Anti-chains of mappings from ω^ω on some BQO

J.Duparc
Equipe de Logique Mathémathique,
CNRS URA 753 et Université Paris VII
U.F.R. de Mathématiques, 2 place Jussieu, 75251 Paris Cedex 05, France.
duparc@logique.jussieu.fr

Issue: Let $(P, \leq_{\boldsymbol{P}})$ be some BQO. Louveau and Saint-Raymond [1] showed that the following structure $(\mathcal{F}, \leq_{\boldsymbol{\mathcal{F}}})$ is also a BQO:

$$\mathcal{F} = \{\phi : \omega^\omega \longmapsto P : \phi \text{ is Borel with countable image}\}$$

with the usual topology on ω^ω and the discrete topology on the BQO P; and $\phi \leq_{\boldsymbol{\mathcal{F}}} \psi$ iff there exists some continuous function $h : \omega^\omega \longmapsto \omega^\omega$ such that for all $x \in \omega^\omega$ $\phi(x) \leq_{\boldsymbol{P}} \psi \circ h(x)$.

The following proposition answers the question of the relation between cardinalities of anti-chains of P and anti-chains of $\mathcal{F}$?

Proposition 1

1. *Every anti-chain in P has cardinality $1 \Longrightarrow$ every anti-chain in $\mathcal{F}$ has cardinality 1*
2. *There exists an anti-chain in P of cardinality 2, but no element of P is incomparable with two different elements $\Longrightarrow$ every anti-chain in $\mathcal{F}$ has cardinality at most 2*
3. *There exists an element in P which is incomparable with two different elements $\Longrightarrow$ there exists anti-chains of any cardinality in $\mathcal{F}$.*

Proof of proposition 1:

1. Obvious since *II* doesn't have a w.s. in $\mathbb{G}_{\leq}(\phi, \psi)$ implies by determinacy that *II* has a w.s. in $\mathbb{G}_{\leq}(\psi, \phi)$; hence $\leq_{\boldsymbol{\mathcal{F}}}$ is total.
2. For any $q \in P$ define $\check{q}$ by

$$\begin{aligned} \check{q} &= q \text{ if } q \text{ is comparable with every other element from } P \\ &= \text{ the unique element in } P \text{ that is incomparable with } q \text{, otherwise.} \end{aligned}$$

And given $\phi \in \mathcal{F}$ we define $\check{\phi}$ by $\check{\phi}(x) = \widecheck{\phi(x)}$.

We say ϕ is self dual whenever $\check{\phi} \equiv_{\boldsymbol{\mathcal{F}}} \phi$ and ϕ is non self dual whenever $\check{\phi} \not\equiv_{\boldsymbol{\mathcal{F}}} \phi$. We remark

Claim 2 $\psi \not\leq_{\mathcal{F}} \phi \Longrightarrow \check{\phi} \leq_{\mathcal{F}} \psi$

Proof of claim 2: Since *II* doesn't have a w.s. in $\mathbb{G}_{\leq}(\psi, \phi)$, by determinacy *I* has a w.s. in the same game. Winning conditions for *I* are to produce a real x that satisfies $\psi x >_{\boldsymbol{p}} \phi(y)$ or $\psi(x) \perp \phi(y)$ against *II*'s run y. Hence *II* has a w.s. in $\mathbb{G}_{\leq}\left(\check{\phi}, \psi\right)$. 2$\square$

Claim 3 *ϕ self dual $\Longrightarrow$ ϕ is comparable with every other mapping in $\mathcal{F}$*

Proof of claim 3: Immediate from claim 2. 3$\square$

Claim 4

$$(\phi \textit{ non self dual } \wedge \ \psi \not\leq_{\mathcal{F}} \phi) \quad \Longrightarrow \quad (\psi \equiv_{\mathcal{F}} \check{\phi} \ \vee \ \psi \geq_{\mathcal{F}} \phi)$$

Proof of claim 4: Assume $\check{\phi} \not\geq_{\mathcal{F}} \psi$, hence *I* has a w.s. in $\mathbb{G}_{\leq}\left(\psi, \check{\phi}\right)$ which is a w.s. for *II* in $\mathbb{G}_{\leq}(\phi, \psi)$. 4$\square$

This leads to:

Proposition 5

$$\psi \perp \phi \implies \psi \equiv_{\mathcal{F}} \check{\phi}$$

showing that antichains in $\mathcal{F}$ have length at most 2.

3. Given ϕ and ψ we define $\phi^{\frown}\!\!\rightarrow\psi$ as the mapping φ such that for all $x \in (\omega \setminus \{0\})^{\omega}$ $\varphi(x) = \phi(x')$ - where $x'(n) = x(n) - 1$ $(\forall n \in \omega)$ - and $\varphi(x) = \psi(y)$ for all $x = u^\frown\langle 0\rangle^\frown y \in \omega^{\omega}$ where $u(i) \neq 0$ $(\forall i < lh(u))$.

   (a) Assume there exists three different elements $a,\ 0,\ 1 \in P$ such that $a \perp 0$, $a \perp 1$ and $0 <_{\boldsymbol{p}} 1$.

   Define:

   $\psi_0 = \overline{a} \,{}^{\frown}\!\!\rightarrow \overline{0} \,{}^{\frown}\!\!\rightarrow \overline{a} \,{}^{\frown}\!\!\rightarrow \overline{0}$,
   $\psi_1 = \overline{a} \,{}^{\frown}\!\!\rightarrow \overline{1}$.

   And for each $u \in 2^{<\omega} \setminus \langle\rangle$ define ϕ_u by

   $$\phi_u = \psi_{u(0)} \,{}^{\frown}\!\!\rightarrow \psi_{u(1)} \,{}^{\frown}\!\!\rightarrow \ldots {}^{\frown}\!\!\rightarrow \psi_{u(lh(u)-1)}$$

   Proposition 6 *Let $u,\ v \in 2^{<\omega}$ with $lh(u) = lh(v)$,*

   $$u \neq v \Longrightarrow \phi_u \perp \phi_v$$

   Proof of proposition 6: The proof relies on the following

   Claim 7 *Given u and v same length,*

I has winning strategies in both $\mathbb{G}_{\leq}\left(\overline{0}^{\frown}\phi_u, \phi_v\right)$ *and* $\mathbb{G}_{\leq}\left(\overline{1}^{\frown}\phi_u, \phi_v\right)$.

Proof of claim 7: The proof is by induction on $lh(u)$

i. $lh(u) = 1$. We need to check 8 different games. In each of them the winning strategy for I is straightforward:

A. $\mathbb{G}_{\leq}\left(\overline{0} ^\frown \overline{a} ^\frown \overline{0} ^\frown \overline{a} ^\frown \overline{0}\ ,\ \overline{a} ^\frown \overline{0} ^\frown \overline{a} ^\frown \overline{0}\right)$

B. $\mathbb{G}_{\leq}\left(\overline{1} ^\frown \overline{a} ^\frown \overline{0} ^\frown \overline{a} ^\frown \overline{0}\ ,\ \overline{a} ^\frown \overline{0} ^\frown \overline{a} ^\frown \overline{0}\right)$

C. $\mathbb{G}_{\leq}\left(\overline{0} ^\frown \overline{a} ^\frown \overline{1}\ ,\ \overline{a} ^\frown \overline{1}\right)$

D. $\mathbb{G}_{\leq}\left(\overline{1} ^\frown \overline{a} ^\frown \overline{1}\ ,\ \overline{a} ^\frown \overline{1}\right)$

E. $\mathbb{G}_{\leq}\left(\overline{0} ^\frown \overline{a} ^\frown \overline{0} ^\frown \overline{a} ^\frown \overline{0}\ ,\ \overline{a} ^\frown \overline{1}\right)$

F. $\mathbb{G}_{\leq}\left(\overline{1} ^\frown \overline{a} ^\frown \overline{0} ^\frown \overline{a} ^\frown \overline{0}\ ,\ \overline{a} ^\frown \overline{1}\right)$

G. $\mathbb{G}_{\leq}\left(\overline{0} ^\frown \overline{a} ^\frown \overline{1}\ ,\ \overline{a} ^\frown \overline{0} ^\frown \overline{a} ^\frown \overline{0}\right)$

H. $\mathbb{G}_{\leq}\left(\overline{1} ^\frown \overline{a} ^\frown \overline{1}\ ,\ \overline{a} ^\frown \overline{0} ^\frown \overline{a} ^\frown \overline{0}\right)$

ii. $lh(u) > 1$. $u = \langle i_u\rangle^\frown u'$, $v = \langle i_v\rangle^\frown v'$. ϕ_u and ϕ_v end up with and $^\frown\overline{0}$ or an $^\frown\overline{1}$. By induction hypothesis I has a w.s. in the following games: $\mathbb{G}_{\leq}\left(\overline{0}^\frown\phi_{u'}, \phi_{v'}\right)$, $\mathbb{G}_{\leq}\left(\overline{1}^\frown\phi_{u'}, \phi_{v'}\right)$, $\mathbb{G}_{\leq}\left(\overline{0}^\frown\phi_{\langle i_u\rangle}, \phi_{\langle i_v\rangle}\right)$, $\mathbb{G}_{\leq}\left(\overline{1}^\frown\phi_{\langle i_u\rangle}, \phi_{\langle i_v\rangle}\right)$. From which we get winning strategies for I in $\mathbb{G}_{\leq}\left(\overline{0}^\frown\phi_u, \phi_v\right)$, $\mathbb{G}_{\leq}\left(\overline{1}^\frown\phi_u, \phi_v\right)$.

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(b) Assume there exists three different elements $a,\ b,\ c \in P$ such that $a \perp b \perp c \perp a$.

Given $u \in 2^{<\omega}$ define the sequence $v_u \in \{a, b, c\}^{lh(u)}$ by

$$\begin{array}{llll}
v_u(0) & = a & \text{iff} & u(0) = 0 \\
 & = b & \text{iff} & u(0) = 1 \\
 & & & \\
v_u(i+1) & = a & \text{iff} & (u(0) = 0 \wedge v_u(i) = c)\ \vee\ (u(0) = 1 \wedge v_u(i) = b) \\
 & = b & \text{iff} & (u(0) = 0 \wedge v_u(i) = a)\ \vee\ (u(0) = 1 \wedge v_u(i) = c) \\
 & = c & \text{iff} & (u(0) = 0 \wedge v_u(i) = b)\ \vee\ (u(0) = 1 \wedge v_u(i) = a)
\end{array}$$

And finally define the mapping ϕ_u by

$$\phi_u = v_u(0) ^\frown v_u(1) ^\frown \ldots ^\frown v_u(lh(u) - 1)$$

Claim 8 *Let $u,\ v \in 2^{<\omega}$ with $lh(u) = lh(v)$,*

$$u \neq v \Longrightarrow \phi_u \perp \phi_v$$

Proof of claim 8: Straightforward. 8□

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References

[1] **A. Louveau and J. Saint-Raymond** [1990] *On the quasi-ordering of Borel linear orders under embeddability* J. Symbolic Logic **55** (1990), no. 2, 537–560;