* $d_m^\circ\,(\mathbf{sup}_{i\in I}\phi_i) = \lambda + 1$ *for some limit ordinal* λ, $cof(\lambda) < \omega_1$.

□ **prop.33** -

(a) is immediate once noticed that in the game $\mathbb{G}_{\leq}\,(\mathbf{sup}_{i\in I}\phi_i, \mathbf{sup}_{i\in I}\phi_i)$ as soon as I stops skipping, $I\!I$ is certain to end the run having produced a sequence y such that $(\mathbf{sup}_{i\in I}\phi_i)(x) < (\mathbf{sup}_{i\in I}\phi_i)(y)$ (with x being the result of I's run).

(b) $\mathbf{sup}_{i\in I}\phi_i > \phi_j$ holds for any $j \in I$. Besides, $\psi < \mathbf{sup}_{i\in I}\phi_i$ implies $\psi \leq \phi_i$ for some $i \in I$. This leads to

$$\begin{aligned} d_t^\circ\,(\mathbf{sup}_{i\in I}\phi_i) &= \sup\{d_t^\circ\,(\psi) + 1 : tg\,(\psi) = 0 \text{ or limit } \wedge\ \psi < \mathbf{sup}_{i\in I}\phi_i\} \\ &= \sup\{d_t^\circ\,(\phi_i) + 1 :\ i \in I\} \\ &= \sup\{d_l^\circ\,(\phi_i) :\ i \in I\}. \end{aligned}$$

(c) Straightforward and left to the reader.

33 □

7 Conciliatory sum

Unexpectedly this operation relies on the conciliatory supremum of countable cofinality defined in the previous section and on the "arrow" operation related to the notion of initializibility.

Definition 34 *Let* $\psi :\ {\Lambda_\psi}^\omega \xrightarrow{\Delta_1^1} \boldsymbol{On}$, $\phi :\ {\Lambda_\phi}^\omega \xrightarrow{\Delta_1^1} \boldsymbol{On}$,

$$\psi {+} \phi \ =_{df}\ \phi^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi).$$

Proposition 35 *Let* $\psi :\ {\Lambda_\psi}^\omega \xrightarrow{\Delta_1^1} \boldsymbol{On}$ *and* $\phi :\ {\Lambda_\phi}^\omega \xrightarrow{\Delta_1^1} \boldsymbol{On}$ *be both Borel,*

$$d_l^\circ\,(\psi {+} \phi) = d_l^\circ\,(\psi) + d_l^\circ\,(\phi)\,.$$

Given T a tree on Λ, the set of all infinite branches of T is denoted $[T]$:

$$[T] = \{x \in \Lambda^\omega :\ \forall n \in \omega\ \ x \restriction n \in T\}.$$

□ **prop.35** - By induction on $d_l^\circ\,(\phi)$:

(a) If $d_l^\circ\,(\phi) = 1$ then $\phi = C_0$ and the following holds:

- $C_0^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi) \equiv \mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi)$
- $\varphi < \mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi)\ \wedge\ tg\,(\varphi) = 0$ or limit $\implies\ \varphi \leq \psi$.

Hence

$$\begin{aligned}
d_l^\circ(\psi+C_0) &= \sup\{d_l^\circ(\varphi)+1:\ tg(\varphi)=0 \text{ or limit } \wedge\ \varphi < \mathbf{sup}_{n\in\mathbb{N}}(\mathcal{S}hift^n\circ\psi)\}\ .\\
&= \sup\{d_l^\circ(\varphi)+1:\ tg(\varphi)=0 \text{ or limit } \wedge\ \varphi \le \psi\}\\
&= d_l^\circ(\psi)+1
\end{aligned}$$

(b) If $d_l^\circ(\phi) > 1$ then

Claim 36

$$tg(\varphi)=0 \textit{ or limit } \wedge\ \varphi < \psi+\phi \implies \exists\varphi' < \phi\ \ \varphi \le \psi+\varphi'\ \wedge\ tg(\varphi')=0 \textit{ or limit.}$$

By the above claim we obtain:

$$\begin{aligned}
d_l^\circ(\psi+\phi) &= \sup\{d_l^\circ(\varphi)+1:\ tg(\varphi)=0 \text{ or limit } \wedge\ \varphi < \psi+\phi\}\\
&= \sup\{d_l^\circ(\psi+\varphi')+1:\ tg(\varphi')=0 \text{ or limit } \wedge\ \varphi' < \phi\}\\
&= \sup\{d_l^\circ(\psi)+d_l^\circ(\varphi')+1:\ tg(\varphi')=0 \text{ or limit } \wedge\ \varphi' < \phi\}\\
&= d_l^\circ(\psi)+\sup\{d_l^\circ(\varphi')+1:\ tg(\varphi')=0 \text{ or limit } \wedge\ \varphi' < \phi\}\\
&= d_l^\circ(\psi)+d_l^\circ(\phi)
\end{aligned}$$

35 □

It just remains to prove the claim.

□ **cl. 36** - Get σ a w.s. for *II* in the strict game $\mathbb{G}_<(\varphi,\psi+\phi)$ and set

$$T=\{u\in\Lambda_\varphi^{<\omega}:\ \forall v\ \ s\text{-variant of } u,\ \ \sigma\star v\in{\Lambda_\phi}^{<\omega}\}$$

i.e. T contains the positions that I can reach without σ forcing *II* to "cross the arrow". Clearly T is closed under subsequences, therefore it is a tree. Set $\varphi_{stay}:\ \Lambda_\varphi^{\le\omega}\longmapsto \boldsymbol{On}$ by

$$\begin{aligned}
\varphi_{stay}(x) &= \varphi(x) && \text{if } x\in T\cup[T]\\
&= 0 && \text{otherwise}
\end{aligned}$$

And $\varphi_{go}:\ \Lambda_\varphi^{\le\omega}\longmapsto \boldsymbol{On}$ by

$$\begin{aligned}
\varphi_{go}(x) &= 0 && \text{if } x\in T\cup[T]\\
&= \varphi(x) && \text{otherwise}
\end{aligned}$$

We remark that:

(i) $\varphi\equiv\varphi_{stay}\overset{\longrightarrow}{}\varphi_{go}$.

(ii) $\varphi_{go}\le\mathbf{sup}_{n\in\mathbb{N}}(\mathcal{S}hift^n\circ\psi)$ - from the very definition of T and φ_{go}.

(iii) This is true because there exists a winning strategy for *II* in the *II*-imposed game $\mathbb{G}_\le(\varphi_{stay},\varphi_{stay})$ where *II* must always remain inside T. This idea combined with the fact that σ is winning for *II* in $\mathbb{G}_<(\varphi_{stay},\phi)$ as long as I stays inside T gives $\varphi_{stay}<\phi$.

Since $\varphi_{stay} < \phi$ holds, by corollary 25 we get $\phi_{stay} \equiv {S_{hift}}^k \circ \varphi'$ for some integer k and mapping φ' verifying $tg\,(\varphi') = 0$ or limit. So we end up with $\varphi \leq \varphi_{stay}{}^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi)$ and

$$\varphi_{stay}{}^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi) \equiv ({S_{hift}}^k \circ \varphi')^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi) \equiv {S_{hift}}^k \circ (\varphi'^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi)),$$

which leads to $\varphi \leq {S_{hift}}^k \circ (\varphi'^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi))$. Since the target of φ is 0 or some limit ordinal and $tg\left({S_{hift}}^k \circ (\varphi'^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi))\right) = tg\left((\varphi'^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi))\right) + k$, by corollary 25 to $\varphi \leq \varphi'^{\longrightarrow}\mathbf{sup}_{n\in\mathbb{N}}({S_{hift}}^n \circ \psi) = \psi \dot{+} \varphi'$. 36□

8 Conciliatory countable multiplication

Now that we have both the conciliatory counterparts of ordinal sum and countable supremum, it is easy to define an operation on mappings which satisfies the property of ordinal countable multiplication when viewed through the looking glass of the degree.

Definition 37 *Let* $\phi : \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$, $tg\,(\phi) = 0$ *or limit and* $0 < \xi < \omega_1$, $\phi \bullet \xi$ *is defined by induction on* ξ*:*

- $\phi \bullet 1 = \phi$
- $\phi \bullet (\xi + 1) = (\phi \bullet \xi) \dot{+} \phi$
- $\phi \bullet \lambda = \mathbf{sup}_{\xi\in\lambda}(\phi \bullet \xi)$ *for* λ *limit ordinal.*

Proposition 38 *Let* $\phi : \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$ *and* $0 < \xi < \omega_1$*. Then:*

$$d_l^\circ\,(\phi \bullet \xi) = d_l^\circ\,(\phi) \cdot \xi.$$

□ **prop.38** - Straigthforward by induction on ξ using proposition 33 for ξ successor and proposition 35 for ξ limit. 38 □

The three operations $\mathbf{sup}$, $\dot{+}$, $\bullet$ all produce non initializable mappings. And these three are just what we need to generate all non initializable functions on the basis of initializable ones. But the main issue remains a way to create these initializable mappings; which means in terms of arithmetical operations that we need powerful tools that simulate the strength of the ω_1 first Veblen functions of base ω_1.

9 Conciliatory Veblen operations: priority arguments and unfolded games

We introduce an operation on finite and infinite sequences: $x \longmapsto x^{\circlearrowleft}$ based on the idea of the priority argument. This operation first appeared in [2] and was extended in [3] as a main tool for the study of of the Wadge hierarchy.

Definition 39 *Let* Λ *be any alphabet,* $\{\twoheadleftarrow_{(\theta)} : 0 < \theta < \omega_1\}$ *a set of different arrows, indexed by* **non zero** *ordinals, disjoint from* Λ*.*

As a convention, given $a \in \Lambda \cup \{\twoheadleftarrow(\theta) : 0 < \theta < \omega_1\}$ *we define* $\mathcal{D}(a)$ *the* **depth** *of a letter in* $\Lambda \cup \{\twoheadleftarrow(\theta) : 0 < \theta < \omega_1\}$ *by*

$$\begin{array}{rcl} \mathcal{D}(a) = 0 & \Longleftrightarrow & a \in \Lambda \\ \mathcal{D}(a) = \theta > 0 & \Longleftrightarrow & a = \twoheadleftarrow(\theta) \end{array}$$

Given $x \in (\Lambda \cup \{\twoheadleftarrow(\theta) : 0 < \theta < \omega_1\})^{\leq\omega}$ and any countable ordinal ξ (possibly 0) we successively define $x^{\looparrowleft\xi}$, $x^{\circlearrowleft\xi} \in \Lambda \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \xi\})^{\leq\omega}$ and $x^{\circlearrowleft} \in \Lambda^{\leq\omega}$. In words, the first of them, $x^{\looparrowleft\xi}$, is just x when every occurrence of the letter $\twoheadleftarrow(\xi)$ is translated into an eraser; which means that, going from left to right and imagining $\twoheadleftarrow(\xi)$ represents the Back Space key (the one familiar to your keyboard), the sequence $x^{\looparrowleft\xi}$ is what remains once the whole erasing work is done; formally:

(1) Definition of $\boldsymbol{x^{\looparrowleft\xi}}$

- $<>^{\looparrowleft\xi} = <>$
- *in case x is finite but non empty, $x = y^\frown < a >$:*

  $$\begin{array}{rll} (y^\frown < a >)^{\looparrowleft\xi} = & y^{\looparrowleft\xi\frown} < a > & \text{if } a \neq \twoheadleftarrow(\xi) \\ = & <> & \text{if } a = \twoheadleftarrow(\xi) \text{ and } lh(y^{\looparrowleft\xi}) \leq 1 \\ = & y^{\looparrowleft\xi} \restriction (k-1) & \text{if } a = \twoheadleftarrow(\xi) \text{ and } lh(y^{\looparrowleft\xi}) = k > 1 \end{array}$$

- *for x infinite:*

  $$x^{\looparrowleft\xi} = \lim_{n\in\omega}\left((x \restriction n)^{\looparrowleft\xi}\right) = \cup\{u : \ \exists m \forall n \geq m \ \ u \subseteq \left((x \restriction n)^{\looparrowleft\xi}\right)\}.$$

  This operation $x \longmapsto x^{\looparrowleft\xi}$ was the first step, now we use this tool to define $x^{\circlearrowleft\xi}$. This next operation consists in iterating the previous one in a downward way that stops when arriving at level ξ.

(2) Definition of $\boldsymbol{x^{\circlearrowleft\xi}}$ *Let* $0 \leq \xi < \omega_1$,
The sequence $x^{\circlearrowleft\xi}$ is an element of $(\Lambda \cup \{\twoheadleftarrow(\theta) : 0 < \theta \leq \xi\})^{\leq\omega}$ *defined by*

if $x = <>$, then $x^{\circlearrowleft\xi} = <>$,

if $x \neq <>$, then depending on the depth of $x(0)$:

- *if $\mathcal{D}(x(0)) = 0$, then $x^{\circlearrowleft\xi} = x(\ /\xi <)$,*
  where $x(\ /\xi <)$ is just x where all letters of the form $\twoheadleftarrow(\theta)$ for $\theta > \xi$ are removed, - in particular if $\xi = 0$ then $x^{\circlearrowleft\xi} = x^{\circlearrowleft 0} = x(\ /\{\twoheadleftarrow(\iota) : 0 < \iota < \omega_1\})$ -
- *if $0 < \mathcal{D}(x(0)) \leq \xi$ i.e. $x(0) = \twoheadleftarrow(\nu)$ for some $\nu \leq \xi$, then $x^{\circlearrowleft\xi} = x(\ /\xi <)$,*
- *if $0 < \xi < \mathcal{D}(x(0))$ and $\mathcal{D}(x(0))$ is successor, i.e. $x(0) = \twoheadleftarrow(\nu+1)$ for some $\nu \geq \xi$, then $x^{\circlearrowleft\xi} = ((x(\ /\nu+1 <))^{\looparrowleft\nu+1})^{\circlearrowleft\xi}$,*
- *if $0 \leq \xi < \mathcal{D}(x(0)) = \delta$ **limit**, then*

  $$x^{\circlearrowleft\xi} = < (x(\ /\delta \leq))_i{}^{\circlearrowleft\xi}(0) / i \in \omega >$$

  where:

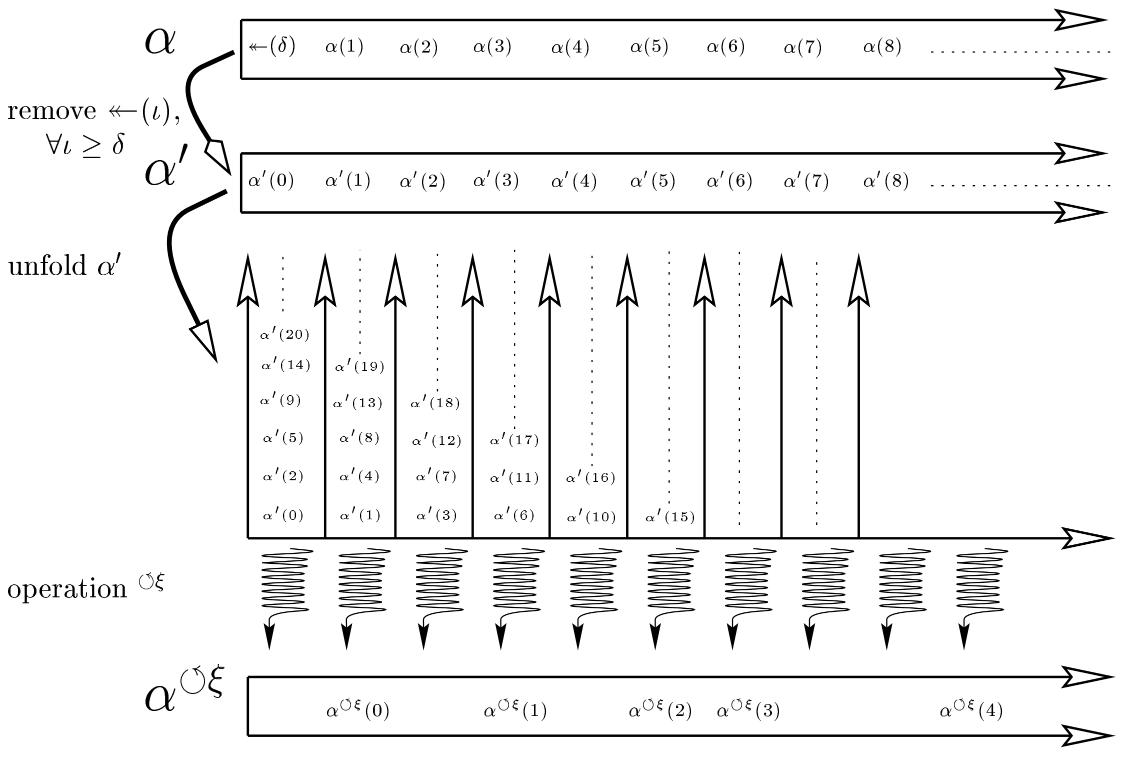
  - $(x(\ /\delta \leq))$ *is x where all letters of the form $\twoheadleftarrow(\theta)$ for $\theta \geq \delta$ are removed and*

- ⋄ *for any x' the sequence ${x'}_i$ is defined by first unfolding x' into ω sequences and secund picking up the one labeled i: ${x'}_i = < x'(\phi(i,j))/j \in \omega >$ for some bijection ϕ between ω^2 and ω. The choice of this function is not important; in order to simplify the presentation we choose the Cantor function $K \; : \; \omega \times \omega \longmapsto \omega$ defined by*

$$K(n,m) = \frac{(n+m)(n+m+1)}{2} + m$$

So to get $x^{\circlearrowleft\xi}$ in case $\mathcal{D}(x(0)) = \delta$ is limit greater than ξ (see figure 5), one has first to

Figure 5:



erase every letter of depth at least δ which gives a new sequence x', then to unfold this sequence x' into ω sequences (the x_i's) and apply the $^{\circlearrowleft\xi}$ operation independently to each of them. And finally to pick up the first value of each resulting sequence and concatenate them:

$$x^{\circlearrowleft\xi} = ({x'}_0^{\circlearrowleft\xi} \upharpoonright 1)^\frown({x'}_2^{\circlearrowleft\xi} \upharpoonright 1)^\frown \ldots {}^\frown({x'}_n^{\circlearrowleft\xi} \upharpoonright 1)^\frown({x'}_{n+1}^{\circlearrowleft\xi} \upharpoonright 1)^\frown \ldots$$

The last step consists in iterating operation $x \longmapsto x^{\circlearrowleft\xi_i}$ along a descending "well chosen" chain of variables ξ_i.

(3) Definition of $x^{\circlearrowleft}$ *(by construction $x^{\circlearrowleft} \in \Lambda^{\leq\omega}$)*

- *if* $x = <>$, *then* $x^{\circlearrowleft} = <>$,
- *if* $\mathcal{D}(x(0)) = 0$ *i.e.* $x(0) \in \Lambda$, *then* $x^{\circlearrowleft} = x^{\circlearrowleft 0}$ ($=_{df} x$ *once every "arrow" is removed).*
- *if* $\mathcal{D}(x(0)) = \nu + 1$ *i.e.* $x(0) = \twoheadleftarrow_{(\nu+1)}$, *then* $x^{\circlearrowleft} = (x^{\circlearrowleft \nu})^{\circlearrowleft}$,
- *if* $\mathcal{D}(x(0)) = \delta$ *limit, it admits a Cantor Normal Form of base* ω*:*

  $$\delta = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0$$

  with

  $$\omega_1 > \xi_k > \ldots > \xi_0 > 0 \text{ and } \omega > n_i > 0 \text{ for each } i \leq k$$

  then

  $$x^{\circlearrowleft} = (x^{\circlearrowleft \delta - \omega^{\xi_0}})^{\circlearrowleft}$$

  where by convention $\delta - \omega^{\xi_0} = \omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.(n_0 - 1)$.

To say it in words, given any finite or infinite sequence x with letters among the alphabet $\Lambda \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota < \omega_1\}$, the decoding process of x denoted $x^{\circlearrowleft}$ goes this way:
To start with, one looks at the first letter of the sequence (if x is the empty sequence then there is no such letter and $x^{\circlearrowleft}$ is just the empty sequence). If $x(0)$ exists the first thing to do is to throw away every arrow indexed by a greater ordinal than $\mathcal{D}(x(0))$ (this guarantees the finiteness of the process) which gives rise to a new sequence x'. Then the depth of $x(0)$ determines the way it is read:

- if $x'(0)$ is a letter that belongs to Λ then $x^{\circlearrowleft}$ is just x'. There is nothing else that can be done, so the decoding process stops.

- If $x'(0)$ is a letter of the form $\twoheadleftarrow_{(\nu+1)}$, the next step applies the erasing operation to x' with $\twoheadleftarrow_{(\nu+1)}$ used as an eraser. Namely this gives x'' which is exactly x' when the letter $\twoheadleftarrow_{(\nu+1)}$ is replaced by the Back Space key and operates from left to right inside x' (considering the sequence x' is also written from left to right: $x' = x(0).x(1).\ldots$). And in order to obtain $x^{\circlearrowleft}$ one must iterate this whole process; that is start all over again but this time with x'' instead of x'. i.e. one gets $x^{\circlearrowleft} = x''^{\circlearrowleft}$.

- If $x'(0) = \twoheadleftarrow_{(\delta)}$ for some δ limit. The aim is to unfold x' into ω sequences and apply the very same decoding operation to each of them and to finally pick up one single information from each of the ω sequences. Gathering up the data into a new sequence. Since x' still contains letters of the form $\twoheadleftarrow_{(\delta)}$ we first need to eliminate all letters of the form $\twoheadleftarrow_{(\delta)}$ and unfold the resulting sequence into ω sequences $(x'_i)_{i \in \omega}$ by use of the Cantor pairing function. The next step is like applying the operation $^{\circlearrowleft}$ to each new sequence; except that in order to sufficiently complexify the resulting sequence $x^{\circlearrowleft}$, we need to repeat this unfolding operation many times. So for each x'_i we need the decoding process to stop at some point in order to gather the information and then start again with the resulting sequence. This point is given by the Cantor Normal Form of base ω of the ordinal δ. We stop at the ordinal $\delta' = \omega^{\delta_k}.n_k + \ldots + \omega^{\delta_0}.(n_0 - 1)$ where the Cantor Normal Form of δ is $\omega^{\delta_k}.n_k + \ldots + \omega^{\delta_0}.n_0$. This means any x'_i is decoded by use of this iterating process until a sequence x''_i is reached that is either the empty sequence or contains a first letter of depth less than or equal to δ'. Then, picking up the first letter of each x''_i (no letter at all when x''_i is the empty sequence) one gathers up this information into a new sequence y which necessarily contains letters of depth at most δ': $y = < x_i''(0)/i < \omega >$. We then repeat the whole process using y instead of x. This process stops after a finite number of steps giving $x^{\circlearrowleft} = y^{\circlearrowleft}$.

This combinatorial manipulation of sequences is entirely devoted to the following definition of Veblen-like operations on mappings.

Definition 40 *Let $\phi : \Lambda_\phi{}^{\leq\omega} \xmapsto{\Delta^1_1} \boldsymbol{On}$, $\xi < \omega_1$, $\{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\} \cap \Lambda_\phi \neq \emptyset$. Set*

(a) $\phi^{\sim\xi} : (\{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\} \cap \Lambda_\phi)^{\leq\omega} \longmapsto \boldsymbol{On}$ *is the mapping defined by* $\phi^{\sim\xi}(x) = \phi(x^{\circlearrowleft})$

(b) $\mathbf{V}_\xi \circ \phi = \phi^{\sim\omega^\xi}$.

We use the mapping composition symbol $\circ$ for conciseness. It would be more appropriate to write $\mathbf{V}_\xi(\phi)$ than $\mathbf{V}_\xi \circ \phi$. But it is just easier to read when it comes to applying iterations of that Veblen-like function.
In particular we see that $\phi^{\sim 0} = \phi$ and $\mathbf{V}_0 \circ \phi = \phi^{\sim 1}$. For instance, for a player in charge of $\phi^{\sim 1}$ in a reduction game, everything goes as if that player was in charge of ϕ and was playing letters in Λ_ϕ but with the additional opportunity of erasing any final move at any time. This way of seeing $\phi^{\sim 1}$ shows that it is initializable.
Such an operation bears an interest only if it preserves the Steel ordering. This means it must satisfy $\phi \leq \psi \implies \mathbf{V}_\zeta \circ \phi \leq \mathbf{V}_\xi \circ \psi$. To show this property, we introduce a relation between countable ordinals which was also designed for the study of the Wadge hierarchy in [3], as well as the results that directly follow.

Definition 41 (remote control of a strategy) *Given ζ, ξ two countable ordinals, we write:*

$$\boldsymbol{\zeta \preceq \xi}$$

if for any strategy τ : $(\Lambda_\phi \cup \{s\})^{<\omega^-} \longmapsto \Lambda_\psi \cup \{s\}$ there exists a strategy:

$$\sigma_\tau^{\zeta,\xi} \;:\; (\Lambda_\phi \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\})^{<\omega^-} \longmapsto \Lambda_\psi \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \xi\}$$

such that $\forall x \in (\Lambda_\phi \cup \{s\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \zeta\})^\omega$:

(a) $(\sigma_\tau^{\zeta,\xi} \star x)(0) = \twoheadleftarrow_{(\xi)}$ *(by convention in case $\xi = 0$ there is no condition (a) at all)*

and

(b) $\exists x' \in \Lambda_\phi \cup \{s\} \quad x(\ /s)^{\circlearrowleft} = x'(\ /s)$ *and* $(\sigma_\tau^{\zeta,\xi} \star x)(\ /s)^{\circlearrowleft} = (\tau \star x')(\ /s)$.

If we regard the strategy $\sigma_\tau^{\zeta,\xi}$ as a strategy for player *II* involved in a conciliatory reduction-game, condition (a) says that player *II* must play at least one letter and his first move must be $\twoheadleftarrow_{(\xi)}$. Condition (b) says that the strategy remote-controls the strategy τ.
The two main results about this relation $\preceq$ are displayed in next lemmma and corollary.

Lemma 42 (remote control lemma) *For all countable ordinals $\zeta \leq \xi$,*

$$\zeta \preceq \xi.$$

Lemma 43

(a) Let $0 < \xi_0 < \ldots < \xi_k$ *and* $\forall i \leq k \;\; 0 < n_i < \omega$,

$$\omega^{\xi_k}.n_k + \ldots + \omega^{\xi_0}.n_0 \preceq \omega^{\xi_k}.n_k$$

(b) Let $\xi,\ \zeta$ *be countable non null ordinals,*

$$\xi + \zeta \preceq \zeta + \xi.$$

□ **lem. 43 & cor. 42** - Both were proved in [3]. The first one is an immediate consequence of what appeared in [3] as lemmas 10 and 15. The second lemma is almost exactly corollary 16 in [3].
43&42 □
The proofs of both previous lemmas would be tedious. Fortunately we can skip them since they appeared in [3], and we can directly state the result about order preserving.

Proposition 44 (remote control) *Let* $\phi :\ {\Lambda_\phi}^{\leq\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$, $\psi :\ {\Lambda_\psi}^{\leq\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$, $\zeta \leq \xi$,

(a) $\phi \leq \psi \implies \phi^{\sim\zeta} \leq \psi^{\sim\xi}$

(b) $\phi < \psi \implies \phi^{\sim\zeta} < \psi^{\sim\xi}$

(c) $\phi \leq \psi \implies \mathbf{V}_\zeta \circ \phi \leq \mathbf{V}_\xi \circ \psi$

(d) $\phi < \psi \implies \mathbf{V}_\zeta \circ \phi < \mathbf{V}_\xi \circ \psi$

(e) $\zeta < \xi \implies \mathbf{V}_\zeta \circ \mathbf{V}_\xi \circ \phi \equiv \mathbf{V}_\xi \circ \phi.$

□ **prop.44** -

(a) This is an immediate consequence of lemma 42. Because given a strategy τ that is winning for *II* in the game $\mathbb{G}_{\leq}(\phi,\psi)$, one gets σ^ξ_τ that ξ-remote controls τ and is a winning strategy for *II* in the game $\mathbb{G}_{\leq}\left(\phi^{\sim\xi},\psi^{\sim\xi}\right)$, which shows $\phi^{\sim\xi} \leq \psi^{\sim\xi}$. Combining the $\phi^{\sim\zeta} \leq \phi^{\sim\xi}$, which is obviou, to the preceding remark gives the result.

(b) Same as (a) turning each game into a strict game.

(c) Immediate from (a).

(d) Immediate from (b).

(e) Immediate from lemma 43(a).

44 □

In order to iterate Veblen-like operations, we introduce a notation.

Notation 45 Let $\psi : \Lambda_\psi{}^{\leq\omega} \xrightarrow{\boldsymbol{\Delta}^1_1} \boldsymbol{On}$, $\lambda < \omega_1$ with Cantor Normal Form of base ω:

$$\lambda = \omega^{\lambda_k}.n_k + \ldots + \omega^{\lambda_0}.n_0$$

the notation $\psi^{\approx\lambda}$ stands for

$$\psi^{\approx\lambda} = \left(\left(\left(\psi^{\sim\omega^{\lambda_0}.n_0}\right)^{\cdots}\right)^{\sim\omega^{\lambda_k}.n_k}\right) = \underbrace{\mathbf{V}_{\lambda_k} \circ \ldots \circ \mathbf{V}_{\lambda_k}}_{n_k} \circ \ldots\ldots \circ \underbrace{\mathbf{V}_{\lambda_0} \circ \ldots \circ \mathbf{V}_{\lambda_0}}_{n_0} \circ \psi$$

And for $x \in \Lambda_\psi \cup \{\twoheadleftarrow_{(\iota_p)} : 0 < \iota_p \leq \omega^{\lambda_p}.n_p, \;\; 0 \leq p \leq k\}$ we write

$$x^{\odot} \text{ for } \left(\left(\ldots\left(x^{0\circlearrowleft}\right)\ldots\right)^{k-1\circlearrowleft}\right)^{k\circlearrowleft}$$

where $y^{p\circlearrowleft}$ is just $y^{\circlearrowleft}$ where the ${}^{\circlearrowleft}$-operation only deals with the arrows of type p. So in particular we obtain $\psi^{\approx\lambda} = \psi(x^{\odot})$.
We also write $\mathcal{A}_\lambda$ for the set of arrows $\{\twoheadleftarrow_{(\iota_p)} : 0 < \iota_p \leq \omega^{\lambda_p}.n_p, \;\; 0 \leq p \leq k\}$.

10 Auxiliary games: a kind of inverse for Veblen operations

In [2] and [3] we designed a sort of inverse to the Veblen operations for the Wadge hierarchy we were studying. It worked as a major help in proving that these operations satisfied the properties we wanted. Following the same method we introduce a kind of inverse for the Veblen-like operations denoted $\mathbf{V}_\xi^{-1}$. As $\mathbf{V}_\xi$ was defined on the basis of ${}^{\sim\xi}$, we will first define ${}^{\backsim\xi}$, an inverse for ${}^{\sim\xi}$ and relate both operations by setting $\mathbf{V}_\xi = {}^{\backsim\omega^\xi}$.
Strictly speaking these operations cannot be the true inverses since $\phi \longmapsto \phi^{\sim\xi}$ is an operation defined on conciliatory mappings, while $\phi \longmapsto \phi^{\backsim\xi}$ is defined on "true" mappings - i.e. of the form $\psi : \Lambda_\psi{}^{\omega} \xrightarrow{\boldsymbol{\Delta}^1_1} \boldsymbol{On}$. This means $\psi^{\backsim\lambda}$ maps **infinite** sequences and is involved in **games** where player I can't skip.
The main effect of ${}^{\backsim\lambda}$ when applied to ϕ is to change the topology of the space Λ_ϕ^ω, in a way that turns some well chosen $\boldsymbol{\Sigma}^\circ_\lambda$ sets into open ones. This is done in a natural way that fits in the game theoretic point of view of our work. As in Martin's proof of Borel Determinacy, changing the topology is done by answering auxiliary questions about the future of the run.

To start with, we need special trees called ξ-$\mathcal{T}$rees for ξ any countable ordinal. Such a ξ-$\mathcal{T}$ree should be seen as the space of all possible positions for a player in charge of $\phi^{\backsim\xi}$. The case $\xi = 0$ is easy since a 0-$\mathcal{T}$ree on Λ_ϕ is isomorphic to Λ_ϕ^ω. But as the ordinal ξ increases, the alphabet of a ξ-$\mathcal{T}$ree becomes more and more complicated. A 1-$\mathcal{T}$ree on on Λ_ϕ is a tree on an alphabet that contains all letters from Λ_ϕ plus all finite sequences of elements from Λ_ϕ. A $\xi + 1$-$\mathcal{T}$ree is built as a 1–$\mathcal{T}$ree on a ξ-tree. And a λ-$\mathcal{T}$ree for some limit ordinal λ is a tree that "contains" the information of particular ξ-$\mathcal{T}$rees for all $\xi < \lambda$.
More precisely, each node of a ξ-$\mathcal{T}$ree is seen as being labeled by some couple (ζ, n) where ζ is either some ordinal strictly less than ξ or -1, and n an integer. Given an infinite branch in a $\xi\mathcal{T}$ree, for each integer n there is one and only node of that branch labeled $(-1, n)$; and the information at this node is exactly the one corresponding to the main run: it is a letter from Λ_ϕ. This means that the infinite sub-branch formed by the concatenation of the information displayed by nodes labeled

by $(-1,0),(-1,1),(-1,2),\ldots$ is an ω-sequence of elements in Λ_ϕ. All other nodes correspond to auxiliary moves. The information linked to nodes labeled (ξ,n) with $\xi\geq 0$ tells us about making a choice about whether or not the foregoing run will remain in a closed subset of a space whose complexity depends on ξ. For example, with $\xi=0$ and from the point of view of a player whose run goes along one branch of a ξ-$\mathcal{T}$ree, at each node labeled $(-1,n)$ this player chooses a letter in Λ_ϕ and at each node labeled $(0,m)$ answers a question whether or not the whole main run will be inside a given closed subset of Λ_ϕ^ω (which particular subset depends on this particular node). So, answering yes or not compels the player to reach some position that "escapes" the closed set; or to never reach some such position.
But, as the construction is made by induction on ξ, this (closed) subset is a subset of a complicated space implicitly described by the nodes of the λ-$\mathcal{T}$ree that are labeled by couples of the form (ζ,n) with $\zeta<\xi$. So inductively, this closed subset can easily be coding information about whether or not the part of the run that belongs to the original space will remain in a $\boldsymbol{\Sigma}^\circ_\xi$ set or not.

When looking at a λ-$\mathcal{T}$ree, we want to be able to see it as the result of an increasing chain of ξ-$\mathcal{T}$rees for any $\xi<\lambda$ (increasing in the sense that it codes more and more information), therefore we begin the formal definition of a ξ-$\mathcal{T}$ree with a notation about a suitable coding of integers.
To not overload the script, given any countable ordinal θ, we write $\boldsymbol{\theta}^-$ instead of $\theta\cup\{-1\}$.

Notation 46 Let $0\leq\lambda<\omega_1$ and $\begin{array}{ccc}\lambda^-\times\omega & \longleftrightarrow & \omega\\ (\zeta,n) & \longmapsto & \prec\zeta,n\succ^\lambda\end{array}$ be any bijection that satisfies the following condition (C^*)

$$(C^*)\qquad\qquad \forall\zeta<\lambda\quad (n\ <\ m\implies\ \prec\zeta,n\succ^\lambda\ \ <\ \ \prec\zeta,m\succ^\lambda)$$

For any ordinal ξ such that $0\leq\xi<\lambda$, we define the bijection $\begin{array}{ccc}\xi^-\times\omega & \longleftrightarrow & \omega\\ (\zeta,n) & \longmapsto & \prec\zeta,n\succ^\xi\end{array}$ as the collapse of the trace of $\prec\succ^\lambda$ on $\xi^-\times\omega$.
Namely, $\prec\succ^\xi$ is the one and only enumeration in increasing order of the following set of integers:

$$\{\prec\zeta,n\succ^\lambda:\ \zeta<\lambda,\ n<\omega\}.$$

Or to say it in an inductive way, this enumeration takes the form:

$$\prec\zeta,n\succ^\xi=0\ \text{ iff }\ \forall\theta<\xi\ \forall m<\omega\ \prec\theta,m\succ^\lambda\ \geq\ \prec\zeta,n\succ^\lambda$$

assume now that $\prec\zeta_0,n_0\succ^\xi=0,\ldots,\prec\zeta_k,n_k\succ^\xi=k$ then

$$k+1=\prec\zeta_{k+1},n_{k+1}\succ^\xi=\inf\Big(\{\prec\theta,n\succ^\lambda:\ \theta<\xi,\ n<\omega\}\smallsetminus\{\prec\zeta_0,n_0\succ^\xi,\ldots,\prec\zeta_k,n_k\succ^\xi\}\Big).$$

By construction, we remark that the function $\prec\succ^\xi$ still satisfies the condition (C^*):

$$\forall\zeta<\xi\quad(n\ <\ m\implies\ \prec\zeta,n\succ^\xi\ \ <\ \ \prec\zeta,m\succ^\xi).$$

Notation 47 Given $0\leq\lambda<\omega_1$ and a mapping from $\lambda^-\times\omega$ 1-1 and onto ω: $(\xi,n)\longmapsto\prec\xi,n\succ^\lambda$, that satisfies condition (C^*): $\forall\xi<\lambda\quad(n\ <\ m\implies\ \prec\xi,n\succ^\lambda\ \ <\ \ \prec\xi,m\succ^\lambda)$, and any Λ

(a) for any $u \in \Lambda^{<\omega}$ and any $\zeta < \lambda$, $\boldsymbol{u^\zeta}$ denotes the subsequence of u defined by induction on the length of u: $<>^\zeta = <>$; and $(u^\frown\langle\alpha\rangle)^\zeta \;=\; u^{\zeta\frown}\langle\alpha\rangle$ if $lh(u) = \prec \theta, n \succ^\lambda$ with $\theta < \zeta$

$\qquad\qquad = u^\zeta$ otherwise.

(b) For any tree on Λ (in fact T will always represent a $\mathcal{T}$ree) and any $\zeta < \lambda$, $\boldsymbol{T^\zeta}$ denotes a tree formed by eliminating from T the nodes that appear at levels $\prec \theta, n \succ^\lambda$ for any $\theta \geq \zeta$. Formally : $T^\zeta = \{u^\zeta : u \in T\}$. We call it a "sub$\mathcal{T}$ree" of T.

Our motivation is that given $\zeta < \xi < \lambda$ and T some λ-$\mathcal{T}$ree, to have the ζ-$\mathcal{T}$ree $(T^\xi)^\zeta$ be exactly the same as T^ζ. Only taking into account codings that satisfy condition (C^*) assures us this property.

(c) Given a tree T, $x \in [T]$, and $\xi < \lambda$ we write $\boldsymbol{x^\xi}$ for the trace of x on $[T^\xi]$: $x^\xi = \bigcup\{u^\xi : u \subsetneq x\}$.

We now come to the definition of a λ-$\mathcal{T}$ree T. This definition is by induction on λ in the sense that for any ordinal $\xi < \lambda$ the "sub$\mathcal{T}$rees" $\boldsymbol{T^\xi}$ **have already been defined as $\boldsymbol{\xi}$-$\mathcal{T}$rees**.

Definition 48 *Given $0 \leq \lambda < \omega_1$ and a mapping 1-1 from $\lambda^- \times \omega$ and onto ω: $(\xi, n) \longmapsto \prec \xi, n \succ^\lambda$, that satisfies condition (C^*): $\forall \xi < \lambda \quad (n < m \implies \prec \xi, n \succ^\lambda \;<\; \prec \xi, m \succ^\lambda)$, and any Λ, A λ-$\mathcal{T}$ree T is some* **non empty pruned tree** *that satisfies for any u in T and any ordinal ζ and integer n such that $\prec \zeta, n \succ^\lambda < lh(u)$:*

if $\boldsymbol{\zeta = -1}$: *then $u(\prec \zeta, n \succ^\lambda) \in \Lambda$ (these are the nodes corresponding to the main run); and*

if $\boldsymbol{0 \leq \zeta}$: *then this corresponds to the auxiliary runs and there are two options :*

Option $\langle 1 \rangle$: *$u(\prec \zeta, n \succ^\lambda) = \langle 1 \rangle^\frown v$ with $v \in T^\zeta$ and $\left(u \restriction \prec \zeta, n \succ^\lambda)\right)^\zeta \subseteq v$. So v is some position in the sub$\mathcal{T}$ree T^ζ that extends the position $\left(u \restriction \prec \zeta, n \succ^\lambda)\right)^\zeta$. But then, any w in T that verifies $u \restriction (\prec \zeta, n \succ^\lambda + 1) \subseteq w$ must satisfy $w^\zeta \subseteq v$ or $v \subseteq w^\zeta$.*

Option $\langle 0 \rangle$: *$u(\prec \zeta, n \succ^\lambda) = \langle 0 \rangle$. This must be regarded as the option to avoid all other nodes with same father as this one. Formally this means that any w in T that verifies $u \restriction (\prec \zeta, n \succ^\lambda + 1) \subseteq w$ must satisfy*

$$v \subseteq w^\zeta \text{ **fails for any** } \boldsymbol{v} \text{ such that } (u \restriction \prec \zeta, n \succ^\lambda)^\frown \left\langle \langle 1 \rangle^\frown v \right\rangle \in T.$$

In order to simplify the arguments in the proofs, we also require T to satisfy an additional condition:

$$\left(u \in T \;\wedge\; u^\frown < \langle 1 \rangle^\frown v > \in T \;\wedge\; u^\frown < \langle 1 \rangle^\frown w > \in T\right) \implies v \perp w.$$

Moreover, T must be maximal for inclusion, which means given any tree S on Λ, either $T \not\subsetneq S$ or S does not verify all other mandatory conditions to be a λ-$\mathcal{T}$ree.

Remarks 49

(a) When T is a λ-$\mathcal{T}$ree, since it is maximal for inclusion we see that $T^0 = (\{-1\} \times \Lambda)^{<\omega}$ which is isomorphic to $\Lambda^{<\omega}$. So, in the sequel, we identify T^0 and $\Lambda^{<\omega}$ and we see $u \in T^0$ and $x \in [T^0]$ as respectively a finite sequence of letters from Λ and an element of Λ^ω.

(b) When ζ is at least 0, one should think of $u(\prec \zeta, n \succ^\lambda)$ as the answer to some question about the future of the run in T^ζ, that asks if it will remain in some closed subset $C_{u\restriction(\prec\zeta,n\succ^\lambda+1)}$ of $[T^\zeta]$, simply characterized by negativity (not choosing option $\langle 1\rangle$):

$$C_{u\restriction(\prec\zeta,n\succ^\lambda+1)} = \left\{x \in [T^\zeta]:\ (u \restriction\prec \zeta, n \succ^\lambda)^\frown \left\langle \langle 1\rangle^\frown v\right\rangle \in T \Longrightarrow\ v \not\subseteq x\right\}$$

- When the answer is $\langle 0\rangle$ meaning **yes**, the player certifies that the sequence $x \in [T]$ he will have produced at the end of the game will satisfy the condition $x^\xi \in C_{u\restriction(\prec\zeta,n\succ^\lambda+1)}$.
- When the answer is $\langle 1\rangle^\frown v$ meaning **no**, the player (while choosing the $\langle 1\rangle$ option) affirms that x^ζ will exit the closed set $C_{u\restriction(\prec\zeta,n\succ^\lambda+1)}$. And by playing the sequence v he indicates the way the run will effectively leave $C_{u\restriction(\prec\zeta,n\succ^\lambda+1)}$.

In any cases, all moves in the future are restricted to the ones that agree with options previously taken.

The λ-$\mathcal{T}$rees are the background of the Veblen-like inverse operation. Given a mapping ϕ, we are now interested in what happens to a player who is in charge of ϕ but must play along some λ-$\mathcal{T}$ree T on Λ_ϕ. He always reaches positions that are in T, so at each move that corresponds to an auxiliary move, he gives his adversary a lot of information about what he will play in the future. So that it is much more difficult for him to win.

Definition 50 *Given* $\phi:\ {\Lambda_\phi}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$, $\lambda < \omega_1$ *and* T *some* λ-$\mathcal{T}$*ree on* Λ, *by* ϕ^T *we denote the mapping* $\phi:\ {\Lambda_T}^\omega \longmapsto \boldsymbol{On}$ *defined by*

- $\Lambda_T = \Lambda_\phi\ \cup\ \{\langle 0\rangle\}\ \cup\ \{\langle 1\rangle^\frown v:\ v \in T^\xi, 0 \leq \xi < \lambda\}$
- *if* $x \in [T]$ *then* $\phi^T(x) = \phi(x^0)$
- *if* $x \notin [T]$ *then* $\phi^T(x) = 0$.

We defined ϕ^T this way so that player *II* has a w.s in the *II*-imposed game $\mathbb{G}_\leq\left(\phi^T, \phi^T\right)$ where *II* must remain inside T. So there is no reason for a player in charge of some ϕ^T to leave the $\mathcal{T}$ree T and we will always assume that a run of a player being in charge of such a mapping remains inside T.

Remarks 51

(a) Assume T is a λ-$\mathcal{T}$ree on an alphabet Λ, with $\lambda = \xi + \zeta$. One can regard T as a ζ-tree on the alphabet of T^ξ. More precisely, there exists a ζ-$\mathcal{T}$ree T^ζ on Λ_{T^ξ} (the alphabet of T^ξ) such that for any mapping $\phi:\ \Lambda^\omega \longmapsto \boldsymbol{On}$, the relation $\phi^T \equiv (\phi^{T^\xi})^{T^\zeta}$ holds. We call T^ζ the "rest" of T when T^ξ has been removed.

(b) Given T and T' two λ-*Trees* on the same alphabet Λ, one may easily define T'' a λ-*Tree* on Λ that codes both T and T', i.e. for all $x \in [T]$ and $x' \in [T']$ verifying $x^0 = x'^0$ there exists $x'' \in [T'']$ such that $x^0 = x'^0 = x''^0$ and for all ξ that satisfies $0 < \xi \leq \lambda$ there are strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$, f_ξ and f'_ξ such that

$$x^\xi = x''^\xi \circ f_\xi \text{ and } x'^\xi = x''^\xi \circ f'_\xi.$$

Or in other words, for any mapping $\phi : \Lambda^\omega \longmapsto \boldsymbol{On}$, player *II* has a w.s. in the four games

$$\mathbb{G}_{\leq}\left(\phi^T, \phi^{T''}\right), \quad \mathbb{G}_{\leq}\left(\phi^{T''}, \phi^T\right), \quad \mathbb{G}_{\leq}\left(\phi^{T'}, \phi^{T''}\right), \quad \mathbb{G}_{\leq}\left(\phi^{T''}, \phi^{T'}\right).$$

Notation 52 Let $\phi : \Lambda_\phi{}^\omega \overset{\boldsymbol{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$ and $\xi \in \boldsymbol{On}$, we write ξ_ϕ for the subset of Λ_ϕ^ω defined by $x \in \xi_\phi \iff {}_{\mathrm{df}}\phi(x) = \xi$.

Here we need to introduce the notion of Wadge game and Wadge reducibility that deals with sets instead of mappings.

Definition 53 *Let $A \subseteq \Lambda_A^{\leq\omega}$ and $B \subseteq \Lambda_B^{\leq\omega}$,*

(a) the Wadge game $\mathbf{W}(A, B)$ is exactly the same game as $\mathbb{G}_{\leq}(\chi_A, \chi_B)$ where χ_A and χ_B are the characteristic functions of respectively A, B; except for the winning conditions that are: II wins iff $\chi_A(x) = \chi_B(y)$.

(b) The relation $A \leq_w B$ means that II has a winning strategy in the game $\mathbf{W}(A, B)$.

In the case where A and B are Borel, the Wadge game is determined, therefore either $A \leq_w B$ or $\Lambda_A^\omega \setminus A \leq_w B$ holds. In that context, the Wadge ordering is a well quasi order, which means it has no infinite descending chain, and its antichains are finite (in fact antichains have length at most 2). Wadge designed the game which bears his name so that a winning strategy for *II* defines a continuous mapping $f : \Lambda_A^\omega \longmapsto \Lambda_B^\omega$ such that $A = f^{-1}B$.

We remark that given T a λ-*Tree* on Λ and $\phi : \Lambda^\omega \longmapsto \boldsymbol{On}$, for each ordinal ξ that belongs to the image of ϕ, player *II* has an obvious w.s. in the Wadge game $\mathbf{W}\left(\xi_{\phi^T}, \xi_{\phi^{T^\xi}}\right)$, which consists of copying I's part of the run that belongs to T^ξ. In particular, taking $\xi = 0$ and bearing in mind that ϕ^{T^0} is isomorphic to ϕ, we see $\xi_{\phi^T} \leq_w \xi_\phi$. This proves that if ϕ is Borel, then ϕ^T is Borel too. Therefore the set of all mappings ϕ^T where T ranges over all λ-*Trees* on Λ_ϕ is wellordered by $\leq$. Hence, there exists one minimal element (up to $\equiv$). So we set

Definition 54 *Given $0 \leq \lambda < \omega_1$ and $\phi : \Lambda_\phi{}^\omega \overset{\boldsymbol{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$,*

$$\boldsymbol{\phi^{\sim\lambda}} = a \leq \textit{-minimal element of } \{\phi^T : T \textit{ is a } \lambda - \textit{Tree on } \Lambda_\phi\}.$$

We insist on the fact that $\phi^{\sim\lambda}$ is just a notational convenience and none of the foregoing proofs involving $\phi^{\sim\lambda}$ require the Axiom of Choice.
By definition of a 0-*Tree* $\phi^{\sim 0}$ is equivalent to ϕ, so to simplify notations in the following, let us assume by convention that $\phi^{\sim 0}$ is exactly ϕ. We also use the notation $\phi^\sim$ for $\phi^{\sim 1}$.

One clearly sees that ${}^{\approx\lambda}$ lowers the Steel degree of a mapping. This allows us, in well chosen cases, to decrease the degree of the mapping we deal with. The final proofs of this paper use this technique for the induction step. But before we come to the last part of this work, we state some general properties of ${}^{\approx\lambda}$.

Operation ${}^{\sim\lambda}$ preserves both relations $\leq$ and $<$. But the quasi-inverse operation ${}^{\approx\lambda}$ preserves $\leq$ but not $<$:

Lemma 55 *Let* $\psi: \ {\Lambda_\psi}^\omega \overset{\Delta^1_1}{\longmapsto} \boldsymbol{On}$, $\phi: \ {\Lambda_\phi}^\omega \overset{\Delta^1_1}{\longmapsto} \boldsymbol{On}$ *and* $0 \leq \lambda < \omega_1$,

(a) II has a w.s. in $\mathbb{G}_{\leq}(\psi, \phi) \implies$ *II has a w.s. in* $\mathbb{G}_{\leq}\left(\psi^{\approx\lambda}, \phi^{\approx\lambda}\right)$.

(b) II has a w.s. in $\mathbb{G}_{<}(\psi, \phi) \implies$ *II has a w.s. in* $\mathbb{G}_{<}\left(\psi^{\approx\lambda}, \phi^{\approx\lambda}\right)$.

$\square$ **lem. 55** - The proof goes by induction on λ. We prove simultaneously both (a) and (b).

$\boldsymbol{\lambda = 0}$: is obvious since $\psi^{\approx 0} = \psi$ and $\phi^{\approx 0} = \phi$.

$\boldsymbol{\lambda > 0}$: let σ be some w.s. for *II* in $\mathbb{G}_{\leq}(\psi, \phi)$ - respectively $\mathbb{G}_{<}(\psi, \phi)$ - and $\phi^{\approx\lambda} = \phi^{T_\phi}$ with T_ϕ a λ-$\mathcal{T}$ree on Λ_ϕ. By induction on ξ $(0 < \xi \leq \lambda)$ we will define

- ${}^{\xi}T_\psi$ a ξ-$\mathcal{T}$ree on Λ_ψ,
- ${}^{\xi}\sigma$ a w.s. for *II* in $\mathbb{G}_{\leq}\left(\psi^{{}^{\xi}T_\psi}, \phi^{T^\xi_\phi}\right)$ - respectively in $\mathbb{G}_{<}\left(\psi^{{}^{\xi}T_\psi}, \phi^{T^\xi_\phi}\right)$.

By minimality, we will get a winning strategy for player *II* in $\mathbb{G}_{\leq}\left(\psi^{\approx\lambda}, \psi^{{}^{\lambda}T_\psi}\right)$. Elsewhere we will find a w.s. for *II* in $\mathbb{G}_{\leq}\left(\psi^{{}^{\lambda}T_\psi}, \phi^{T_\phi}\right)$ - respectively in $\mathbb{G}_{<}\left(\psi^{{}^{\lambda}T_\psi}, \phi^{T_\phi}\right)$. Since $\phi^{T_\phi} = \phi^{\approx\lambda}$, this altogether will provide a w.s. for *II* in $\mathbb{G}_{\leq}\left(\psi^{\approx\lambda}, \phi^{\approx\lambda}\right)$ - respectively in $\mathbb{G}_{<}\left(\psi^{\approx\lambda}, \phi^{\approx\lambda}\right)$.

We also want $\left({}^{\xi}T_\psi\right)^\zeta = {}^{\zeta}T_\psi$ to hold for all $\zeta < \xi$; and the "restriction" of ${}^{\xi}\sigma$ to ${}^{\zeta}T_\psi$ to be exactly ${}^{\zeta}\sigma$ in the following sense: if $u \in T^\xi$ and $lh(u) - 1 = \prec \theta, n \succ^\xi$ for some $\theta < \xi$ and $n < \omega$, then ${}^{\xi}\sigma(u) = {}^{\nu}\sigma(u^\nu)$ is true for all $\theta < \nu \leq \xi$.

Roughly speaking ${}^{\lambda}T_\psi$ is the pullback of T_ϕ along the strategy σ and ${}^{\lambda}\sigma$ is the strategy induced by this pullback.

We define ${}^{\xi}T_\psi$ and ${}^{\xi}\sigma$ by induction on ξ:

- $\boldsymbol{\xi = 0}$: ${}^{0}T_\psi = {\Lambda_\phi}^{<\omega}$ and ${}^{0}\sigma = \sigma$.
- $\boldsymbol{\xi}$ **limit**: this inductive step is entirely determined by the cases of lower degrees.
- $\boldsymbol{\xi = \zeta + 1 > 0}$: we just have to define $u(\prec \zeta, n \succ^\xi)$ and ${}^{\xi}\sigma(u)$ with $u \in {}^{\xi}T_\psi$.
  We assume that $lh(u) - 1 = \prec \zeta, n \succ^\xi$. we also assume that the values ${}^{\xi}\sigma(u')$ have already been defined for any $u' \subsetneq u$ and that $\left({}^{\xi}\sigma * (u \restriction lh(u) - 1)\right)(\ /s) = v$ with $lh(v) = \prec \theta, n \succ^\xi$. There are two cases:
  (a) if $\theta < \zeta$ then ${}^{\xi}\sigma(u) = s =$skip: player *II* must wait and see until player *I* (who can't skip) reaches a position $u^\frown u'$ such that $lh(u^\frown u') - 1 = \prec \theta, n' \succ^\xi$. At that moment, player *II* just applies ${}^{\theta}\sigma$.

(b) if $\theta = \zeta$ then next move for *II* is to answer about wether or not the foregoing part of the run related to T^ζ_ϕ will reach some position belonging to a certain set $Q_v \subseteq T^\zeta_\phi$. Let Q_u be the set of positions for I in ${}^\zeta T_\psi$ defined by

$$Q_u = \{u'' \in {}^\zeta T_\psi :\ ({}^\zeta\sigma * u'')(\ /s) \in Q_v\}.$$

Let $u \in {}^\xi T$ be the position where I is questioned about the set of positions Q_u, and ${}^\xi\sigma(u)$ is defined by:

- if I answers yes (i.e. takes option $\langle 0\rangle$), then *II* answers also yes by taking option $\langle 0\rangle$.
- if I answers no (i.e. takes option $\langle 1\rangle^\frown u'$), then *II* also takes option $\langle 1\rangle$ by answering $\langle 1\rangle^\frown v'$ with $v' = ({}^\zeta\sigma * u')(\ /s)$.

55□

Corollary 56 Let $\psi :\ \Lambda_\psi{}^\omega \overset{\Delta^1_1}{\longmapsto} \boldsymbol{On}$, $\phi :\ \Lambda_\phi{}^\omega \overset{\Delta^1_1}{\longmapsto} \boldsymbol{On}$ and $0 \leq \lambda \leq \delta < \omega_1$,

$$\psi \leq \phi \Longrightarrow \psi^{\sim\delta} \leq \phi^{\sim\lambda}.$$

□ **cor. 56** - The case $\delta = \lambda$ is a direct consequence of lemma 55(a). Cases $\delta > \lambda$ are immediate since $\psi^{\sim\delta} \leq \psi^{\sim\lambda}$ holds by minimality.

56□

Operation ${}^{\sim\lambda}$ satisfies an elementary property very similar to arithmetical exponentiation:

Proposition 57 *Let* $\psi :\ \Lambda_\psi{}^\omega \overset{\Delta^1_1}{\longmapsto} \boldsymbol{On}$, $0 \leq \zeta, \xi < \omega_1$,

$$\psi^{\sim(\zeta+\xi)} \equiv \left(\psi^{\sim\zeta}\right)^{\sim\xi}.$$

□ **prop.57** - Assume $\psi^{\sim(\zeta+\xi)} = \psi^T$ so that T is some $\zeta + \xi$-$\mathcal{T}$ree on Λ_ϕ. Then $\psi^T \equiv \left(\psi^{T^\zeta}\right)^{T_\xi}$ where T_ξ is the "rest" of T when T^ζ has been removed from T. So T_ξ is some ξ-$\mathcal{T}$ree on Λ_{T^ξ} (=the alphabet of T^ζ). By minimality one obtains

$$\psi^{\sim(\zeta+\xi)} = \psi^T \equiv \left(\psi^{T^\zeta}\right)^{T_\xi} \geq \left(\psi^{T^\zeta}\right)^{\sim\xi} \geq \left(\psi^{\sim\zeta}\right)^{\sim\xi}.$$

On the other direction, assume $\left(\psi^{\sim\zeta}\right)^{\sim\xi} = \left(\psi^{T_\zeta}\right)^{T_\xi}$. Clearly from T_ζ and T_ξ one builds T a $\zeta + \xi$-$\mathcal{T}$ree on Λ_ϕ such that $\left(\psi^{T_\zeta}\right)^{T_\xi} \geq \psi^T$. And by minimalit,y $\psi^T \geq \psi^{\sim(\zeta+\xi)}$ is true. 57
□

In order to establish the essential properties of $\mathbf{V}_\xi^{-1}$ we prove:

Lemma 58 *Let* $\psi :\ \Lambda_\psi{}^{\leq\omega} \overset{\Delta^1_1}{\longmapsto} \boldsymbol{On}$ *with* ψ *Borel, and* $0 < \lambda < \omega_1$,

(a) $\left(\psi^{\sim\lambda}\right)^{*\sim\lambda} \leq \psi^*$

(b) $\left(\psi^{\approx\lambda}\right)^{*\sim\lambda} \leq \psi^*$.

This lemma is all we need to get

Proposition 59 *Let* $\psi : \Lambda_\psi{}^{\leq\omega} \xrightarrow{\Delta^1_1} \boldsymbol{On}$ *with* ψ *Borel, and* $0<\lambda<\omega_1$,

$$\mathbf{V}_\lambda^{-1} \circ (\mathbf{V}_\lambda \circ \psi)^* \ \leq \psi^*.$$

□ **prop.59** - Immediate from lemma 58. 59 □

We come to the proof of the lemma.

□ **lem. 58** -

(a) It is enough to show for each ordinal λ there exists a λ-$\mathcal{T}$ree T_λ such that $((\psi^{\sim\lambda})^*)^{T_\lambda} \leq \psi^*$. The proof goes by induction on λ. The induction hypothesis is a little bit stronger than what is required: we will show that there exists T_λ a λ-$\mathcal{T}$ree on the alphabet of $(\psi^{\sim\lambda})^*$ such that for all $x \in [T_\lambda]$, for each integer n and each letter $a \in \Lambda_\phi \cup \{*\}$ that **appears in** x^0, the sequence x "answers" each question: will $(x^0)^{\circlearrowleft}(n)$ be the letter a ?.

By "answers" each question, we mean for all n and $a \in \Lambda_\phi \cup \{*\}$ appearing in x^0 there exists an integer p such that $x(p)$ is the answer to the question: will $(x^1)^{\circlearrowleft}(n)$ be the letter a ? ($\langle 0\rangle$ meaning **no** and $\langle 1\rangle^\frown v$ meaning **yes**). So, from that position, the w.s. for player *II* in the underlying game is obvious: wait (which means play $*$) until *I* answers **yes** to some question $(x^1)^{\circlearrowleft}(0) = a_0$?, then play a_0 and wait until *I* answers **yes** to some question $(x^1)^{\circlearrowleft}(1) = a_1$?, then play a_1, etc...

Assume $\boldsymbol{\lambda = 0}$: this is obvious since the identity works.

Assume $\boldsymbol{\lambda = 1}$:

(i) Set, for each non empty sequence $u \in \Lambda_\psi^{<\omega}$ and integer n,

$$C_{(u,n)} = \{x \in (\Lambda_\psi \cup \{\twoheadleftarrow_{(1)}, *\})^\omega : \forall m \geq n \quad u \subseteq (x(\ /*) \restriction m)^{\circlearrowleft}\}$$

Clearly $C_{(u,n)}$ is closed in $(\Lambda_\psi \cup \{\twoheadleftarrow_{(1)}, *\})^\omega$. Let T_1 be any 1-$\mathcal{T}$ree on Λ_ψ such that for all $x \in [T_1]$, for all finite sequence $u \subseteq \{x^0(k) :\ k \in \omega\} \cap \Lambda_\psi$ and for all integer n there exists $i \in \omega$ such that $x^1(i)$ is the answer to the question whether or nor the main run x^0 will remain inside $C_{(u,n)}$.

Now player *II* has a w.s. in $\mathbb{G}_{\leq}\left((\psi^{\sim})^{*T_1}, \psi^*\right)$ that consists in playing according to *I*'s positive answers to the questions formulated by T_1: Player *II* first waits until *I* answers "yes" to some question about being in $C_{(\langle a_0\rangle, n_0)}$ for some $a_0 \in \Lambda_\psi$ and $n_0 \in \omega$; then *II* plays a_0 and waits until *I* answers "yes" to some other question $C_{(\langle a_0,a_1\rangle, n_1)}$ for some $a_1 \in \Lambda_\psi$ and $n_1 \in \omega$, then *II* plays a_1, etc... By minimality of $(\psi^{\sim})^{*\curvearrowright}$ we obtain:

$$(\psi^{\sim})^{*\curvearrowright} \leq (\psi^{\sim})^{*T_1} \leq \psi^*$$

Assume $\boldsymbol{\lambda \neq \omega^\xi.n}$ for all $\boldsymbol{\xi \geq 0}$ and $\boldsymbol{n > 0}$: then for some $\nu < \lambda$ one has $\lambda \preceq \nu$. So there exists a w.s. σ for *II* in the game $\mathbb{G}_{\leq}\left((\psi^{\sim\lambda})^*, (\psi^{\sim\nu})^*\right)$ such that:

$$\forall x \in (\Lambda_\phi \cup \{*\} \cup \{\twoheadleftarrow_{(\iota)} : 0 < \iota \leq \lambda\})^\omega \quad (x(\ /*))^{\circlearrowleft} = ((\sigma \star x)(\ /*))^{\circlearrowleft}.$$

By induction hypothesis there exists a ν-$\mathcal{T}$ree T_ν that fits the conditions of the induction hypothesis. Now get T_λ any λ-$\mathcal{T}$ree that "contains" the pullback of T_ν along the strategy σ. Clearly the $\mathcal{T}$ree T_λ works.

Assume $\lambda = \omega^\xi$ with $\xi > 0$: we describe a λ-$\mathcal{T}$ree T_λ by giving the questions player *II* asks pllayer *I* in the game $\mathbb{G}_{\leq}\left(\left(\left(\psi^{\sim\omega^\xi}\right)^*\right)^{T_\lambda}, \psi^* \right)$:

First, as long as *II* waits for the letter $(x^0(\ /*))(0)$, *II* keeps on playing $*$. In that case the possible questions along the $\mathcal{T}$ree are totally irrelevant. This may go until the end of the run, but if *I* finally plays the letter $(x^0(\ /*))(0)$ there are three cases:

(i) if $(x^0(\ /*))(0) \in \Lambda_\phi$ then everything is easy since it is the same as if *I* was simply in charge of ψ^*.

(ii) if $(x^0(\ /*))(0) = \twoheadleftarrow_{(\theta)}$ with $\theta < \omega^\xi$ then everything goes as if *I* was in charge of $\left(\psi^{\sim\theta}\right)^*$ instead of $\left(\psi^{\sim\omega^\xi}\right)^*$. By induction hypothesis there exists a θ-$\mathcal{T}$ree T_θ that works, so any λ-$\mathcal{T}$ree "containing" T_θ is suitable.

(iii) if $(x^0(\ /*))(0) = \twoheadleftarrow_{(\omega^\xi)}$, the sequence $x^0(\ /*)(\ /\twoheadleftarrow_{(\omega^\xi)})$ is unfolded into ω sequences $(x_n)_{n\in\omega}$ (we keep in mind that $x^{0\circlearrowleft} = \langle x_n^\circlearrowleft(0)/n \in \omega >)$. So during *I*'s run, player *II* imagines the sequences x_n. Of course x_n is a finite or infinite sequence on the alphabet $\Lambda_\phi \cup \{\twoheadleftarrow_{(\iota)} :\ 0 < \iota < \omega^\xi\}$, but continuously *II* may replace x_n by an infinite sequence x'_n that belongs to $(\Lambda_\phi \cup \{*\} \cup \{\twoheadleftarrow_{(\iota)} :\ 0 < \iota < \omega^\xi\})^\omega$ and that is exactly x_n once the letters $*$ are removed.

Now consider each n such that x_n is not the empty sequence and $\theta_n = \mathcal{D}(x_n(0))$. By induction hypothesis there exists a θ_n-$\mathcal{T}$ree S_n which, for each integer m and for each letter $a \in \Lambda_\phi$ played by *I* during the run, gives an answer to each question of the form "will $x_n^\circlearrowleft(m)$ be a ?". So if $x_n^\circlearrowleft$ is not the empty sequence, climbing along S_n we will know what the value $x_n^\circlearrowleft(0)$ will be. But it is easy to build a θ_n+1-$\mathcal{T}$ree T_n that verifies $T_n^\theta = S_n$ and T_n gives also the answer to the question "will $x_n^\circlearrowleft$ be the empty sequence ?".

At last, we can "mix" these trees $(T_n)_{n\in\omega}$ in order to get T an ω^ξ-$\mathcal{T}$ree such that for all integer n, T^{θ_n+1} "contains" the θ_n+1-$\mathcal{T}$ree T_n. Therefore T will provide an answer to each following questions

(A) will x_n be the empty sequence ?

(B) will $x_n^\circlearrowleft$ be the empty sequence ?

(C) what will $x_n^\circlearrowleft(0)$ be ? (in case T answers "no" to (A) and (B)).

And this is exactly what is sufficient to answer any questions "will $x^\circlearrowleft(n)$ be a ?" for any a in Λ_ϕ that is played by *I* during the run, provided that effectively $x^\circlearrowleft$ will be long enough.

Assume $\lambda = \omega^\xi.n$ with $0 \leq \xi$ and $1 < n$: this case is relatively easy because:

$$\psi^{\sim\omega^\xi.n} \equiv_c \left(\psi^{\sim\omega^\xi.(n-1)}\right)^{\sim\omega^\xi}$$

and by induction hypothesis there exists both:

- a ω^ξ-$\mathcal{T}$ree T_{ω^ξ} on the alphabet of $\left(\left(\psi^{\sim\omega^\xi.(n-1)}\right)^{\sim\omega^\xi}\right)^*$ that fits the conditions of the induction hypothesis. Therefore there exists σ, a w.s. for *II* in the game

$$\mathbb{G}_{\leq}\left(\left(\left(\left(\psi^{\sim\omega^\xi.(n-1)}\right)^{\sim\omega^\xi}\right)^*\right)^{T_{\omega^\xi}}, \left(\psi^{\sim\omega^\xi.(n-1)}\right)^* \right).$$

- a $\omega^{\xi}(n-1)$-*Tree* $T_{\omega^{\xi}(n-1)}$ on the alphabet of $\left(\psi^{\sim\omega^{\xi}.(n-1)}\right)^*$ that also fits the conditions of the induction hypothesis and

$$\left(\left(\psi^{\sim\omega^{\xi}.(n-1)}\right)^*\right)^{T_{\omega^{\xi}(n-1)}} \leq \psi^*$$

Let T' be a pullback of $T_{\omega^{\xi}(n-1)}$ by the strategy σ. Now from $T_{\omega^{\xi}}$ and T' we easily get a $\omega^{\xi}.n$-*Tree* T_{λ} that works.

(b) Assume the Cantor Normal Form of base ω of λ is

$$\lambda = \omega^{\lambda_k}.n_k + \ldots + \omega^{\lambda_0}.n_0$$

then

$$\psi^{\approx\lambda} = \left(\left(\left(\psi^{\sim\omega^{\lambda_0}.n_0}\right)^{\cdots}\right)^{\sim\omega^{\lambda_k}.n_k}\right)$$

and by proposition 57

$$\left(\psi^{\approx\lambda}\right)^{*\nsim\lambda} \leq \left(\ldots\left(\left(\psi^{\approx\lambda}\right)^{*\nsim\omega^{\lambda_k}.n_k}\right)^{\cdots}\right)^{\nsim\omega^{\lambda_0}.n_0}$$

and by (b)

$$\left(\ldots\left(\left(\left(\left(\left(\psi^{\sim\omega^{\lambda_0}.n_0}\right)^{\cdots}\right)^{\sim\omega^{\lambda_k}.n_k}\right)^*\right)^{\nsim\omega^{\lambda_k}.n_k}\right)^{\cdots}\right)^{\nsim\omega^{\lambda_0}.n_0} \leq \psi^*$$

58□

11 Target and inverse of Veblen-like operations

Of course we would like to apply ${}^{\nsim\lambda}$ to conciliatory mappings of the form $\phi^{\sim\lambda}$, which we cannot. But anyhow this is not a big problem since we may apply it to "true" mappings like $(\phi^{\sim\lambda})^*$. What causes trouble is that working with conciliatory functions we were sure to always get conciliatory mappings, hence mappings that necessarily admit a target. Target or no target is a big issue here. We don't know whether or not $\phi^{\sim\lambda}$ is target-free or not. There is no reason why it should admit a target. And in fact in most cases it does not. So we now determine the conditions that ϕ must satisfy to ensure that $\psi^{\nsim\lambda}$ has a target.

Proposition 60 *Let* $\psi : \Lambda_{\psi}{}^{\omega} \xrightarrow{\boldsymbol{\Delta}^1_1} \boldsymbol{On}$ *be Borel,*

$$\psi \text{ is initializable} \iff \psi^{\nsim 1} \text{ has a target.}$$

□ **prop.60 -**

($\Longrightarrow$) : Assume T is some 1-$\mathcal{T}$ree on Λ_ψ such that $\psi^T = \psi^{\sim 1}$. To get the result it is enough to produce a sequence $x \in {\Lambda_T}^\omega$ that verifies $\psi^{\sim 1}[x \restriction n] \equiv \psi^{\sim 1}$ for each integer n. For this proves that $\mathcal{I}nit_\psi$ is illfounded, hence ψ has a target.

For each finite sequence of even length $u \in {\Lambda_\psi}^{<\omega}$, we denote F_u the closed set about which any position extending u of length $lh(u)+1$ deals with: that is the closed set to which the main run is restricted once played $u^\frown\langle\langle 1\rangle\rangle$.

We inductively define x by:

- $x(0)$ is anything so that $\langle x(0)\rangle \in \mathcal{I}nit_\psi$.
- assume $\langle x(0), x(1), \ldots, x(2n)\rangle$ has been defined and the promise made states that the run will take place in the closed set F_n **and** the next mandatory moves are going to be v_n. So the commitment about the run at that point can be summed up by

$$\langle x(0), x(2), \ldots, x(2n)\rangle^\frown v_n \subsetneq x^0 \in F_n$$

where:

$$F_n = \bigcap_{i<n,\ \ x(2i+1)=\langle 0\rangle} F_{\langle x(0), x(1), \ldots, x(2i)\rangle}$$

and

$$\begin{aligned} G_n &= \langle x(0), x(2), \ldots, x(2n)\rangle^\frown v_n{}^\frown {\Lambda_\psi}^\omega \\ &= \bigcap_{i<n,\ \ x(2i+1)=\langle 1\rangle^\frown w_i} \langle x(0), x(2), \ldots, x(2i)\rangle^\frown w_i{}^\frown {\Lambda_\psi}^\omega \end{aligned}$$

(By convention $v_0 = \langle\rangle$, $G_0 = F_0 = {\Lambda_\psi}^\omega$). From that position, legal moves are now restricted to the closed set $F_n \cap G_n$.

Assume that, by construction, the following holds:

$$[\mathcal{I}nit_\psi] \cap G_n \subseteq F_n \ \text{ and } \ G_n \cap [\mathcal{I}nit_\psi] \neq \emptyset \text{ (i.e. } \langle x(0), x(2), \ldots, x(2n)\rangle^\frown v_n \in \mathcal{I}nit_\psi)$$

We set

(a) $x(2n+1) = \langle 1\rangle^\frown w_n$ iff there exists $w_n \in {\Lambda_\psi}^{<\omega}$ such that $v_n \subseteq w_n$ **and** $\langle x(0), x(2), \ldots, x(2n)\rangle^\frown w_n \in \mathcal{I}nit_\psi$. This gives

$$G_{n+1} = \langle x(0), x(2), \ldots, x(2n)\rangle^\frown w_n{}^\frown {\Lambda_\psi}^\omega \text{ and } F_{n+1} = F_n.$$

(b) $x(2n+1) = \langle 0\rangle$ otherwise. we notice that $[\mathcal{I}nit_\psi] \cap G_n \subseteq F_{\langle x(0), x(1), \ldots, x(2n)\rangle}$ is necessarily true. We then get

$$G_{n+1} = G_n \text{ and } F_{n+1} = F_n \cap F_{\langle x(0), x(1), \ldots, x(2n)\rangle}.$$

We remark that in both cases the following requirement holds:

$$G_{n+1} \cap [\mathcal{I}nit_\psi] \subseteq F_{n+1} \text{ and } G_{n+1} \cap [\mathcal{I}nit_\psi] \neq \emptyset$$

- On even moves, we let $x(2n+2)$ be anything such that

$$\langle x(0), x(2), \dots, x(2n), x(2n+2)\rangle^\frown {\Lambda_\psi}^\omega \subseteq [\mathcal{I}nit_\psi] \cap G_{n+1}.$$

We now show that x works i.e. $\forall k \in \omega \quad \psi^\sim \leq \psi^\sim[x \restriction k]$. It is enough to show that the property holds for k even. Set $k = 2n$.

First we remark that ψ being initializable there exists a strategy $\mathcal{T}$ that **always remains inside** $\mathcal{I}nit_\psi$ and is winning for $I\!I$ in the game

$$\mathbb{G}_\leq \left(\psi \ , \ \psi[\langle x(0), x(2), \dots, x(2n)\rangle^\frown v_n] \right).$$

This holds since the sequence $\langle x(0), x(2), \dots, x(2n)\rangle^\frown v_n$ belongs to $\mathcal{I}nit_\psi$ therefore

$\psi \leq \psi[\langle x(0), x(2), \dots, x(2n)\rangle^\frown v_n]$ holds. The mapping ψ being initializable there exists a w.s. for $I\!I$ in the game

$$\mathbb{G}_\leq \left(\psi^\frown \psi \ , \ \psi[\langle x(0), x(2), \dots, x(2n)\rangle^\frown v_n] \right).$$

But in that game, as long as I does not "cross the arrow" (that is plays a letter that is not in Λ_ψ) the w.s. requires $I\!I$'s run to remain inside $\mathcal{I}nit_\psi$ (otherwise I would have a w.s. then). From that w.s. we easily derive a strategy that remains in $\mathcal{I}nit_\psi$ and is winning for $I\!I$ in the game

$$\mathbb{G}_\leq \left(\psi \ , \ \psi[\langle x(0), x(2), \dots, x(2n)\rangle^\frown v_n] \right).$$

Hence there exists a strategy $\mathcal{T}$ winning for $I\!I$ in

$$\mathbb{G}_\leq \left(\psi \ , \ \psi[\langle x(0), x(2), \dots, x(2n)\rangle] \right)$$

such that $\forall \beta \in \Lambda_\phi^\omega \quad \mathcal{T} \star \beta \in G_n \cap [\mathcal{I}nit_\psi]$.

We set T' = the pullback of T by the strategy $\mathcal{T}$. Clearly $I\!I$ has a w.s. in the game

$$\mathbb{G}_\leq \left(\psi^{T'} \ , \ \psi^T[\langle x(0), x(1), \dots, x(2n)\rangle] \right)$$

By minimality of $\psi^\sim$ this shows

$$\psi^\sim \leq \psi^{T'} \leq \psi^T[\langle x(0), x(1), \dots, x(2n)\rangle] = \psi^\sim[x \restriction 2n+1]$$

($\Longleftarrow$) : We show that if ψ is not initializable then $\psi^\sim$ is target-free.

(a) If ψ has no target, then there exists σ, a w.s. for $I\!I$ in $\mathbb{G}_<(\psi, \psi)$. We assume T is the 1-$\mathcal{T}$ree such that $\psi^T = \psi^\sim$ and S is the 1-$\mathcal{T}$ree defined as the pullback of T by σ. From σ, T and S we derive $\mathcal{T}$ a strategy that is winning for $I\!I$ in $\mathbb{G}_< \left(\psi^S, \psi^T\right)$. (The strategy $\mathcal{T}$ is simply σ on even moves and the correct answers on odd moves. By minimality $I\!I$ has a winning strategy in $\mathbb{G}_<(\psi^\sim, \psi^\sim)$.

(b) If ψ has a target, the proof goes by induction on the degree of ψ. The $<$-least such ψ is $\mathbf{2}_\omega = (\mathbf{sup}_{n\in\mathbb{N}} C_n)^*$ and clearly $\mathbf{2}_\omega{}^{\sim} \equiv \sum_{n\in\mathbb{N}} C_n^*$ which does not have a target.

We now assume that $\psi > \mathbf{2}_\omega$ holds.

Claim 61 *Let* $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \boldsymbol{On}$,

$$(\phi^{\frown}\phi)^{\sim} \leq \phi^{\sim}.$$

□ **cl. 61** - Let T be a 1-*Tree* on Λ_ϕ such that $\phi^T = \phi^{\sim}$, set T' a 1-*Tree* on the alphabet of Λ_ϕ as the *Tree* that first of all asks the question whether or not the run is going to "cross the arrow" (this is a closed condition) and then, depending on the answer, asks the questions which are in T relatively to "ϕ before the arrow" or to "ϕ after the arrow". The result is then a direct consequence of the minimality of the *Tree* T' such that $(\phi^{\frown}\phi)^{T'} = (\phi^{\frown}\phi)^{\sim}$. 61□

We know that $\psi \equiv \psi_{Init}{}^{\frown}\psi_{Exit}$ is verified and, since ψ is not initializable, $\psi > \psi_{Init}$ and $\psi_{Init} < \psi_{Exit}$ are both true.

(i) If $\psi_{Exit} < \psi$ holds: then

$$\psi^{\sim} \equiv (\psi_{Init}{}^{\frown}\psi_{Exit})^{\sim} \leq (\psi_{Exit}{}^{\frown}\psi_{Exit})^{\sim} \leq \psi_{Exit}{}^{\sim} \leq \psi^{\sim}.$$

Since ψ is not initializable ψ_{Exit} is not initializable either; moreover $\psi_{Exit} < \psi$ holds. So by induction hypothesis we know that $\psi_{Exit}{}^{\sim}$ does not have a target. Hence $\psi^{\sim}$ which is equivalent to $\psi_{Exit}{}^{\sim}$ does not have a target either.

(ii) If $\psi_{Exit} \geq \psi$ holds: then clearly $\psi_{Exit} \equiv (\mathbf{sup}_{i\in I}\varphi_i)^*$ for some countable family of mappings $(\varphi_i)_{i\in I}$ that does not contain its upper bound and verifies $\varphi_i{}^* < \psi$. We assume that the alphabet of $\mathbf{sup}_{i\in I}\varphi_i$ is $\Lambda \supseteq I$. We know that $\sum_{\langle a\rangle\in\Lambda^1} \varphi_{\langle a\rangle}{}^*$ (where $\varphi_{\langle a\rangle}$ is φ_a when $a \in I$ and is C_0 otherwise) does not have a target.

Claim 62

$$\left(\sum_{\langle a\rangle\in\Lambda^1} \varphi_{\langle a\rangle}{}^*\right)^{\sim} \equiv (\mathbf{sup}_{i\in I}\varphi_i)^{*\sim}.$$

By the previous case we get the result since $\sum_{\langle a\rangle\in\Lambda^1} \varphi_{\langle a\rangle}{}^*$ does not have a target and is equivalent to $\psi^{\sim}$. 60 □

□ **cl. 62** - It is enough to prove $\left(\sum_{\langle a\rangle\in\Lambda^1} \varphi_{\langle a\rangle}{}^*\right)^{\sim} \geq (\mathbf{sup}_{i\in I}\varphi_i)^{\sim}$ since the fact that $^{\sim}$ preserves $\leq$ takes care of the other direction. Assume T is a 1-*Tree* such that $\left(\sum_{\langle a\rangle\in\Lambda^1} \varphi_{\langle a\rangle}{}^*\right)^{\sim} = \left(\sum_{\langle a\rangle\in\Lambda^1} \varphi_{\langle a\rangle}{}^*\right)^{T}$. Define T', a 1-*Tree* on the alphabet Λ of $(\mathbf{sup}_{i\in I}\varphi_i)^*$, as the *Tree* that asks as its first question: "Is the main run going to be outside the open set $\bigcup_{n\in\omega, i\in I} \{\langle\underbrace{*,\ldots,*}_{n}, i\rangle^{\frown}\gamma : \gamma \in \Lambda\}$ or not". And then depending on the answer:

- If the answer is "yes", non of the next questions on T' matters.
- If the answer is "no", then it takes the form $\langle\underbrace{*,\ldots,*}_{n}, i\rangle$ for some integer n and some letter $i \in I$. the next questions on T' are exactly the ones of the part of T that corresponds to the choice of $\phi^*_{\langle i\rangle}$.

62□

We have just settled the case $\phi \longmapsto \phi^{\propto\lambda}$ for $\lambda = 1$. We will now generalize this to any countable λ.

Proposition 63 *Let* $\psi : \Lambda_\psi{}^\omega \overset{\Delta^1_1}{\longmapsto} \boldsymbol{On}$ *be a Borel mapping that* **admits a target**, $1 < \lambda < \omega_1$

$$(\forall \xi < \lambda \;\; \psi \leq \psi^{\propto \xi}) \Longrightarrow \psi^{\propto\lambda} \;\; \textit{has a target.}$$

The converse is false.

□ **prop.63** - We prove it by induction on λ.

Assume $\boldsymbol{\lambda = \nu + 1}$: the result is direct since ϕ has a target and the following holds

$$\psi^{\propto \nu+1} \equiv (\psi^{\propto\nu})^{\propto} \equiv \psi^{\propto} \equiv \psi.$$

Assume $\boldsymbol{\lambda}$ is limit: then the result is a consequence of the axiom of Dependant Choice and the following

Claim 64 *Assume* $(\forall \xi < \lambda \;\; \psi \leq \psi^{\propto\xi})$ *and* $\psi^{\propto\lambda} = \psi^T$, $u \in T$ *(*$u \in T^\zeta$ *for some* ζ *large enough (*$\zeta < \lambda$*)) and* u *satisfies*

$$\forall \xi \;\; \left(\zeta \leq \xi < \lambda \implies \psi^{T^\xi}[u] \geq \psi^{T^\xi}\right)$$

then:

(a) $\psi^T[u] \geq \psi^T$

(b) $\exists\, t \;\; u^\frown\langle t\rangle \in T,\; u^\frown\langle t\rangle \in T^\nu$ *(for some* ν *large enough that verifies* $\zeta \leq \nu < \lambda$*) such that* t *satisfies*

$$\forall \mu \;\; \left(\nu \leq \mu < \lambda \implies \psi^{T^\mu}[u^\frown\langle t\rangle] \geq \psi^{T^\mu}\right).$$

□ **cl. 64** - (a) relies on the fact $\psi^T[u] \geq \left(\psi^{T^\zeta}[u]\right)^{T'}$ where T' is some θ-Tree (with $\zeta + \theta = \lambda$) which is the "rest" of T when T^ζ has been removed. Thus

$$\psi^T[u] \geq \left(\psi^{T^\zeta}[u]\right)^{T'} \geq \left(\psi^{T^\zeta}[u]\right)^{\propto\theta} \geq \left(\psi^{T^\zeta}\right)^{\propto\theta} \geq \left(\psi^{\propto\zeta}\right)^{\propto\theta} \equiv \psi^{\propto(\zeta+\theta)} = \psi^{\propto\lambda} = \psi^T.$$

(b) Let ν be the least ordinal such that every sequence of T of length exactly $lh(u) + 1$ is in T^ν. One obtains

$$\psi \geq \psi^{T^\nu}[u] \geq \psi^{T^\nu} \geq \psi$$

so that $\psi^{T^\nu}[u] \equiv \psi$. Therefore there exists t such that

$$\left(\psi^{T^\nu}[u]\right)[\langle t\rangle] \equiv \psi, \text{ thus } \psi^{T^\nu}[u^\frown\langle t\rangle] \equiv \psi$$

and for any $\mu > \nu$ ($\mu < \lambda$)

$$\psi^{T^\mu}[u^\frown\langle t\rangle] \geq \left(\psi^{T^\nu}[u^\frown\langle t\rangle]\right)^{T''}$$

where T'' is the "rest" of T^μ when T^ν has been removed; T'' is some δ-Tree with δ satisfying $\nu + \delta = \mu$. Hence

$$\psi^{T^\mu}[u^\frown\langle t\rangle] \geq \left(\psi^{T^\nu}[u^\frown\langle t\rangle]\right)^{T''} \geq \left(\psi^{T^\nu}[u^\frown\langle t\rangle]\right)^{\sim\delta} \geq \psi^{\sim\delta} \geq \psi \geq \psi^{T_\mu}.$$

64$\square$

Now consider $\mathcal{I}nit_{\phi T} = \{u \in T : \psi^T[u] \geq \psi^T\}$. This set is non empty since the empty sequence belongs to it. And by application of the above claim, if u is in $\mathcal{I}nit_{\phi T}$ then there exists a strict extension of u that is still in $\mathcal{I}nit_{\phi T}$. In other words $\mathcal{I}nit_{\phi T}$ is a non empty pruned tree. By Dependant Choice, let x be an infinite branch of $\mathcal{I}nit_{\phi T}$, it satisfies the following

$$\forall n \in \omega \quad \psi^{\sim\lambda}[x \restriction n] = \psi^T[x \restriction n] \geq \psi^T = \psi^{\sim\lambda}.$$

63 $\square$

12 Eliminating inverse Veblen-like operations with Veblen conciliatory ones

We designed the inverse Veblen operation so that it lowers the degree of a Borel mapping. This will allow us to proceed by induction on the degree. Given a mapping of lower degree (of the form $\phi^{*\sim\xi}$) we will be able to apply the induction hypothesis to this function as long as it is equivalent to some conciliatory one. Last section settled that issue, by providing the sufficient conditions on ξ and ϕ that guarantee this equivalence. Then, in order to go back from $\phi^{*\sim\xi}$ to ϕ, we need to find a way to "eliminate" the auxiliary moves. This section is devoted to show that ω^ξ erasers are just what is needed to "neutralize" auxiliary moves in a ω^ξ-Tree.

Proposition 65 *Let $\psi : \Lambda_\psi{}^\omega \overset{\mathbf{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$, $\phi : \Lambda_\phi{}^{\leq\omega} \overset{\mathbf{\Delta}^1_1}{\longmapsto} \boldsymbol{On}$, ψ and ϕ^* both Borel, $0 \leq \lambda < \omega_1$ and T some λ-Treeon Λ_ψ,*

- *(a)* $\psi^T \leq \phi^* \Longrightarrow \psi \leq \left(\phi^{\approx\lambda}\right)^*$
- *(b)* $\psi^T < \phi^* \Longrightarrow \psi < \left(\phi^{\approx\lambda}\right)^*$
- *(c)* $\mathbf{V}_\lambda^{-1} \circ \psi \leq \phi^* \Longrightarrow \psi \leq (\mathbf{V}_\lambda \circ \phi)^*$
- *(d)* $\mathbf{V}_\lambda^{-1} \circ \psi < \phi^* \Longrightarrow \psi < (\mathbf{V}_\lambda \circ \phi)^*$.

The proof is a consequence of claim 67 below based on another game which was originally introduced to study the Wadge hierarchy in [3]:

Definition 66 *Given λ some countable ordinal and T some λ-Tree on Λ, the game $\mathbf{G}_\lambda(T)$ is a two player infinite game where*

- *I starts and plays letters in Λ, hence builds the sequence $x \in \Lambda^\omega$.*
- *II plays letters in $\Lambda_T \cup \mathcal{A}_\lambda$ (Λ_T denotes the alphabet of the Tree T and $\mathcal{A}_\lambda$ is the set of arrows of different types involved by operation ${}^{\approx\lambda}$). So II produces $y \in (\Lambda_T \cup \mathcal{A}_\lambda)^\omega$*

$$
G_\lambda(T) \quad
\begin{array}{c|ccccccccccc|c}
\mathrm{I} & x(0) & & x(1) & & \ldots\ldots\ldots & & x(p) & & \ldots\ldots & & & x \\
 & & \searrow \nearrow & & \searrow \nearrow & & \nearrow & & \searrow & & & & \\
\mathrm{II} & & x(0) & & x(1) & \ldots\ldots\ldots & x(p\text{-}1) & & & x(p) & \ldots & & x
\end{array}
$$

$$\text{\textit{II} wins } \mathbf{G}_\lambda(T) \text{ iff } ((y^\odot)^0 = x \textbf{ and } y^\odot \in [T])$$

I wins otherwise.

Claim 67 *Given T any λ-Tree on Λ,*

$$\textit{II} \quad \text{has a w.s. in } \mathbf{G}_\lambda(T).$$

□ **cl. 67** - This was proved in [3]; where it appears as Claim 33.

67□

□ **prop.65** -

(a) Let δ be a w.s. for *II* in the game $\mathbb{G}_\leq\left(\psi^T, \phi^*\right)$. By claim 67 player *II* has a w.s. ${}^{\Lambda,T}\sigma_\lambda$ in the game $\mathbf{G}_\lambda(T)$. Let δ^λ be a strategy that "λ-remote controls" δ. To be more precise, assumming the C.N.F of λ is $\lambda = \omega^{\lambda_k}.n_k + \ldots + \omega^{\lambda_0}.n_0$, let $\delta^{\omega^{\lambda_0}.n_0}$ be a strategy that $\omega^{\lambda_0}.n_0$-remote controls δ, then $\delta^{\omega^{\lambda_1}.n_1+\omega^{\lambda_0}.n_0}$ a strategy that $\omega^{\lambda_1}.n_1$-remote controls $\delta^{\omega^{\lambda_0}.n_0}$,..., and finally $\delta^{\omega^{\lambda_k}.n_k+\ldots+\omega^{\lambda_0}.n_0}$ a strategy that $\omega^{\lambda_k}.n_k$-remote controls $\delta^{\omega^{\lambda_{k-1}}.n_{k-1}+\ldots+\omega^{\lambda_1}.n_1+\omega^{\lambda_0}.n_0}$. The strategy $\delta^\lambda \circ {}^{\Lambda,T}\sigma_\lambda$ is winning for *II* in $\mathbb{G}_\leq\left(\psi, \left(\phi^{\approx\lambda}\right)^*\right)$ because when *I* plays x and *II* replies with $\delta^\lambda \circ {}^{\Lambda,T}\sigma_\lambda * x = y$ one has by definition of ${}^{\Lambda,T}\sigma_\lambda$:

$$({}^{\Lambda,T}\sigma_\lambda * x)^\odot \in [T] \ \wedge \ \left(({}^{\Lambda,T}\sigma_\lambda * x)^{\circlearrowleft 0}\right) = x \ \wedge \ \left(({}^{\Lambda,T}\sigma_\lambda * x)^{\circlearrowleft 0}\right) = \delta{\star}x'.$$

for x' some s-variant of x. Thus $\phi^{\approx\lambda}(y) = \phi(\delta{\star}\gamma) \geq \psi(x)$.

(b) The same argument with the game $\mathbb{G}_<\left(\psi^T, \phi^*\right)$ instead of $\mathbb{G}_\leq\left(\psi^T, \phi^*\right)$ works.

(c) and (d) are immediate from respectively (a) and (b).

65 □

13 Where the wild things are: reaching Borel mappings with conciliatory Veblen operations

We answer the question of how high conciliatory conciliatory Veblen operations can "boost" a mapping inside the Borel hierarchy. We first establish that it doesn't exceed the Borel domain.

Proposition 68 *Let* $\phi: \Lambda_\phi^{\leq\omega} \xrightarrow{\mathbf{\Delta}^1_1} \boldsymbol{On}$, $0 < \delta < \omega_1$

$$\phi \text{ is Borel} \implies \mathbf{V}_\delta \circ \phi \text{ is Borel.}$$

$\square$ **prop.68** - In [3] notation 36 and definition 17, we defined an operation on sets of the form $A \subseteq \Lambda_A^{\leq\omega}$ where Λ_A was a set of any cardinality less than some regular cardinal κ. Given any countable ordinal λ, it was denoted $A \longmapsto {}^{\kappa}V_c^{\lambda}(A)$; the set ${}^{\kappa}V_c^{\lambda}(A)$ being a subset of $(\Lambda_A \cup \mathcal{A}_\lambda)^{\leq\omega}$. In the context of this paper, we are only interested in $\kappa = \omega_1$ and for clarity we rename it $\mathbf{W}^\lambda$ (i.e. ${}^{\omega_1}V_c^{\lambda}(A) = \mathbf{W}^\lambda(A)$).
We recall that χ_S stands for the characteristic function of the set S. The very definition of $\mathbf{W}^\lambda(A)$ gives the following relation: $\chi_{\mathbf{W}^\lambda(A)} = \mathbf{V}_\lambda \circ \chi_A$. In [3], theorem 2 and proposition 41 established that $\mathbf{W}^\lambda(A)$ is Borel whenever A is Borel. Translated in the vocabulary of this paper, this says that if χ_A is Borel, $\mathbf{V}_\lambda \circ \chi_A$ is Borel too; a fact that we now use to get the result.
Clearly $\mathcal{I}m(\mathbf{V}_\delta \circ \phi) = \mathcal{I}m(\phi)$. Let $\xi \in \mathcal{I}m(\phi)$,

$$x \in \xi_{\mathbf{V}_\delta \circ \phi} \iff \mathbf{V}_\delta \circ \phi(x) = \xi \iff \phi(x^{\circlearrowleft}) = \xi \iff \chi_{\xi_\phi}(x^{\circlearrowleft}) = 1 \iff \mathbf{V}_\delta \circ \chi_{\xi_\phi}(x) = 1$$

By [3], $\mathbf{W}^\delta \xi_\phi$ is Borel which gives the result. 68 $\square$
Now that we know that $\mathbf{V}_\lambda \circ$ operations do not produce non Borel mappings starting from Borel ones, we show that they give simple bounds for any Borel function.

Proposition 69 *Let* $\phi : \Lambda_\phi{}^{\leq\omega} \xrightarrow{\mathbf{\Delta}_1^1} \delta$ *with* δ *some limit ordinal of countable cofinality.*

$$\phi \text{ is Borel} \implies \exists \nu < \omega_1 \quad \phi \leq \mathbf{V}_\nu \circ \mathbf{2}_\delta.$$

Where $\mathbf{2}_\delta$ stands for $\mathbf{sup}_{n\in\mathbb{N}} C_{\delta_n}$ for any sequence $(\delta_n)_{n\in\mathbb{N}}$ that is cofinal in δ.
$\square$ **prop.69** - By definition ϕ is Borel iff ξ_ϕ is Borel for all $\xi \in \mathcal{I}m(\phi)$ (*stricto sensu* $\xi_\phi{}^*$ is Borel in $\Lambda_\phi{}^*$). Hence there exists some $\zeta < \omega_1$ such that $\xi_\phi \in \mathbf{\Sigma}_\zeta^\circ$ holds for any $\xi \in \mathcal{I}m(\phi)$. It is very easy to see by induction on ζ that, for each $\xi \in \mathcal{I}m(\phi)$, there exists a ζ-$\mathcal{T}$ree T'_ξ such that $(\chi_{\xi_\phi^*})^{T'_\xi}$ is open. So, for each $\xi \in \mathcal{I}m(\phi)$, there exists a $\zeta+1$-tree T_n such that $(\chi_{\xi_\phi^*})^{T'_\xi}$ is clopen **and** one of the "questions" that T_n "decides" is "will the run ultimately be in $\xi_\phi{}^*$ or not ?". Then get T, any $\zeta+1$-$\mathcal{T}$ree that combines all $\mathcal{T}$rees T_ξ. That is, for each $\xi \in \mathcal{I}m(\phi)$, there is an embedding from T_ξ into T.
Player *II* clearly has a winning strategy in the game $\mathbb{G}_{\leq}\left(\phi^T, \mathbf{2}_\delta\right)$ which consists in waiting until *I* answers *yes* to one of the relevant questions of T about the $\xi_\phi{}^*$. This way he knows after a finite number of skips how big he must play to win. In other words, there is necessarily a position reached by *I* that corresponds to committing to play x such that $\phi(x^0) = \xi$ for some $\xi \in \mathcal{I}m(\phi)$. Player *II* then stops skipping and plays any y that satisfies $\mathbf{2}_\delta(y) > \xi$.
Therefore $\phi^{*T} \leq \mathbf{2}_\delta{}^*$ holds, and prop 65 gives $\phi \leq \mathbf{2}_\delta{}^{\approx\zeta+1}$. Finally any ν such that $\omega^\nu \geq \zeta+1$ gives the result $\phi \leq \mathbf{V}_\nu \circ \mathbf{2}_\delta$.

69 $\square$

Now that we know $\mathbf{V}_\xi \circ \phi$ is Borel whenever ϕ is Borel, we can state the result showing that $\mathbf{V}_\xi^{-1}$ stands as the quasi-inverse of $\mathbf{V}_\xi$.

Proposition 70 *Let* $\phi : \Lambda_\phi{}^{\leq\omega} \xrightarrow{\mathbf{\Delta}_1^1} \boldsymbol{On}$ *be Borel and* $0 \leq \xi < \omega_1$

$$\mathbf{V}_\xi^{-1} \circ (\mathbf{V}_\xi \circ \phi)^* \equiv \phi^*.$$

□ **prop.70 -** $\mathbf{V}_\xi^{-1} \circ (\mathbf{V}_\xi \circ \phi)^* \leq \phi^*$ comes from proposition 59. Towards a contradiction, assume

$$\mathbf{V}_\xi^{-1} \circ (\mathbf{V}_\xi \circ \phi)^* \not\geq \phi^*$$

By proposition 68, the mapping $\mathbf{V}_\xi^{-1} \circ (\mathbf{V}_\xi \circ \phi)^*$ is Borel. And by proposition 65(d)

$$\mathbf{V}_\xi^{-1} \circ (\mathbf{V}_\xi \circ \phi)^* < \phi^* \implies (\mathbf{V}_\xi \circ \phi)^* < (\mathbf{V}_\xi \circ \phi)^*.$$

A contradiction.

70 □

14 The isomorphism between ordinals and conciliatory Veblen operations

In this section we establish the ordinal behavior of the conciliatory Veblen operations. All the tedious previous work around the notion of $\mathcal{T}$rees reveals its motivation in this result.

Proposition 71 *Let $\phi: \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta_1^1} \boldsymbol{On}$ be Borel with $tg\,(\phi) = 0$ or limit*

$$d_l^\circ\,(\mathbf{V}_\xi \circ \phi) = V_\xi(d_l^\circ\,(\phi)).$$

We postpone the proof of this proposition which relies entirely on the following lemma.

Lemma 72 *Let $cof(\delta) = \omega$, $\phi: \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta_1^1} \delta$ be Borel, unbounded in δ and $tg\,(\phi) = 0$ or limit*

(a) $\{\psi : \psi$ borel $\wedge\ \psi < \mathbf{V}_\xi \circ \phi\}$ is closed under $Shift$, $+$, countable $\mathbf{sup}$, $\bullet\,\nu$ $(0 < \nu < \omega_1)$, and $\mathbf{V}_\zeta \quad (\zeta < \xi)$

(b) $\psi < \mathbf{V}_\xi \circ \phi \implies \exists \varphi \in \mathcal{C}\,(\mathbf{V}_\xi \circ \phi) \quad \psi \leq \varphi$
where $\mathcal{C}\,(\mathbf{V}_\xi \circ \phi)$ stands for the closure of $\{\mathbf{V}_\xi \circ \rho : \rho < \phi,\ tg\,(\rho) = 0$ or limit$\}$ by the operations $shift$, $+$, countable $\mathbf{sup}$, $\bullet\,\nu$ $(0 < \nu < \omega_1)$, and $\mathbf{V}_\zeta \quad (\zeta < \xi)$.

□ **lem. 72 -**

(a) Assume $\psi < \mathbf{V}_\xi \circ \phi$, we need to show that $S_{hift} \circ \psi < \mathbf{V}_\xi \circ \phi$ is still true. But $tg\,(\mathbf{V}_\xi \circ \phi) = tg\,(\phi) = 0$ or limit and $S_{hift} \circ \psi \leq \mathbf{V}_\xi \circ \phi$ both hold. Since $tg\,(\mathbf{V}_\xi \circ \phi) \neq tg\,(S_{hift} \circ \psi)$ we conclude that $S_{hift} \circ \psi < \mathbf{V}_\xi \circ \phi$

(b) Assume $\psi_i < \mathbf{V}_\xi \circ \phi$ (for any $i \in I$), we need to show that $\mathbf{sup}_{i \in I} \psi_i < \mathbf{V}_\xi \circ \phi$ still holds which is obvious since $\mathbf{V}_\xi \circ \phi$ is unbounded and initializable.

(c) Assume $\psi_0,\ \psi_1 < \mathbf{V}_\xi \circ \phi$ we need to show that $\psi_1 + \psi_0 = \psi_0{}^\frown \mathbf{sup}_n S_{hift}{}^n \circ \psi_1$. We already know by (a) and (b) that $\mathbf{sup}_n S_{hift}{}^n \circ \psi_1 < \mathbf{V}_\xi \circ \phi$ holds. Initializability of $\mathbf{V}_\xi \circ \phi$ gives the result.

(d) Assume $\psi < \mathbf{V}_\xi \circ \phi$, we need to show that $\psi \bullet \nu < \mathbf{V}_\xi \circ \phi$ is true. Here too (b), (c) and the initializability of $\mathbf{V}_\xi \circ \phi$ give the result.

(e) Assume $\psi < \mathbf{V}_\xi \circ \phi$, we need to show that $\mathbf{V}_\zeta \circ \psi < \mathbf{V}_\xi \circ \phi$. By remote control (proposition 44(c)) we have $\mathbf{V}_\zeta \circ \psi < \mathbf{V}_\zeta \circ \mathbf{V}_\xi \circ \phi$, then by proposition 44(e) we know that $\mathbf{V}_\zeta \circ \mathbf{V}_\xi \circ \phi \equiv \mathbf{V}_\xi \circ \phi$, thus we obtain the result.

The proof goes by induction on $d_l^\circ(\psi)$. If $d_l^\circ(\psi) = 1$ then $\psi = C_0$ and $C_0 = \mathbf{V}_\xi \circ C_0$ belongs to $\mathcal{C}(\mathbf{V}_\xi \circ \phi)$.

If ψ is not initilizable then $\psi \equiv \psi_{Init} \overset{\frown}{\rightarrow} \psi_{Exit}$ with $\psi_{Init} < \psi$

(a) Assume $\psi_{Exit} < \psi$. Then by induction hypothesis there exists $\psi_0,\ \psi_1 \in \mathcal{C}(\mathbf{V}_\xi \circ \phi)$ such that $\psi_0 \geq \psi_{Init}$ and $\psi_1 \geq \psi_{Exit}$. We get

$$\psi \leq \psi_0 \overset{\frown}{\rightarrow} \psi_{Exit} \leq \psi_1 + \psi_0 \in \mathcal{C}(\mathbf{V}_\xi \circ \phi)$$

(b) Assume $\psi_{Exit} \geq \psi$. Then there exists $\{\psi_i \ :\ i \in I\}$, $card(I) < \omega_1$, each $\psi_i < \psi_{Exit}$ such that $\psi_{Exit} \equiv \mathbf{sup}_{i \in I} \psi_i$. By induction hypothesis $\psi_i \leq \varphi_i$ holds for some φ_i belonging to $\mathcal{C}(\mathbf{V}_\xi \circ \phi)$. We get

$$\psi \leq \psi_{Exit} \leq \mathbf{sup}_{i \in I} \psi_i \leq \mathbf{sup}_{i \in I} \varphi_i \in \mathcal{C}(\mathbf{V}_\xi \circ \phi).$$

If ψ is initilizable

(a) Either ϕ is equivalent to some constant function: $\phi \equiv C_0$ or $\phi \equiv C_\delta$ for some limit ordinal δ. In which case $\mathbf{V}_\xi \circ \phi \equiv \phi$ holds and $\psi < \mathbf{V}_\xi \circ \phi$ iff $\psi < \phi$; hence $\psi \in \mathcal{C}(\mathbf{V}_\xi \circ \phi)$.

(b) Or ϕ is unbounded in some limit ordinal of countable cofinality $\delta \leq \eta$.

- If ψ is bounded in δ then $\psi \in \mathcal{C}(\mathbf{V}_\xi \circ \phi)$.
- If ψ is unbounded in δ then, by propositions 68 and 69, since ψ is Borel, there exists some countable ordinal λ such that $\mathbf{V}_\xi \circ \psi \leq \mathbf{V}_\lambda \circ \mathbf{2}_\delta$. Moreover, since $\mathbf{2}_\delta \leq \psi$ holds, we obtain

$$(\mathbf{V}_\xi \circ \psi)^{\backsim(\omega^\lambda+1)} \leq (\mathbf{V}_\lambda \circ \mathbf{2}_\delta)^{\backsim(\omega^\lambda+1)} \leq \left((\mathbf{V}_\lambda \circ \mathbf{2}_\delta)^{\backsim\omega^\lambda}\right)^{\backsim 1} \leq \mathbf{2}_\delta{}^{\backsim} < \mathbf{2}_\delta \leq \psi$$

Therefore there exists a $<$-least ordinal ζ such that $\psi^{\backsim\zeta} < \psi$. By propositions 60 and 63 $\psi^{\backsim\zeta}$ does have a target. Three cases appear:

(i) Assume $\zeta < \omega^\xi$. By induction hypothesis there exists $\varphi \in \mathcal{C}(\mathbf{V}_\xi \circ \phi)$ such that $\psi^{\backsim\zeta} \leq \varphi$. Thus, there exists an integer n and an ordinal $\theta < \xi$ such that $\zeta < \omega^\theta \cdot n < \omega^\xi$. By proposition 65(b) we end up with

$$\psi \leq \varphi^{\approx\zeta} \leq \underbrace{\mathbf{V}_\theta \circ \ldots \mathbf{V}_\theta}_{n} \circ \varphi \ \in \mathcal{C}(\mathbf{V}_\xi \circ \phi)$$

(ii) Assume $\zeta = \omega^\xi$. Then $\psi^{\backsim\zeta} \equiv \mathbf{V}_\xi^{-1} \circ \psi$. By proposition 55(b) player *II* has a winning strategy in the strict game $\mathbb{G}_<\left(\mathbf{V}_\xi^{-1} \circ \psi, \mathbf{V}_\xi^{-1} \circ \mathbf{V}_\xi \circ \phi\right)$. Since both payoff mappings $\mathbf{V}_\xi^{-1} \circ \psi$ and $\mathbf{V}_\xi^{-1} \circ \mathbf{V}_\xi \circ \phi$ have a target, this implies

$$\mathbf{V}_\xi^{-1} \circ \psi < \mathbf{V}_\xi^{-1} \circ \mathbf{V}_\xi \circ \phi \equiv \phi.$$

By corollary 25 there exists some mapping $\varphi < \phi$ with $tg(\varphi) = 0$ or limit and some integer n such that $\mathbf{V}_\xi^{-1} \circ \psi \equiv \mathcal{S}_{hift}{}^n \circ \varphi$ holds. By proposition 44(d) (remote control) and proposition 65(c) we obtain:

$$\mathbf{V}_\xi^{-1} \circ \psi < \phi \implies \mathcal{S}_{hift}{}^n \circ \varphi < \phi \implies \mathbf{V}_\xi \circ \mathcal{S}_{hift}{}^n \circ \varphi < \mathbf{V}_\xi \circ \phi$$

And

$$\mathbf{V}_\xi^{-1} \circ \psi \equiv \mathcal{S}_{hift}{}^n \circ \varphi \implies \psi \leq \mathbf{V}_\xi \circ \mathcal{S}_{hift}{}^n \circ \phi$$

But

$$\mathbf{V}_\xi \circ \mathcal{S}_{hift}{}^n \circ \varphi = \mathcal{S}_{hift}{}^n \circ \mathbf{V}_\xi \circ \varphi \in \mathcal{C}(\mathbf{V}_\xi \circ \phi).$$

(iii) Assume $\zeta > \omega^\xi$. Then $\mathbf{V}_\xi^{-1} \circ \psi \equiv \psi$ holds and by proposition 65(d), since player *II* has a winning strategy in the strict game $\mathbb{G}_<(\psi, \mathbf{V}_\xi \circ \phi)$, this player still has a winning strategy in $\mathbb{G}_<\left(\mathbf{V}_\xi^{-1} \circ \psi, \mathbf{V}_\xi^{-1} \circ \mathbf{V}_\xi \circ \phi\right)$. The fact both payoff mappings have a target leads to

$$\psi \equiv \mathbf{V}_\xi^{-1} \circ \psi < \mathbf{V}_\xi^{-1} \circ \mathbf{V}_\xi \circ \phi \equiv \phi$$

Hence ψ already belongs to $\mathcal{C}(\mathbf{V}_\xi \circ \phi)$.

72□

In order to prove proposition 71, we have to compute the degree of the constant functions. So we state

Proposition 73 *Let $\delta > 0$,*

$$d_l^\circ(C_{\omega\cdot\delta}) = V_{\omega_1}(1+\delta).$$

And we prove both propositions 71 and 73 together.

□ **pro. 71 & 73** - For the case $\eta < \omega$ everything is obvious since the only mapping with target 0 or limit is C_0.

We assume $\omega \leq \eta = \omega \cdot \delta + n$ with $n \in \mathbb{N}$. We clearly see that the constant function $C_{\omega\cdot\delta}$ is the $<$-least mapping above all Borel mapping with image included in $\omega \cdot \delta$. We also prove by induction on δ that

$$d_l^\circ(C_{\omega\cdot\delta}) = \sup\{d_l^\circ(\phi) : \phi : \Lambda_\phi{}^{\leq\omega} \xrightarrow{\Delta_1^1} \omega \cdot \delta \text{ Borel } \wedge \ tg(\phi) = 0 \text{ or limit}\} = V_{\omega_1}(1+\delta)$$

(a) If $d_l^\circ(\phi) = 1$ then $\phi = C_0 = \mathbf{V}_\xi \circ C_0 = \mathbf{V}_\xi \circ \phi$. Thus $d_l^\circ(\mathbf{V}_\xi \circ \phi) = 1 = V_\xi(1) = V_\xi(d_l^\circ(\phi))$.

(b) If $d_l^\circ(\phi) > 1$ and there exists $\gamma \leq \omega \cdot \delta$ of countable cofinality such that $\mathcal{I}m(\phi) \subseteq \gamma$ and ϕ is unbounded in γ. Let $\mathcal{C}losure(V_\xi(\theta))$ denote the smallest set that contains all ordinals less than θ and is closed under ordinal addition, countable multiplication, supremum for sequences of countable cofinality and all Veblen operations V_ζ for any $\zeta < \xi$. In other words, $\mathcal{C}losure(V_\xi(\theta))$ is exactly the **set** $V_\xi(\theta)$.
By induction on ξ and for ξ fixed by induction on $d_l^\circ(\phi)$ we apply lemma 72(a):

$$\begin{array}{rcl} d_l^\circ(\mathbf{V}_\xi \circ \phi) & = & \sup\{(d_l^\circ(\psi)) + 1 : \ tg(\psi) = 0 \text{ or limit } \wedge \ \psi < \mathbf{V}_\xi \circ \phi\} \\ & \leq & \sup\{(d_l^\circ(\psi)) + 1 : \ tg(\psi) = 0 \text{ or limit } \wedge \ \psi \in \mathcal{C}(\mathbf{V}_\xi \circ \phi)\} \\ & \leq & \sup\{(d_l^\circ(\psi)) + 1 : \ tg(\psi) = 0 \text{ or limit } \wedge \ d_l^\circ(\psi) \in \mathcal{C}losure(V_\xi(d_l^\circ(\phi)))\} \\ & \leq & \sup\{(d_l^\circ(\psi)) + 1 : \ tg(\psi) = 0 \text{ or limit } \wedge \ d_l^\circ(\psi) < V_\xi(d_l^\circ(\phi))\} \\ & \leq & V_\xi(d_l^\circ(\phi)). \end{array}$$

For the other direction we apply lemma 72(b). Since $\mathbf{V}_\xi \circ \phi$ is closed under operations $+$, countable **sup**, $\bullet\, \nu$ $(0 < \nu < \omega_1)$, and $\mathbf{V}_\zeta$ $(\zeta < \xi)$ we have

$$d_l^\circ\,(\mathbf{V}_\xi \circ \phi) \geq \sup\,(\mathcal{C}losure\,(V_\xi(d_l^\circ\,(\phi)))) = V_\xi(d_l^\circ\,(\phi)).$$

(c) If $d_l^\circ\,(\phi) > 1$ and ϕ is bounded inside $\mathcal{I}m(\phi)$. Then $\phi \equiv C_\lambda$ for some limit ordinal λ. And $\mathbf{V}_\xi \circ \phi = \mathbf{V}_\xi \circ C_\lambda = C_\lambda$. So we just have to compute $d_l^\circ\,(C_\lambda)$ depending on the cofinality of λ.

If $\boldsymbol{cof(\lambda) = \omega}$: By propositions 69 and 68

$$\begin{aligned} d_l^\circ\,(C_\lambda) &= \sup\{d_l^\circ\,(\psi) + 1 :\ \psi \text{ Borel } \wedge\ tg\,(\psi) = 0 \text{ or limit } \wedge\ \psi < C_\lambda\} \\ &= \sup\{d_l^\circ\,(\psi) + 1 :\ \psi \text{ Borel } \wedge\ \mathcal{I}m(\psi) \subseteq \lambda\ \wedge\ tg\,(\psi) = 0 \text{ or limit}\} \\ &= \sup\{d_l^\circ\,(\mathbf{V}_\xi \circ \mathbf{2}_\lambda) + 1 :\ \xi < \omega_1\} \\ &= \sup\{V_\xi(d_l^\circ\,(\mathbf{2}_\lambda)) + 1 :\ \xi < \omega_1\} \end{aligned}$$

It just remains to compute $d_l^\circ\,(\mathbf{2}_\lambda)$ in order to get $d_l^\circ\,(C_\lambda)$. The proof goes by induction on λ. We distinguish three cases:

(i) If $\lambda = \omega$ then $\mathbf{2}_\lambda = \mathbf{2}_\omega \equiv \sup_{n\in\mathbb{N}} C_n$. Hence $d_l^\circ\,(\mathbf{2}_\lambda) = d_l^\circ\,(\sup_{n\in\mathbb{N}} C_n) = 2$ and

$$\begin{aligned} d_l^\circ\,(C_\omega) &= \sup\{d_l^\circ\,(\psi) + 1 :\ \psi \text{ Borel } \wedge\ tg\,(\psi) = 0 \text{ or limit } \wedge\ \psi < C_\omega\} \\ &= \sup\{d_l^\circ\,(\psi) + 1 :\ \psi \text{ Borel } \wedge\ \mathcal{I}m(\psi) \subseteq \mathbb{N}\ \wedge\ tg\,(\psi) = 0 \text{ or limit}\} \\ &= \sup\{d_l^\circ\,(\mathbf{V}_\xi \circ \mathbf{2}_\omega) + 1 :\ \xi < \omega_1\} \\ &= \sup\{V_\xi(d_l^\circ\,(\mathbf{2}_\omega)) + 1 :\ \xi < \omega_1\} \\ &= \sup\{V_\xi(2) + 1 :\ \xi < \omega_1\} \\ &= V_{\omega_1}(2) = V_{\omega_1}(1+1) \end{aligned}$$

(ii) If $\lambda = \omega \cdot (\alpha + 1)$ then clearly $\mathbf{2}_\lambda \equiv \sup_{n\in\mathbb{N}} C_{\omega\cdot\alpha+n}$. and it is the $<$-least mapping with target 0 or limit above $C_{\omega\cdot\alpha}$. Hence $d_l^\circ\,(\mathbf{2}_\lambda) = d_l^\circ\,(C_{\omega\cdot\alpha}) + 1$. By induction hypothesis $d_l^\circ\,(C_{\omega\cdot\alpha}) = V_{\omega_1}(1+\alpha)$, thus $d_l^\circ\,(\mathbf{2}_\lambda) = V_{\omega_1}(1+\alpha) + 1$. And

$$\begin{aligned} d_l^\circ\left(C_{\omega\cdot(\alpha+1)}\right) &= \sup\{d_l^\circ\,(\psi) + 1 :\ \psi \text{ Borel } \wedge\ tg\,(\psi) = 0 \text{ or limit } \wedge\ \psi < C_{\omega\cdot(\alpha+1)}\} \\ &= \sup\{d_l^\circ\,(\psi) + 1 :\ tg\,(\psi) = 0 \text{ or limit } \wedge\ \psi :\ \Lambda_\psi{}^{\leq\omega} \xmapsto{\mathbf{\Delta}_1^1} \omega \cdot (\alpha+1)\} \\ &= \sup\{d_l^\circ\left(\mathbf{V}_\xi \circ \mathbf{2}_{\omega\cdot(\alpha+1)}\right) + 1 :\ \xi < \omega_1\} \\ &= \sup\{V_\xi(d_l^\circ\,(\mathbf{2}_\lambda)) + 1 :\ \xi < \omega_1\} \\ &= \sup\{V_\xi(V_{\omega_1}(1+\alpha) + 1) + 1 :\ \xi < \omega_1\} \\ &= \sup\{V_\xi(V_{\omega_1}(1+\alpha) + 1) :\ \xi < \omega_1\} \\ &= V_{\omega_1}(1 + (\alpha+1)) \end{aligned}$$

(iii) If $\lambda = \omega \cdot \gamma$ for some limit γ then clearly $\mathbf{2}_{\omega\cdot\gamma} \equiv \sup_{n\in\mathbb{N}} C_{\omega\cdot\gamma_n}$ for any sequence $(\gamma_n)_{n\in\mathbb{N}}$ cofinal in γ. So that it is the $<$-least mapping among those with target 0 or limit that are located in the Steel hierarchy strictly above all constant functions $C_{\omega\cdot\theta}$ $(\theta < \gamma)$. Hence by induction hypothesis

$$\begin{aligned} d_l^\circ\,(\mathbf{2}_{\omega\cdot\gamma}) &= \sup\{d_l^\circ\,(\psi) + 1 :\ \psi \text{ Borel } \wedge\ \psi < \mathbf{2}_{\omega\cdot\gamma}\} \\ &= \sup\{d_l^\circ\,(\psi) + 1 :\ \psi \text{ Borel } \wedge\ \exists\theta < \gamma\ \ \psi \leq \mathbf{2}_{\omega\cdot\theta}\} \\ &= \sup\{d_l^\circ\,(C_{\omega\cdot\theta}) + 1 :\ \theta < \gamma\} \\ &= \sup\{V_{\omega_1}(1+\theta) + 1 :\ \theta < \gamma\} \\ &= \sup_{\theta<\gamma} V_{\omega_1}(\theta) \end{aligned}$$

And

$$
\begin{aligned}
d_l^\circ(C_{\omega\cdot\gamma}) &= \sup\{d_l^\circ(\psi)+1:\ \psi \text{ Borel } \wedge\ tg(\psi)=0 \text{ or limit } \wedge\ \psi < C_{\omega\cdot\gamma}\}\\
&= \sup\{d_l^\circ(\psi)+1:\ \psi \text{ Borel } \wedge\ \mathcal{Im}(\psi)\subseteq \omega\cdot\gamma\ \wedge\ tg(\psi)=0 \text{ or limit}\}\\
&= \sup\{d_l^\circ(\mathbf{V}_\xi\circ \mathbf{2}_{\omega\cdot\gamma})+1:\ \xi<\omega_1\}\\
&= \sup\{V_\xi(d_l^\circ(\mathbf{2}_{\omega\cdot\gamma}))+1:\ \xi<\omega_1\}\\
&= \sup\{V_\xi(\sup_{\theta<\gamma} V_{\omega_1}(\theta))+1:\ \xi<\omega_1\}\\
&= \sup\{V_\xi(V_{\omega_1}(\sup_{\theta<\gamma} V_{\omega_1}(\theta)):\ \xi<\omega_1\}\\
&= V_{\omega_1}(\gamma)\\
&= V_{\omega_1}(1+\gamma)
\end{aligned}
$$

If $cof(\boldsymbol{\lambda}) > \boldsymbol{\omega}$: Then $\omega\cdot\lambda=\lambda$ and for any ψ Borel with countable image inside λ, there exists some limit ordinal of countable cofinality $\mu_\psi<\lambda$ such that $\psi < C_{\omega\cdot\mu_\psi}$ holds. We conclude with

$$
\begin{aligned}
d_l^\circ(C_\lambda) &= \sup\{d_l^\circ(\psi)+1:\ \psi \text{ Borel } \wedge\ tg(\psi)=0 \text{ or limit } \wedge\ \psi < C_\lambda\}\\
&= \sup\{d_l^\circ(\psi)+1:\ \psi \text{ Borel } \wedge\ \wedge\ tg(\psi)=0 \text{ or limit}\}\ \exists\mu<\lambda\ \ \psi < C_{\omega\cdot\mu}\}\\
&= \sup\{d_l^\circ(C_{\omega\cdot\mu})+1:\ \mu<\lambda\}\\
&= \sup\{V_{\omega_1}(1+\mu)+1:\ \mu<\lambda\}\\
&= \sup\{V_{\omega_1}(1+\mu):\ \mu<\lambda\}\\
&= V_{\omega_1}(\lambda)\\
&= V_{\omega_1}(1+\lambda).
\end{aligned}
$$

71&73 □

We have everything ready to give a proof of theorem 7 which gives the computation of the order type of the Borel Steel Hierarchy. We recall that this theorem states that the order type of that Hierarchy on $\eta=\omega\cdot\delta+n$ with $n\in\mathbb{N}$ is $OT_{\mathcal{B}_\eta}=V_{\omega_1}(1+\delta)$ in case η is infinite, and it is $OT_{\mathcal{B}_\eta}=n=\eta$ otherwise. Here is the proof:
□ **thm. 7** -

$\boldsymbol{\eta}$ **finite** : clearly the only mappings up to $\equiv$ are the constant functions C_k for all $k<\eta$. Hence $OT_{\mathcal{B}_\eta}=\eta$

$\boldsymbol{\eta}$ **infinite** : clearly in case $n>0$, the only mappings that admit a target and are strictly above the constant function $C_{\omega\cdot\delta}$ are $C_{\omega\cdot\delta+1}, C_{\omega\cdot\delta+2},\dots,C_{\omega\cdot\delta+(n-1)}$. All the rest of the hierarchy consists of functions with image inside $\omega\cdot\delta$. This shows $OT_{\mathcal{B}_\eta}=OT_{\mathcal{B}_{\omega\cdot\delta}}+n$.

So, it just remains to compute $OT_{\mathcal{B}_{\omega\cdot\delta}}$. By previous results it is immediate that $OT_{\mathcal{B}_{\omega\cdot\delta}}=d_t^\circ(C_{\omega\cdot\delta})$. And we already know that for any $\phi:\ {\Lambda_\phi}^\omega \stackrel{\boldsymbol{\Delta}_1^1}{\longmapsto} \omega\cdot\delta$ that admits a target, if $tg(\phi)=0$ or limit, then $d_t^\circ(\phi)=\omega\cdot(-1+d_l^\circ(\phi))$. Therefore we obtain

$$OT_{\mathcal{B}_{\omega\cdot\delta}}=d_t^\circ(C_{\omega\cdot\delta})=\omega\cdot(-1+d_l^\circ(C_{\omega\cdot\delta}))=\omega\cdot(-1+V_{\omega_1}(1+\delta))=\omega\cdot V_{\omega_1}(1+\delta)=V_{\omega_1}(1+\delta)$$

which gives the result $OT_{\mathcal{B}_\eta}=V_{\omega_1}(1+\delta)+n$.

7□

15 Canonical Borel mappings

Now that we have conciliatory operations that behave like ordinal sum, countable sup, countable multiplication and all countable Veblen functions; and knowing exactly the degrees of the constant mappings, we can now define Borel mappings by only use of constant functions.

Definition 74 *Let $\xi \in \boldsymbol{On}$, we define $\mathbb{T}(\xi)$ by mean of the Euclidean division of ξ by ω:*

$$\xi = \omega \cdot (-1+\alpha) + n \quad \textit{with } n \in \mathbb{N} \textit{ and } \alpha > 0$$

($-1+\alpha$ stands for $\alpha - 1$ when α is finite and α otherwise). We set

$$\mathbb{T}(\xi) = S_{\textit{hift}}{}^{n} \circ \mathbb{L}(\alpha)$$

Where $\mathbb{L}(\alpha)$ is defined by mean of the Cantor Normal Form of base ω_1 of α:

$$\alpha = {\omega_1}^{\alpha_n}.\nu_n + \ldots + {\omega_1}^{\alpha_0}.\nu_0$$

with

$$\alpha_n > \ldots > \alpha_0 \geq 0 \quad \textit{and} \quad \forall i \leq n \quad 0 < \nu_i < \omega_1$$

we set:

$$\mathbb{L}(\alpha) = \mathbb{L}({\omega_1}^{\alpha_n}) \bullet \nu_n + \ldots\ldots + \mathbb{L}({\omega_1}^{\alpha_0}) \bullet \nu_0$$

where $\mathbb{L}({\omega_1}^{\alpha_i})$ is defined by:

$$\begin{array}{ll}
\alpha_i = 0 & \mathbb{L}({\omega_1}^{\alpha_i}) = C_0 \\
\textit{cof}(\alpha_i) = \omega & \mathbb{L}({\omega_1}^{\alpha_i}) = \mathbf{sup}_{\theta \in \Theta_{\alpha_i}} \mathbb{L}({\omega_1}^{\theta}) \\
\exists \zeta \ \ V_{\omega_1}(1+\zeta) = \alpha_i & \mathbb{L}({\omega_1}^{\alpha_i}) = C_{\omega \cdot \zeta} \\
\left.\begin{array}{l} \alpha_i \textit{ successor} \\ \textit{cof}(\alpha_i) > \omega \ \wedge \ \forall \zeta \ \ V_{\omega_1}(1+\zeta) \neq \alpha_i \end{array}\right\} & \mathbb{L}(\omega_1^{\alpha_i}) = \mathbf{V}_{\nu} \circ \mathbb{L}(\alpha_i)
\end{array}$$

where ν is the least ordinal such that there exists $\gamma < {\omega_1}^{\alpha_i}$ satisfying ${\omega_1}^{\alpha_i} = V_\nu(\gamma)$ - i.e. γ is not a fixpoint of cofinality ω_1 of the function: $x \longmapsto V_\nu(x)$. And $\beta \longmapsto \Theta_\beta$ is a fixed map for limit $\beta < OT_{\mathcal{B}_\eta}$ of countable cofinality, with Θ_β a countable set cofinal in β.

From the whole study, we conclude that

Theorem 75

(a) For all ordinal α, $\mathbb{T}(\alpha)$ is Borel with countable image.

(b) Let $0 < \alpha \in \boldsymbol{On}$,

$$d^\circ_l\,(\mathbb{L}(\alpha)) = \alpha$$

*(c) Let $\alpha \in$ **On**,*

$$d_t^\circ(\mathbb{T}(\alpha)) = \alpha$$

(d) For all $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta_1^1} \boldsymbol{On}$ with countable image,

$$\phi \textit{ Borel} \wedge tg(\phi) \textit{ exists} \implies \exists! \alpha \in \boldsymbol{On} \;\; \phi \equiv \mathbb{T}(\alpha).$$

□ **thm. 75** -

(a) Easily by induction on α we verify that $\mathbb{T}(\alpha)$ has countable image and that it is a Borel mapping. This relies on the fact constant functions are Borel and operations involved in the definition of $\mathbb{T}(\alpha)$: $shift$, $+$, countable $\mathbf{sup}_{,}$ $\bullet\,\nu$ $(0 < \nu < \omega_1)$, and $\mathbf{V}_\zeta$ $(\zeta < \omega_1)$ are all Borel.

(b) The result simply relies on the relation between arithmetical and conciliatory operations stated in propositions 33, 35, 38, 71.

(c) Follows from (b), propositions 23(a) (which states that regarding d_t°, the conciliatory operation $Shift$ is the equivalent of the ordinal successor operation) and 24 (relates the degree of a function with its target).

(d) Immediate from previous case.

75□

This last theorem achieves the program announced in the introduction of this paper. It gives a complete description of the Steel Hierarchy in its conciliatory form which is isomorphic to the Steel hierarchy unloaded of the target-free functions.

16 Target and Wadge degree: the topological complexity of the cotarget

In this section, we are only concerned with Borel targeted-mappings with image inside the integers. We regard mappings as partitions of the reals. We investigate the issue of the topological complexity of the element of the partition that corresponds to the target.

16.1 Setting the problem

We call co-target of the mapping ϕ the inverse image of the target of ϕ.

Definition 76 *Let $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta_1^1} \mathbb{N}$, $tg(\phi) = n$*

$$cotg(\phi) = \mathcal{n}_\phi = \{x \in \Lambda_\phi^\omega : \; \phi(x) = n\}.$$

A first idea would be to try to find what topological complexity the co-target of ϕ must have. And since the Wadge hierarchy is the most refined topological one, this amounts to describing its precise rank in this hierarchy, also called its **Wadge degree**. What would be nice, is to find a relation between the Steel degree of the mapping ϕ and the Wadge degree of the co-target of ϕ. But this is hopeless, because given any ordinal $\omega \leq \alpha < OT_{\mathcal{B}}\omega = V_{\omega_1}(2)$, there exists infinitely many mappings, each of them of Steel degree precisely α with no two co-targets being Wadge equivalent. Moreover, given any mapping ϕ, any Borel set A satisfying $A \geq_w cotg\,(\phi)$, it is possible to find a mapping ψ that still has the same Steel degree as ϕ but with co-target Wadge equivalent to A. This means, considering all mappings of degree precisely α, the set of the Wadge degrees of their co-targets is some final segment of the sequence of ordinals below $OT_{\mathcal{B}}\omega$.

Proposition 77 *Let* $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta^1_1} \mathbb{N}$ *be Borel and unbounded,* $A \subseteq \Lambda^\omega_\phi$ *Borel*

$$A >_w cotg\,(\phi) \implies \exists \psi :\ \Lambda^\omega_\phi \xrightarrow{\Delta^1_1} \mathbb{N} \quad \psi \equiv \phi \ \wedge\ cotg\,(\psi) \equiv_w A.$$

□ **prop.77** - By hypothesis $card(\Lambda_\phi) \geq 2$, get $a,\ b$ two different letters in Λ_ϕ and set

$$\begin{array}{rcl} \psi(\langle a\rangle^\frown x) & = & \phi(x) \\ \psi(\langle b\rangle^\frown x) & = & tg\,(\phi) \text{ if } x \in A \\ \psi(\langle b\rangle^\frown x) & = & tg\,(\phi) + 1 \text{ if } x \notin A \\ \psi(\langle c\rangle^\frown x) & = & tg\,(\phi) + 1 \text{ if } c \notin \{a, b\} \end{array}$$

Clearly $\psi \equiv \phi$ is verified and the following holds:

$$cotg\,(\psi) = \{\langle a\rangle^\frown x :\ x \in cotg\,(\phi)\} \cup \{\langle b\rangle^\frown x :\ x \in A\} \equiv_w A$$

77 □

16.2 "Canonical" Borel mappings with minimum cotarget

The last proposition showed that the set of Wadge degrees of possible co-targets for mappings of given Steel degree forms a final sequence inside $OT_{\mathcal{B}}\omega$ which is not only the order type of the Steel hierarchy of Borel mappings with integer values, but also the order type of the Wadge hierarchy of Borel subsets of reals. Therefore, if there is a link between the Steel degree of a mapping and the Wadge degree of its co-target, we can only find it in the relation between Steel degree and **minimum** Wadge degree of the corresponding set of co-targets. Thus, the problem becomes the relation between α and $\inf\{d^\circ_w(cotg\phi) :\ d^\circ_t\,(\phi) = \alpha\}$; where $d^\circ_w(A)$ is the Wadge degree of A. In other words, its rank in the Wadge hierarchy.
For this purpose, we first define another version of conciliatory Borel mappings, then show that they are minimal when it comes to the complexity of their co-targets, and finally we compute the Wadge degrees of these co-targets.

Definition 78 *Let* $0 < \alpha < OT_{\mathcal{B}_\omega}$ *and* $n \in \mathbb{N}$, *we inductively define* $\mathbb{T}'(\omega \cdot (-1+\alpha) + n)$ *by:*

$$\mathbb{T}'(\omega \cdot (-1+\alpha) + n) = S_{hift}{}^n \circ \mathbb{L}'(\alpha)$$

Where $\mathbb{L}'(\alpha)$ *is defined by:*

If α=1 : $\mathbb{L}'(\alpha) = C_0$.

If $1 < \alpha = \beta + 1$: $\mathbb{L}'(\alpha) = \mathbf{sup}_{n\in\mathbb{N}} S_{hift}{}^{(n+1)} \circ \mathbb{L}'(\beta)$.

If $cof(\boldsymbol{\alpha}) = \boldsymbol{\omega}$ *:* $\mathbb{L}'(\alpha) = \mathbf{sup}_{\alpha_\xi \in \Theta_\alpha} S_{hift} \circ \mathbb{L}'(\alpha_\xi)$.

If $cof(\boldsymbol{\alpha}) = \boldsymbol{\omega_1}$ *: let the Cantor Normal Form of base ω_1 of α be*

$$\alpha = {\omega_1}^{\alpha_n} \cdot \nu_n + \ldots + {\omega_1}^{\alpha_0} \cdot \nu_0$$

If $\boldsymbol{\alpha > \omega_1^{\alpha_0}}$ *:* $\mathbb{L}'(\alpha) = \mathbb{L}'({\omega_1}^{\alpha_0}) \overset{\frown}{} S_{hift} \circ \mathbf{sup}_{k \in \mathbb{N}} \left({S_{hift}}^k \circ \mathbb{L}(\alpha - \omega_1^{\alpha_0}) \right)$ *where $\alpha - \omega_1^{\alpha_0}$ stands for ${\omega_1}^{\alpha_n} \cdot \nu_n + \ldots + {\omega_1}^{\alpha_0} \cdot (\nu_0 - 1)$*

If $\boldsymbol{\alpha = \omega_1^{\alpha_0}}$ *:* $\mathbb{L}'(\alpha) = \mathbf{V}_\xi \circ \mathbb{L}'(\beta)$ *where* $\alpha = V_\xi(\beta)$ *(for some minimal $\xi < \omega_1$ and some $\beta < \alpha$ such that $\alpha = V_\xi(\beta)$). And And $\beta \longmapsto \Theta_\beta$ is a fixed map for limit $\beta < OT_{\mathcal{B}_\eta}$ of countable cofinality, with Θ_β a countable set cofinal in β.*

We show $d_t^\circ(\mathbb{T}'(\alpha)) = \alpha$ which relies on

Lemma 79 *Let* $0 < \alpha < OT_{\mathcal{B}_\omega}$*,*

$$\mathbb{L}'(\alpha) \equiv \mathbb{L}(\alpha).$$

$\square$ **lem. 79** - By induction on α.

(a) If $\alpha = 1$: $\mathbb{L}'(\alpha) = \mathbb{L}(\alpha)$.

(b) If $\alpha = \beta + 1 > 1$:

$$\begin{array}{rcl} d_t^\circ(\mathbb{L}'(\alpha)) & = & d_t^\circ\left(\mathbf{sup}_{n\in\mathbb{N}} {S_{hift}}^{(n+1)} \circ \mathbb{L}'(\beta)\right) \\ & = & \sup_{n\in\mathbb{N}}(d_t^\circ(\mathbb{L}(\beta)) + (n+1)) \\ & = & d_t^\circ(\mathbb{L}(\beta)) + \omega \\ & = & d_t^\circ(\mathbb{L}(\beta+1)) \\ & = & d_t^\circ(\mathbb{L}(\alpha)) \end{array}$$

Hence $\mathbb{L}'(\alpha) \equiv \mathbb{L}(\alpha)$.

(c) If $cof(\alpha) = \omega$:

$$\begin{array}{rcl} d_t^\circ(\mathbb{L}'(\alpha)) & = & d_t^\circ\left(\mathbf{sup}_{\alpha_\xi\in\Theta_\alpha} S_{hift} \circ \mathbb{L}'(\alpha_\xi)\right) \\ & = & \sup_{\alpha_\xi\in\Theta_\alpha} d_t^\circ(S_{hift} \circ \mathbb{L}'(\alpha_\xi)) \\ & = & \sup_{\alpha_\xi\in\Theta_\alpha}(d_t^\circ(\mathbb{L}'(\alpha_\xi)) + 1) \\ & = & \sup_{\alpha_\xi\in\Theta_\alpha} d_t^\circ(\mathbb{L}'(\alpha_\xi)) \\ & = & \alpha \end{array}$$

Hence $\mathbb{L}'(\alpha) \equiv \mathbb{L}(\alpha)$.

(d) If $cof(\alpha) = \omega_1$: assume $\alpha = {\omega_1}^{\alpha_n} \cdot \nu_n + \ldots + {\omega_1}^{\alpha_0} \cdot \nu_0$ is the Cantor Normal Form of base ω_1 of α.

(i) If $\alpha > \omega_1^{\alpha_0}$: $\mathbb{L}'(\alpha) = \mathbb{L}'({\omega_1}^{\alpha_0}) \overset{\frown}{} S_{hift} \circ \mathbf{sup}_{k\in\mathbb{N}} \left({S_{hift}}^k \circ \mathbb{L}(\alpha - \omega_1^{\alpha_0})\right)$.
By induction hypothesis $\mathbb{L}'({\omega_1}^{\alpha_0}) \equiv \mathbb{L}(\alpha)$. And clearly $\mathbb{L}(\alpha) \leq \mathbb{L}'(\alpha)$ holds. For the other direction we have:

$$\begin{array}{rcl} S_{hift} \circ \mathbf{sup}_{k\in\mathbb{N}}\left({S_{hift}}^k \circ \mathbb{L}(\alpha - \omega_1^{\alpha_0})\right) & \leq & \mathbf{sup}_{n\in\mathbb{N}} {S_{hift}}^n \circ \mathbf{sup}_{k\in\mathbb{N}}\left({S_{hift}}^k \circ \mathbb{L}(\alpha - \omega_1^{\alpha_0})\right) \\ & \leq & \mathbb{L}(\alpha - \omega_1^{\alpha_0}) \mathbf{+2}_\omega \\ & \leq & \mathbb{L}((\alpha - \omega_1^{\alpha_0}) + 2) \end{array}$$

Hence $\mathbb{L}'(\alpha) \leq \mathbb{L}((\alpha - \omega_1^{\alpha_0}) + 1) + \mathbb{L}(\omega_1^{\alpha_0}) \equiv \mathbb{L}((\alpha - \omega_1^{\alpha_0}) + 1 + \omega_1^{\alpha_0}) = \mathbb{L}(\alpha)$

(ii) If $\alpha = \omega_1^{\alpha_0}$: $\mathbb{L}'(\alpha) = \mathbf{V}_\xi \circ \mathbb{L}'(\beta)$ where $\alpha = V_\xi(\beta)$ (for some $\xi < \omega_1$ and $\beta < \alpha$). By induction hypothesis $\mathbb{L}(\beta) \equiv \mathbb{L}'(\beta)$. Thus

$$\mathbb{L}(\alpha) = \mathbf{V}_\xi \circ \mathbb{L}(\beta) \equiv \mathbf{V}_\xi \circ \mathbb{L}'(\beta) = \mathbb{L}'(\alpha)$$

79□

Lemma 80 *Let* $\phi : \Lambda_\phi{}^\omega \xrightarrow{\Delta_1^1} \mathbb{N}$ *be Borel with a target,*

$$d_t^\circ(\phi) = \omega \cdot (-1 + \alpha) + n \implies cotg\,(\phi) \geq_w cotg\,(\mathbb{L}'(\alpha))$$

□ **lem. 80** - The proof goes by induction on $d_t^\circ(\phi)$.
We notice that $\phi \equiv S_{hift}{}^n \circ \mathbb{L}'(\alpha)$ and $cotg\,(S_{hift}{}^n \circ \mathbb{L}'(\alpha)) = cotg\,(\mathbb{L}'(\alpha))$. Cases $\alpha = 1$, $\alpha > 1$ successor and $cof(\alpha) = \omega$ are all obvious. The only remaining case is $cof(\alpha) = \omega_1$. Assume $\alpha = \omega_1{}^{\alpha_n} \cdot \nu_n + \ldots + \omega_1{}^{\alpha_0} \cdot \nu_0$ is the Cantor Normal Form of base ω_1 of α.

(a) If $\alpha > \omega_1^{\alpha_0}$, clearly $cotg\,(\mathbb{L}'(\alpha)) \equiv_w cotg\,(\mathbb{L}'(\omega_1^{\alpha_0}))$ holds because $\mathbb{L}'(\omega_1^{\alpha_0})$ is unbounded and initializable.

We define $\phi'_{Init} : \Lambda_\phi^\omega \longmapsto \mathbb{N}$ as a slight variant of ϕ_{Init}:
if $x \in [\mathcal{I}nit_\phi]$ then $\phi'_{Init}(x) = \phi(x)$
if $x \notin [\mathcal{I}nit_\phi]$ then get the greatest integer k such that $x \restriction k \in \mathcal{I}nit_\phi$ and set $\phi'_{Init}(x) = \inf\{\phi((x \restriction k)^\frown y) : ((x \restriction k)^\frown y) \in [\mathcal{I}nit_\phi]\}$. We remark that the set

$$\{\phi((x \restriction k)^\frown y) : ((x \restriction k)^\frown y) \in [\mathcal{I}nit_\phi]\}$$

is non empty because ϕ having a target implies $\mathcal{I}nit_\phi$ is a non empty pruned tree. Clearly $\phi'_{Init} \equiv \phi_{Init}$ holds.

Claim 81

$$\phi'_{Init} \equiv S_{hift}{}^n \circ \mathbb{L}(\omega_1^{\alpha_0}).$$

□ **cl. 81** - We prove both directions of the equivalence.

- Towards a contradiction, we assume that $\phi'_{Init} < S_{hift}{}^n \circ \mathbb{L}(\omega_1^{\alpha_0})$. We consider the run of the game $\mathbb{G}_\leq\,(S_{hift}{}^n \circ \mathbb{L}(\alpha), \phi)$ where player I, while remaining in the "$S_{hift}{}^n \circ \mathbb{L}(\omega_1^{\alpha_0})$" part of $\mathbb{L}(\alpha)$, applies a w.s. against II as long as II's run remains inside $\mathcal{I}nit_\phi$ and II applies a w.s. against I. Necessarily II must leave $\mathcal{I}nit_\phi$ at some point reaching a position $u \notin \mathcal{I}nit_\phi$ which shows that II has a winning strategy in the game $\mathbb{G}_\leq\,(S_{hift}{}^n \circ \mathbb{L}(\alpha), \phi[u])$; a contradiction.
- Towards a contradiction, assume that $\phi'_{Init} > S_{hift}{}^n \circ \mathbb{L}(\omega_1^{\alpha_0})$. Then we consider the run of the game $\mathbb{G}_\leq\,(\phi, S_{hift}{}^n \circ \mathbb{L}(\alpha))$ where player I while remaining in $\mathcal{I}nit_\phi$ applies a w.s. against II as long as II's run remains inside the $S_{hift}{}^n \circ \mathbb{L}(\omega_1^{\alpha_0})$ part of $\mathbb{L}(\alpha)$. During that time player II applies a w.s. against I. Necessarily II must exit the $S_{hift}{}^n \circ \mathbb{L}(\omega_1^{\alpha_0})$ part of $\mathbb{L}(\alpha)$. At that moment I's run still being inside $\mathcal{I}nit_\phi$, which gives the following contradiction
$$\phi \leq \mathbb{L}(\alpha - \omega_1^{\alpha_0}) + C_0 \equiv \mathbb{L}(\alpha - \omega_1^{\alpha_0} + 1) < \mathbb{L}(\alpha).$$

81$\square$

By the previous claim and induction hypothesis $cotg\left(\phi'_{Init}\right) \geq_w cotg\left(\mathbb{L}'\left(\omega_1^{\alpha_0}\right)\right)$. Thus, it simply remains to check that $cotg\left(\phi\right) \geq_w cotg\left(\phi'_{Init}\right)$ holds which is obvious.

(b) If $\alpha = \omega_1^{\alpha_0}$, then $\phi \equiv \mathcal{S}_{hift}{}^n \circ \mathbf{V}_\xi \circ \mathbb{L}'(\beta)$. By proposition 70 we obtain

$$\mathbf{V}_\xi^{-1} \circ \phi \equiv \mathbf{V}_\xi^{-1} \circ \mathcal{S}_{hift}{}^n \circ \mathbf{V}_\xi \circ \mathbb{L}'(\beta) \equiv \mathbf{V}_\xi^{-1} \circ \mathbf{V}_\xi \circ \mathcal{S}_{hift}{}^n \circ \mathbb{L}'(\beta) \equiv \mathbb{L}'(\beta)$$

In [3], we used $\mathcal{T}$rees to lower the degree of a set. Given $A \subseteq \Lambda_A^\omega$ and T a ξ-$\mathcal{T}$ree on Λ_A, the set A^T was defined as the subset of Λ_T^ω that contains all sequences $x \in [T]$ such that $x \in A$. We showed that $\{A^T : T \text{ a } \mathcal{T}\text{ree on } \Lambda_A\}$ admits a $\leq_w$-minimum element. Let $\mathbf{W}_\xi^{-1}(A)$ denote the minimal element of $\{A^T : T \text{ a } \omega^\xi\text{-}\mathcal{T}\text{ree on } \Lambda_A\}$.

Let T' and T'' be some ω^ξ-$\mathcal{T}$rees such that $\phi^{T'} \equiv \mathbf{V}_\xi^{-1} \circ \phi$ and $\left(cotg\left(\phi\right)\right)^{T''} \equiv_w \mathbf{W}_\xi^{-1}\left(cotg\left(\phi\right)\right)$. Get also some ω^ξ-$\mathcal{T}$rees T that "combines" both T' and T'' (i.e. T' and T'' both embed into T. By minimality we have

$$\phi^T \equiv \mathbf{V}_\xi^{-1} \circ \phi \text{ and } \left(cotg\left(\phi\right)\right)^T \equiv_w \mathbf{W}_\xi^{-1}\left(cotg\left(\phi\right)\right)$$

By induction hypothesis we know that $cotg\left(\phi^T\right) \geq_w cotg\left(\mathbb{L}'(\beta)\right)$. Besides we see that

$$tg\left(\phi^T\right) = tg\left(\mathbf{V}_\xi^{-1} \circ \phi\right) = tg\left(\mathcal{S}_{hift}{}^n \circ \mathbb{L}'(\beta)\right) = n = tg\left(\phi\right).$$

$$\begin{aligned} \text{Hence,} \quad \left(cotg\left(\phi\right)\right)^T &= \{x \in \Lambda_T{}^\omega : \phi(x_0) = tg\left(\phi^T\right)\} \\ &= \{x \in \Lambda_T{}^\omega : \phi(x_0) = tg\left(\phi\right)\} \\ &= \left(cotg\left(\phi\right)\right)^T \end{aligned}$$

So we just obtained $\mathbf{W}_\xi^{-1}\left(cotg\left(\phi\right)\right) \equiv_w \left(cotg\left(\phi\right)\right)^T \geq_w cotg\left(\mathbb{L}'(\beta)\right)$. By the above claim we get $cotg\left(\phi\right) \geq_w \mathbf{W}_\xi\left(cotg\left(\mathbb{L}'(\beta)\right)\right)$.

Given $B \subseteq \Lambda_B^{\leq\omega}$, we write B^* for the set of all sequences $x \in (\Lambda_B \cup \{*\})^\omega$ that belong to B once every occurence of $*$ has been removed, and ${}^-B$ for the complement of B.

Claim 82 *Let $A \subseteq \Lambda_A^\omega$, $B \subseteq \Lambda_B^{\leq\omega}$ A and B^* both Borel; $\zeta < \omega$,*

$$B^* \leq_w \mathbf{W}_\zeta^{-1}(A) \implies \mathbf{W}_\zeta(B)^* \leq_w A.$$

$\square$ **cl. 82** - For any set S we write ${}^-S$ for the complement of S. Towards a contradiction we assume that $\mathbf{W}_\zeta(B) \not\leq_w A$. By determinacy this leads to $A \leq_w {}^-\mathbf{W}_\zeta(B)^*$. From proposition 26 in [3] we obtain $\mathbf{W}_\zeta^{-1}(A) \leq_w \mathbf{W}_\zeta^{-1}\left({}^-\mathbf{W}_\zeta(B)^*\right)$ and from proposition 28 in [3] we get $\mathbf{W}_\zeta^{-1}\left({}^-\mathbf{W}_\zeta(B)^*\right) \leq_w {}^-B$. Therefore we obtain $\mathbf{W}_\zeta^{-1}(A) \leq_w {}^-B^*$. Combined with the hypothesis we get $B^* \leq_w {}^-B^*$ which is a contradiction since I has an obvious w.s. in the underlying Wadge game.

82$\square$

Finally set $\Lambda =$ the alphabet of $\mathbb{L}'(\beta)$, we get

$$
\begin{aligned}
\mathbf{W}_\xi\,(cotg\,(\mathbb{L}'(\beta))) &= \{x \in (\Lambda_{\mathbb{L}'(\beta)} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^\xi\})^{\leq\omega} :\ x^{\circlearrowleft} \in cotg\,(\mathbb{L}'(\beta))\} \\
&= \{x \in (\Lambda_{\mathbb{L}'(\beta)} \cup \{\twoheadleftarrow(\iota) : 0 < \iota \leq \omega^\xi\})^{\leq\omega} :\ \mathbb{L}'(\beta)(x^{\circlearrowleft}) = 0\} \\
&= \{x \in \Lambda_{\mathbb{L}'(\beta)})^{\leq\omega} :\ \mathbf{V}_\xi \circ \mathbb{L}'(\beta)(x^{\circlearrowleft}) = 0\} \\
&= cotg\,(\mathbf{V}_\xi \circ \mathbb{L}'(\beta)).
\end{aligned}
$$

So we have proved $cotg\,(\phi) \geq_w cotg\,(\mathbf{V}_\xi \circ \mathbb{L}'(\beta)) \equiv_w cotg\,(\mathbb{L}'(\alpha))$.

80□

16.3 Computing the minimum topological complexity of the cotarget

This last step is to determine the topological complexity of the co-target of $\mathbb{L}'(\alpha)$, which solves the issue.

Definition 83 *The function $\mathcal{H} :\ OT_{\mathcal{B}_\omega} \smallsetminus \{0\} \longmapsto \omega_1$ is inductively defined by*

if $\alpha = 1$	$\mathcal{H}(\alpha) = 0$
if α *is successor* > 1	$\mathcal{H}(\alpha) = 1$
if $cof(\alpha) = \omega$	$\mathcal{H}(\alpha) = 1$
if $cof(\alpha) = \omega_1$	$\mathcal{H}(\alpha) = (1 + \omega^\xi) + (-1 + \mathcal{H}(\beta))$

where β is the unique ordinal such that there exists some minimal $\xi < \omega_1$ such that $\beta < V_\xi(\beta) = \omega_1^{\alpha_0}$; and α_0 comes from the CNF of base ω_1 of α:

$$\alpha = {\omega_1}^{\alpha_n} \cdot \nu_n + \ldots + {\omega_1}^{\alpha_0} \cdot \nu_0$$

with

$$\omega_1 > \alpha_n > \ldots > \alpha_0 > 0 \text{ and } \omega_1 > \nu_i > 0 \quad (\text{each } i \leq n).$$

We have the following amazing result.

Theorem 84 *Let $n \in \mathbb{N}$ and $0 < \alpha < OT_{\mathcal{B}_\omega}$,*
there exists $\phi :\ {\Lambda_\phi}^\omega \overset{\mathbf{\Delta}^1_1}{\longmapsto} \mathbb{N}$, $d_t^\circ(\phi) = \omega \cdot (-1 + \alpha) + n$ with a minimum cotarget i.e. for all ψ,

$$\psi \equiv \phi \implies cotg\,(\psi) \geq_w cotg\,(\phi)$$

$$\textit{Moreover } cotg\,(\phi) \textit{ is } \mathbf{\Pi}^\circ_{\mathcal{H}(\alpha)}\textit{-complete}$$

with the convention $\mathbf{\Sigma}^\circ_\circ = \{\emptyset\}$.

□ **thm. 84** - We show by induction on α that $cotg\,(\mathbb{L}'(\alpha))$ is $\mathbf{\Pi}^\circ_{\mathcal{H}(\alpha)}$-complete. Cases $\alpha = 1$, $\alpha > 1$ successor and $cof(\alpha) = \omega$ are all obvious. The only interesting case is $cof(\alpha) = \omega_1$. Assume $\alpha = {\omega_1}^{\alpha_n} \cdot \nu_n + \ldots + {\omega_1}^{\alpha_0} \cdot \nu_0$ is the Cantor Normal Form of base ω_1 of α.

(a) If $\alpha > \omega_1^{\alpha_0}$, then $cotg\,(\mathbb{L}'(\alpha)) \equiv_w cotg\,(\mathbb{L}'(\omega_1^{\alpha_0}))$ which is $\mathbf{\Pi}^\circ_{\mathcal{H}(\omega_1^{\alpha_0})}$-complete and $\mathcal{H}(\alpha) = \mathcal{H}(\omega_1^{\alpha_0})$ is definition.

(b) If $\alpha = \omega_1^{\alpha_0}$: $\mathbb{L}'(\alpha) = \mathbf{V}_\xi \circ \mathbb{L}'(\beta)$ where $\alpha = V_\xi(\beta)$ with $\beta < \alpha$. And we have:

$$cotg\,(\mathbf{V}_\xi \circ \mathbb{L}'(\beta)) = \mathbf{W}_\xi\,(cotg\,(\mathbb{L}'(\beta)))\,.$$

By induction hypothesis, $cotg\,(\mathbb{L}'(\beta))$ is $\mathbf{\Pi}^\circ_{\mathcal{H}(\beta)}$-complete. Since $\beta > 1$, $\mathcal{H}(\beta)$ is strictly positive. Besides, we proved in [3] proposition 43, that given any $\mathbf{\Pi}^\circ_1$ complete set C and any countable ordinal λ, with $\omega^{\lambda_k} \cdot n_k + \ldots + \omega^{\lambda_0} \cdot n_0 = \lambda$ as CNF of base ω, the set

$$\underbrace{\mathbf{W}_{\lambda_k} \circ \ldots \circ \mathbf{W}_{\lambda_k}}_{n_k} \circ \ldots \circ \underbrace{\mathbf{W}_{\lambda_0} \circ \ldots \circ \mathbf{W}_{\lambda_0}}_{n_0}(C) \text{ is } \mathbf{\Pi}^\circ_{1+\lambda}\text{-complete.}$$

we set $\mathcal{H}(\beta) = 1 + \lambda$. Since $cotg\,(\mathbb{L}'(\beta))$ is $\mathbf{\Pi}^\circ_{\mathcal{H}(\beta)}$-complete we have

$$cotg\,(\mathbb{L}'(\beta)) \equiv_w \underbrace{\mathbf{W}_{\lambda_k} \circ \ldots \circ \mathbf{W}_{\lambda_k}}_{n_k} \circ \ldots \circ \underbrace{\mathbf{W}_{\lambda_0} \circ \ldots \circ \mathbf{W}_{\lambda_0}}_{n_0}(C)$$

hence

$$\mathbf{W}_\xi\,(cotg\,(\mathbb{L}'(\beta))) \equiv_w \mathbf{W}_\xi \circ \underbrace{\mathbf{W}_{\lambda_k} \circ \ldots \circ \mathbf{W}_{\lambda_k}}_{n_k} \circ \ldots \circ \underbrace{\mathbf{W}_{\lambda_0} \circ \ldots \circ \mathbf{W}_{\lambda_0}}_{n_0}(C).$$

(i) If $\xi < \lambda_k$ then the sets $\mathbf{W}_\xi \circ \underbrace{\mathbf{W}_{\lambda_k} \circ \ldots \circ \mathbf{W}_{\lambda_k}}_{n_k} \circ \ldots \circ \underbrace{\mathbf{W}_{\lambda_0} \circ \ldots \circ \mathbf{W}_{\lambda_0}}_{n_0}(C)$ and $\underbrace{\mathbf{W}_{\lambda_k} \circ \ldots \circ \mathbf{W}_{\lambda_k}}_{n_k} \circ \ldots \circ \underbrace{\mathbf{W}_{\lambda_0} \circ \ldots \circ \mathbf{W}_{\lambda_0}}_{n_0}(C)$ are Wadge equivalent.
So $cotg\,(\mathbf{V}_\xi \circ \mathbb{L}'(\beta))$ is $\mathbf{\Pi}^\circ_{\mathcal{H}(\beta)}$-complete. Besides, since $\xi < \lambda_k$ holds, $\mathcal{H}(\beta)$ is infinite; thus $(-1 + \mathcal{H}(\beta)) = \lambda = \mathcal{H}(\beta)$ and $\omega^\xi + \lambda = \lambda$ and finally $(1 + \omega^\xi) + (-1 + \mathcal{H}(\beta)) = \mathcal{H}(\beta) = \mathcal{H}(\alpha)$.

(ii) If $\xi \geq \lambda_k$ then

$$\mathbf{W}_\xi \circ \underbrace{\mathbf{W}_{\lambda_k} \circ \ldots \circ \mathbf{W}_{\lambda_k}}_{n_k} \circ \ldots \circ \underbrace{\mathbf{W}_{\lambda_0} \circ \ldots \circ \mathbf{W}_{\lambda_0}}_{n_0}(C) \text{ is } \mathbf{\Pi}^\circ_{1+\omega^\xi+\lambda}\text{-complete}$$

thus

$$cotg\,(\mathbf{V}_\xi \circ \mathbb{L}'(\beta)) \text{ is } \mathbf{\Pi}^\circ_{(1+\omega^\xi)+(-1+\mathcal{H}(\beta))}\text{-complete}$$

by definition, $(1 + \omega^\xi) + (-1 + \mathcal{H}(\beta)) = \mathcal{H}\alpha$; so we have proved that $cotg\,(\mathbb{L}'(\alpha))$ is $\mathbf{\Pi}^\circ_{\mathcal{H}(\alpha)}$-complete.

84$\square$

For any mapping $\phi : \Lambda_\phi{}^\omega \overset{\mathbf{\Delta}^1_1}{\longmapsto} \mathbb{N}$, let $\phi^{\bmod 2}$ denote the $\{0,1\}$ valued function defined by $\phi^{\bmod 2}(x) = 0$ if $\phi(x)$ is even and $\phi^{\bmod 2}(x) = 1$ if $\phi(x)$ is odd. An amazing thing in our canonical description of the mappings is that the Wadge degree of the set whose characteristic function is $\mathbb{L}(\alpha)^{\bmod 2}$ is exactly α. But in general this property is false. Given ϕ such that $d^\circ_l(\phi) = \alpha$, the function $\phi^{\bmod 2}$ can be the characteristic function of a set of almost any Wadge degree.
In fact, our operation on mappings have the following properties. If χ_A and χ_B are the characteristic functions of Borel sets respectively A and B, then

- $\chi_A + \chi_B$ is the characteristic function of a set whose Wadge degree is the sum of those of A and B
- $\chi_A \bullet \xi$ is the characteristic function of a set whose Wadge degree is the Wadge degree of A multiplied by ξ.
- $\mathbf{V}_\xi \circ \chi_A$ is the characteristic function of a set whose Wadge degree is $V_\xi(d_w^\circ(A))$.

A. Khelif showed that given any Borel mapping ϕ from 2^ω **onto** $\mathbb{N}$, the set

$$\{cotg\,(p \circ \phi) :\ p \text{ a permutation on } \mathbb{N}\}$$

is finite. Is it possible to determine this set ? More precisely, given a Borel partition $\pi = \{A_i :\ i \in \mathbb{N}\}$ of 2^ω, let F_π be the set such that $i \in F_\pi$ iff there is a permutation on $\mathbb{N}$, p such that A_i is the cotarget of the function ϕ_p defined by $\phi_p(x) = n \iff x \in A_{p(n)}$. Is there a way to characterize this set F_π ?

It would be nice if we could find an isomorphism between the Wadge hierarchy of Borel sets and the Steel hierarchy of Borel integer-valued mappings. This would allow us to deduce the properties and description of a hierarchy from the ones of the other. More generally, could the methods we used in the case of the Wadge hierarchy and in the case of the Steel hierarchy be generalized to the hierarchy of functions from the Baire space into some other BQO ?

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