Wadge degrees of Louveau's (Wadge) classes

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Recall Louveau's descriptions D_1 is the least satisfying:

(a) If $u(0) = 0$, then $u \in D_1$ and

$$\Gamma_u = [\emptyset]$$

(b) If $u(0) = \xi \geq 1$, $u(1) = 1$ and $u(2) = \eta \geq 1$ then $u \in D_1$ and

$$\Gamma_u = D_\eta \mathbf{\Sigma}^\circ_\xi$$

(c) If $u = \xi^\frown 2^\frown \eta^\frown u^*$ with $\xi \geq 1$, $\eta \geq 1$, $u^* \in D_1$ and $u^* > \xi$, then $u \in D_1$ and

$$\Gamma_u = Sep(D_\eta \mathbf{\Sigma}^\circ_\xi, \Gamma_{u^*})$$

(d) If $u = \xi^\frown 3^\frown \eta^\frown < u_0, u_1 >$ with $\xi \geq 1$, $\eta \geq 1$, $u_0, u_1 \in D_1$ and $u_0(0) > \xi$, $u_1(0) \geq \xi$ or $u_1(0) = 0$, and $\Gamma_{u_1} <_w \Gamma_{u_0}$, then $u \in D_1$ and

$$\Gamma_u = Bisep(D_\eta \mathbf{\Sigma}^\circ_\xi, \Gamma_{u_0}, \Gamma_{u_1})$$

(e) If $u = \xi^\frown 4^\frown < u_n : n \in \omega >$ where $\xi \geq 1$, , $u_n \in D_1$ $\forall n$ $\Gamma_{u_n} <_w \Gamma_{u_{n+1}}$, and either for all n, $u_n(0) = \xi' > \xi$ or $(u_n(0))_{n\in\omega}$ is strictly increasing with $\sup \xi_n > \xi$, then $u \in D_1$ and

$$\Gamma_u = SU(\mathbf{\Sigma}^\circ_\xi, \bigcup_{n\in\omega} \Gamma_{u_n})$$

(f) If $u = \xi^\frown 5^\frown \eta^\frown < u_0, u_1 >$ with $\xi \geq 1$, $\eta \geq 1$, $u_0, u_1 \in D_1$ and $u_0(0) = \xi$, $u_0(1) = 4$, $u_1(0) \geq \xi$ or $u_1(0) = 0$, and $\Gamma_{u_1} <_w \Gamma_{u_0}$, then $u \in D_1$ and

$$\Gamma_u = SD_\eta(\Gamma_{u_0}, \Gamma_{u_1})$$

We give for, each Louveau's class Γ_u, a Γ_u-complete set A_u, that is $[A_u^b] = \Gamma_u$, in our terminology:

First of all, in order to simplify the notations, we consider the following, given any conciliatory set A:

- For any non zero countable ordinal η, we write $A \circledast \eta = A \bullet (1+\eta)$; where
  - $\diamond$ $A \circledast (1+\eta) = A \bullet (\eta + 1)$ iff η is an integer
  - $\diamond$ $A \circledast (1+\eta) = A \bullet \eta$ is transfinite
- Given any non zero countable ordinal ξ with Cantor Normal Form of base $\boldsymbol{\omega}$:

$$\xi = \omega^{\alpha_k}.\nu_k + \ldots + \omega^{\alpha_0}.\nu_0$$

$$\text{with } \omega_1 > \alpha_k > \ldots > \alpha_0 \geq 0 \ \text{ and } \ \forall i \leq k,\ 0 < \nu_i < \omega.$$

  Define:

  - $\diamond$ $A^{\approx 1} = A$
  - $\diamond$ $A^{\approx 1+\xi} = A^{\underbrace{\sim\alpha_0\sim\alpha_0\cdots\sim\alpha_0}_{n_0}\underbrace{\sim\alpha_1\sim\alpha_1\cdots\sim\alpha_1}_{n_1}\cdots\cdots\underbrace{\sim\alpha_k\sim\alpha_k\cdots\sim\alpha_k}_{n_k}}$

(a) As $\Gamma_u = [\emptyset]$,

$$A_u = \ominus$$

(b) As $\Gamma_u = D_\eta \boldsymbol{\Sigma}^\circ_\xi$,

$$A_u = (\ominus \circledast \eta)^{\approx \xi}$$

(c) As $\Gamma_u = Sep(D_\eta \boldsymbol{\Sigma}^\circ_\xi, \Gamma_{u^*})$ with $\Gamma_{u^*} = [{A_*}^{\approx \xi}]$

$$A_u = (A_* \circledast \eta)^{\approx \xi}$$

(d) As $\Gamma_u = Bisep(D_\eta \mathbf{\Sigma}^\circ_\xi, \Gamma_{u_0}, \Gamma_{u_1})$ with $\Gamma_{u_0} = [{A_0}^{\approx\xi}]$, and $\Gamma_{u_1} = [{A_1}^{\approx\xi}]$
(or $\Gamma_{u_1} = [\ominus]$ but in that case that as $[\ominus] = [\ominus^{\approx\xi}]$ we consider it as $\Gamma_{u_1} = [{A_1}^{\approx\xi}]$ with $A_1 = \ominus$; so that there is only one case remaining)

$$A_u = (A_0 \bullet \eta + A_1)^{\approx\xi}$$

(e) As $\Gamma_u = SU(\mathbf{\Sigma}^\circ_\xi, \bigcup_{n\in\omega} \Gamma_{u_n})$ with $\Gamma_{u_n} = [{A_n}^{\approx\xi}]$ for all $n \in \omega$

$$A_u = (\sup_{n\in\omega} A_n)^{\approx\xi}$$

(f) As $\Gamma_u = SD_\eta(\Gamma_{u_0}, \Gamma_{u_1})$ with $u_0 = \xi^\frown 4^\frown < v_n : n \in \omega >$, so that $\Gamma_{v_n} = [{A_n}^{\approx\xi}]$ for all $n \in \omega$ and $A_{u_0} = (\sup_{n\in\omega} A_n)^{\approx\xi}$, and $\Gamma_{u_1} = [B^{\approx\xi}]$
(or $\Gamma_{u_1} = [\ominus]$ but in that case, as in (d) above, $[\ominus] = [\ominus^{\approx\xi}]$ we consider it as $\Gamma_{u_1} = [{A_1}^{\approx\xi}]$ with $A_1 = \ominus$; so that there is only one case remaining)

$$A_u = \left((\sup_{n<\omega} A_n) \bullet \eta + B \right)^{\approx\xi}$$

We come now to the Wadge degree of A_u

First, we define the strictly increasing ordinal following functions:

- $\mathcal{W}_0$ is very close to the exponentiation of base ω_1: $\alpha \longmapsto {\omega_1}^\alpha$ for $\mathcal{W}_0(\alpha) = {\omega_1}^{\alpha+\epsilon}$ where
  - $\epsilon = 1$ if $\alpha = \lambda + n$ with λ limit of cofinality ω and $n \in \omega$
  - $\epsilon = 0$ else
- $\mathcal{W}_{\xi+1}$ is the function that enumerates the fixpoints of the function $\mathcal{W}_\xi$ with cofinality ω_1. i.e. $\mathcal{W}_{\xi+1}(\mu)$ is the μ^{th} fixpoint of cofinality ω_1 of $\mathcal{W}_\xi$
- $\mathcal{W}_\lambda$, for λ limit is the functions that enumerates the ordinals of cofinality ω_1 which are fixpoints of each function $\mathcal{W}_\xi$ for $\xi < \lambda$. i.e. $\mathcal{W}_\lambda(\mu)$ is the μ^{th} fixpoint of cofinality ω_1 of every $\mathcal{W}_\xi$ for $\xi < \lambda$.

And for any countable ordinal $\xi \neq 0$, we define the ordinal function: $\alpha \longmapsto \mathbf{W}_\xi(\alpha)$

- If $\xi = 1$: $\mathbf{W}_1(\alpha) = 1 + \alpha$

○ If $\xi > 1$ and $-1+\xi$ admits a Cantor Normal Form of base $\boldsymbol{\omega}$:

$$-1+\xi = \omega^{\alpha_k}.\nu_k + \ldots + \omega^{\alpha_0}.\nu_0$$

$$\text{with } \omega_1 > \alpha_k > \ldots > \alpha_0 \geq 0 \ \text{ and } \ \forall i \leq k,\ 0 < \nu_i < \omega$$

$$\mathbf{W}_\xi(\alpha) = \underbrace{\mathcal{W}_{\alpha_k} \circ \ldots \circ \mathcal{W}_{\alpha_k}}_{\nu_k} \circ \ldots \circ \underbrace{\mathcal{W}_{\alpha_0} \circ \ldots \circ \mathcal{W}_{\alpha_0}}_{\nu_0}(\alpha)$$

(a) $A_u = \ominus$, so $d_c^\circ A_u = 1$

(b) $A_u = (\ominus \circledast \eta)^{\approx\xi}$, so $d_c^\circ A_u = \mathbf{W}_\xi(\eta)$

(c) $A_u = (A_* \circledast \eta)^{\approx\xi}$, so $d_c^\circ A_u = \mathbf{W}_\xi(d_c^\circ A_*.\eta)$

(d) $A_u = (A_0 \bullet \eta + A_1)^{\approx\xi}$, so $d_c^\circ A_u = \mathbf{W}_\xi(d_c^\circ A_0.\eta + d_c^\circ A_1)$

(e) $A_u = (\sup_{n\in\omega} A_n)^{\approx\xi}$, so $d_c^\circ A_u = \mathbf{W}_\xi(\sup_{n\in\omega} d_c^\circ A_n)$

(f) $A_u = ((\sup_{n<\omega} A_n) \bullet \eta + B)^{\approx\xi}$, so $d_c^\circ A_u = \mathbf{W}_\xi((\sup_{n\in\omega} d_c^\circ A_n).\eta + d_c^\circ B)$

Louveau's Class	Complete Set	Order type under $<_w$
$[\emptyset]$	$\ominus$	1
$D_\eta \boldsymbol{\Sigma}^\circ_\xi$	$(\ominus \circledast \eta)^{\approx\xi}$	$\mathbf{W}_\xi\,(\eta)$
$Sep(D_\eta \boldsymbol{\Sigma}^\circ_\xi, \Gamma_{u^*})$	$(A_* \circledast \eta)^{\approx\xi}$	$\mathbf{W}_\xi\,(d^\circ_c A_*.\eta)$
$Bisep(D_\eta \boldsymbol{\Sigma}^\circ_\xi, \Gamma_{u_0}, \Gamma_{u_1})$	$(A_0 \bullet \eta + A_1)^{\approx\xi}$	$\mathbf{W}_\xi\,(d^\circ_c A_0.\eta + d^\circ_c A_1)$
$SU(\boldsymbol{\Sigma}^\circ_\xi, \cup_n \Gamma_{u_n})$	$(\sup_{n\in\omega} A_n)^{\approx\xi}$	$\mathbf{W}_\xi\,(\sup_{n\in\omega} d^\circ_c A_n)$
$SD_\eta(\Gamma_{u_0}, \Gamma_{u_1})$	$((\sup_{n<\omega} A_n) \bullet \eta + B)^{\approx\xi}$	$\mathbf{W}_\xi\,((\sup_{n\in\omega} d^\circ_c A_n).\eta + d^\circ_c B)$

Louveau's Class	Complete Set	Order type under $<_w$
$[\emptyset]$	$\circleddash$	1
$D_\eta \mathbf{\Sigma}^\circ_\xi$	$\left(\circleddash \bullet (1+\eta)\right)^{\approx\xi}$	$\mathbf{W}_\xi\,(\eta)$
$Sep(D_\eta \mathbf{\Sigma}^\circ_\xi, \Gamma_{u^*})$	$\left(A_* \bullet (1+\eta)\right)^{\approx\xi}$	$\mathbf{W}_\xi\,(d^\circ_c A_*.\eta)$
$Bisep(D_\eta \mathbf{\Sigma}^\circ_\xi, \Gamma_{u_0}, \Gamma_{u_1})$	$(A_0 \bullet \eta + A_1)^{\approx\xi}$	$\mathbf{W}_\xi\,(d^\circ_c A_0.\eta + d^\circ_c A_1)$
$SU(\mathbf{\Sigma}^\circ_\xi, \cup_n \Gamma_{u_n})$	$(\sup_{n\in\omega} A_n)^{\approx\xi}$	$\mathbf{W}_\xi\,(\sup_{n\in\omega} d^\circ_c A_n)$
$SD_\eta(\Gamma_{u_0}, \Gamma_{u_1})$	$((\sup_{n<\omega} A_n) \bullet \eta + B)^{\approx\xi}$	$\mathbf{W}_\xi\,((\sup_{n\in\omega} d^\circ_c A_n).\eta + d^\circ_c B)$