A normal form for Borel sets
(extended abstract)

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Abstract

For each Borel set A of reals, or more generally of infinite strings from a countable alphabet, we obtain a "normal form" of A, by finding a Borel set Ω of maximum simplicity, such that A and Ω continuously reduce to each other. Ω only depend on the equivalence class of A modulo inter-reducibility, moreover, the map: $A \longmapsto \Omega$ is defined in a simple way (in $ZF + DC$). In case of Borel sets of finite rank, we prove the above result essentially by defining simple Borel operations which are homomorphic to ordinal sum, to multiplication by a countable ordinal, and to ordinal exponentiation of base ω_1 (under the map which sends every Borel set A of finite rank to its Wadge degree). Extension to transfinite ranks is provided by iterating both ordinal exponentiation and its Borel counterpart. And finally, we show that all these results can be extended to all Borel subsets of X^ω when the alphabet X is of any cardinal.

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1 Introduction

By a **Borel set** we mean a Borel subset A of Σ_A^ω, where Σ_A is some atmost countable alphabet, and the topology on Σ_A^ω is the product of the discrete topology on Σ_A. A **continuous reduction** of A to B is a continuous function f: $\Sigma_A^\omega \longrightarrow \Sigma_B^\omega$ such that $A = f^{-1}B$. It is denoted $A \leq_w B$; intuitively it means that A is less complicated than B -strictly if there is no reduction of B to A; which is denoted $<_w$. If $A \leq_w B \leq_w A$, in other words, A and B continuously reduce to each other, then A is said **Wadge-equivalent** to B, denoted $\equiv_w$.
We recall the definition of a self-dual set: a set A that satisfies $A \equiv_w {}^-A$ where ${}^-A$ stands for the complement of A. The Wadge degree of A, denoted $d_w^\circ\ A$, for A non self-dual, is the largest ordinal α such that there is a sequence $\langle \Omega(\beta)/0 < \beta \leq \alpha \rangle$ of Borel non self-dual sets which satisfies $\beta < \gamma \leq \alpha \longrightarrow \Omega(\beta) <_w \Omega(\gamma) \leq_w A$; and for A self-dual, it is the least Wadge degree of non self-dual sets $>_w$ than A . Wadge and Martin proved that two Borel sets have same Wadge degree iff they are Wadge-equivalent or Wadge-equivalent to the complement of each other. Thus the Wadge degree of A is a measure of the complexity of A, just as the Borel

rank of A. But the former is a much finer measure: to begin with, let ϵ^1 denote the first fixpoint of ordinal exponentiation of base ω_1: $\epsilon^1 = sup_{n<\omega}\omega_1^{(n)}$, where $\omega_1^{(0)} = 1$, $\omega_1^{(n+1)} = \omega_1^{\omega_1^{(n)}}$. Then the degree of Borel sets of finite rank range from 1 to ϵ^1, while the Borel rank only goes from 0 to ω. And things get even much worse for Borel sets of infinite rank: the degree of sets of Borel rank exactly ω, range from ϵ^1 to ϵ^{ω_1} (the ω_1^{th} fixpoint of: $x \longmapsto \omega_1^x$). Thus the single transfinite Borel rank ω refines to ϵ^{ω_1}distinct degrees...

As a consequence, it is quite a problem to provide for each Wadge degree α a description of a Borel set $\Omega(\alpha)$ of degree precisely α. A solution was given essentially by Wadge [19], and confirmed after ten years of work culminating with Louveau [6] and Louveau & Saint-Raymond [7]. But it remains complicated. We present a new solution, its characteristic feature is: to give a Borel definition of $\Omega(\alpha)$ which is isomorphic to the canonical description of the ordinal α itself. As a consequence, the definition of $\Omega(\alpha)$ is remarkably simple; in particular, the map: $\alpha \longmapsto \Omega(\alpha)$ is definable in $ZF + DC$ - as soon as the alphabet of $\Omega(\alpha)$ is allowed to depend on the ordinal α.
Let us explain this solution in case of Borel sets of finite rank, hence of Wadge degree $\alpha < \epsilon^1$. So α has an "iterated Cantor normal form of base ω_1" which describes α, using countable ordinals as parameters and only "arithmetical" operations on ordinals:
the sum: $(\beta, \gamma) \longmapsto \beta + \gamma$, the product: $(\beta, \nu) \longmapsto \beta.\nu$ (for each ν, $0 < \nu < \omega_1$)
and exponentiation: $\beta \longmapsto \omega_1^\beta$.
To each of these operations we associate below an operation defined on all sets $A \subseteq \Sigma_A^\omega$, that maps borel sets to Borel sets:
sum: $(A, B) \longmapsto A \pm B$, multiplication: $(A, \nu) \longmapsto A \bullet \nu$ (for each ν, $0 < \nu < \omega_1$)
and exponentiation: $A \longmapsto A^\sim$.

We forget all self-dual sets (those that reduce to their complement), thus we restrict the Wadge hierarchy to non self-dual sets. Starting with $d_w^\circ \emptyset = 1$, we show:

Theorem 1

(a) $d_w^\circ A \pm B = d_w^\circ A + d_w^\circ B$

(b) $d_w^\circ A \bullet \nu = d_w^\circ A.\nu$

(c) $d_w^\circ A^\sim = \omega_1^{d_w^\circ A + \epsilon}$ *Where* $\epsilon = -1, 0, 1$ *depending on the ordinal* $d_w^\circ A$.

For every $\alpha < \epsilon^1$, this yields the following way to define a Borel set $\Omega(\alpha)$ of degree α:
α has an iterated Cantor normal form: $\alpha = \sum_{i=0}^n \omega_1^{\xi_i}.\nu_i$, with $\epsilon^1 > \xi_0 > \ldots > \xi_n$ and $0 < \nu_i < \omega_1$; and where ξ_i itself is written in normal form, and so on. Then, in order to get $\Omega(\alpha)$, **evaluate** this "iterated polynomial of ordinals" when the ordinal 1 is replaced by the Borel set $\emptyset$, and the ordinal operations are replaced by their above set theoretic counterparts. The above results (1), (2) and (3) guarantee that the resulting Borel set $\Omega(\alpha)$ is of degree α, provided that the correction introduced in (3) by the variable value $\epsilon \in \{-1, 0, 1\}$ is duly taken into account. Thus given any Borel set A of finite rank, knowing the degree α of A we have a trivial procedure for constructing a maximally simple Borel set $\Omega(\alpha)$ such that

$$\forall\ x \in A \longleftrightarrow f(x) \in \Omega(\alpha)$$

for some continuous f.

The very same solution applies to the transfinite Borel ranks; only the Wadge degrees get much higher: up to the "Veblen ordinal of base ω_1". Then one has to replace the above iterated Cantor normal form by the **Veblen** normal form, and extend the operations $A+B$, $A \bullet \nu$, $A^{\sim}$ so as to make them correspond to the operations on ordinals used by Veblen.

Last but not least, we will make a major simplification to the procedure we just sketched. It consists in defining the operation $A+B$, $A \bullet \nu$, $A^{\sim}$, etc, not on subsets of Σ^ω but on subsets of $\Sigma^{\leq\omega}$, i.e. in allowing finite strings too. As a result, the operations become **Borel** functions of canonical character - see next § for explanation.

Remarks 2 Next we give the definitions of $A+B$, $\bullet\nu$, $A^{\sim}$; after which we discuss these definitions to show that these operations are natural. But as just said, we define these operations for sets A, B of **finite or** infinite strings: $A \subseteq \Sigma_A^{\leq\omega}$, not only Σ_A^ω.
In order to define countable multiplication $A \bullet \nu$ $(0 < \nu < \omega^1)$, we define below a **countable sup** operation: $(A_i)_{i\in I} \longmapsto \sup_i A_i$ when $(A_i)_{i\in I}$ is a countable family of sets $A_i \subseteq \Sigma_{A_i}^{\leq\omega}$. This sup operation is easily seen to satisfy:

$$d_w^\circ \sup(A_i)_{i\in I} = \sup_{i\in I} d_w^\circ A_i.$$

So that we can define $A \bullet \nu$ from it, by induction:

$$A \bullet 1 = A, \quad A \bullet (\nu+1) = (A \bullet \nu) + A \;\text{ and }\; A \bullet \nu = \sup_{\alpha<\nu} A_\alpha \;\text{ for } \nu \text{ limit}$$

Definition 3 *(a) Assume Σ_A contains Σ_B and at least two additional symbols e_+, e_- (Rem.4(b) below shows there is no loss of generality). Then*

$$A+B = B \cup (\Sigma_B^*.e_+.A) \cup (\Sigma_B^*.e_-.{}^{\neg}A)$$

(b) Assume $\forall i \in I$, $0 \in \Sigma_{A_i} = \Sigma$ and $0 \notin I \subseteq \Sigma$ (again, there is no loss of generality by Rem.4(b))

$$\sup_{i\in I} A_i = \bigcup_{i\in I, n\in\omega} 0^n i.A_i$$

(c)

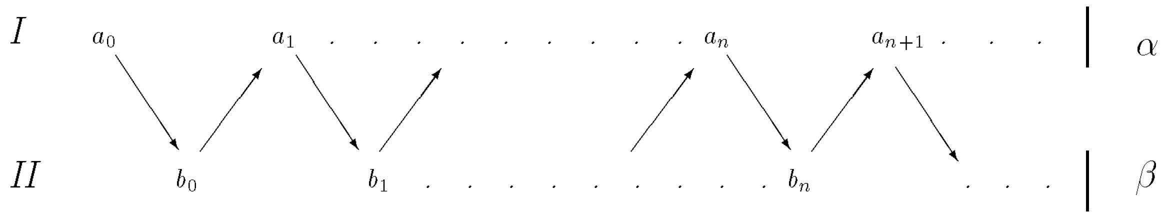
$$A^{\sim} = \{\alpha \in (\Sigma_A^{<\omega})^\omega : \lim_{n\in\omega} \alpha(n) \in A\}$$

$$\textit{where } \beta = \lim_{n\in\omega} \alpha(n) \iff \forall m \exists n \forall p > n \; \beta(m) \textit{ is an initial segment of } \alpha(p)$$

Remarks 4 (a) In fact, all notions we define are extrinsic. i.e. they are properties of (A, Σ_A) rather than A. But, by abuse of language, we omit Σ_A.

(b) In most cases, $\Sigma_A \subsetneq \Sigma \Longrightarrow (A, \Sigma_A) \equiv_w (A, \Sigma)$. But when it is not the case, we replace A by $A^{\triangleleft}$, the set of all sequences in Σ^ω which belong to A when omitting every letter not in Σ_A; i.e. $A^{\triangleleft}$ is the subset of Σ^ω which gives rise to A when looking at it while blinded to any letter not in Σ_A. Thus guaranteing $(A, \Sigma_A) \equiv_w (A^{\triangleleft}, \Sigma)$

Definition 5 *Given $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$, the Wadge game $\boldsymbol{W}(A, B)$ is the following two players game, where players take turn (except that II is allowed to skip), producing two infinite sequences: $\alpha \in \Sigma_A^\omega$ for I and $\beta \in \Sigma_B^\omega$ for II.*



And II wins this game iff ($\alpha \in A \longleftrightarrow \beta \in B$), I wins in the other case.

Then:

$$A \leq_w B \iff \textit{II has a winning strategy in } \boldsymbol{W}(A, B)$$

Remarks 6 (a) In the context of the Wadge game, the definition of $A + B$, $\sup A_i$, $A^{\sim}$ becomes most natural:

- if a player is in charge of $A + B$ in a Wadge game, then he begins the game as if he was playing with $A \subseteq \Sigma_A^{\omega}$. As long as he doesn't play e_+ nor e_-, everything goes as if he was in charge of A. But during the run, he can change his mind and decide to be in charge of B or ^{-}B. Playing e_+ he becomes in charge of B, and playing e_-, he becomes in charge of ^{-}B. But there is no way back, that change is done for ever.
- A player in charge of $\sup A_i$ can choose the set he wants to be in charge of. The first letter "i"$\in I$ that he plays, makes him be in charge of A_i. As for the previous operation, there is no way back: the choice is irreversible (the letters he can play before i allow him to wait before deciding for i).
- For player in charge of $A^{\sim}$, everything goes as if he was in charge of A, but with an extra ability which consists in, at any moment of the run, erasing any terminal sequence played before. This is the same as saying he's got an additional letter which is an eraser, so that he can change his mind as long as he wants to.

(b) The definition of $A^{\sim}$, given above is the one when $A \subseteq \Sigma_A^{\leq\omega}$, more precisely, it is the "conciliatory" definition - as explained below. Some changes are needed when dealing only with the Wadge hierarchy, that is with $A \subseteq \Sigma_A^{\omega}$ (see thm.7 and 9(b) below).

2 Away with self-dual sets: the conciliatory Wadge hierarchy

For the purpose of considering only non self-dual sets of finite rank, it is enough to replace sets of ω-sequences by sets of $\leq \omega$-sequences. Hence, we say a set A is **conciliating** if it is a subset of $\Sigma_A^{\leq\omega}$, where Σ_A is at most countable. And given two conciliating sets A, B, we define the conciliatory version $\mathbf{C}(A, B)$ of the Wadge game W(A,B), which is this game when the sequences α, β produced by the players may be finite. Remember that in the Wadge game $\mathbf{W}(A, B)$, I can't skip while II is allowed to skip even ω times, provided he also plays (a letter in Σ_B) ω times. In the conciliatory form $\mathbf{C}(A, B)$, **both** players are allowed to skip, even indefinitely if they want so. Thus their final choices α, β may be finite: $\alpha \in \Sigma_A^{\leq\omega}, \beta \in \Sigma_B^{\leq\omega}$. In any case, we say that II wins $\mathbf{C}(A, B)$ iff $\alpha \in A \longleftrightarrow \beta \in B$. We write $A \leq_c B$ if II has a winning strategy in the conciliatory Wadge game $\mathbf{C}(A, B)$. As it is for the Wadge notions $\equiv_w$ and $<_w$ we define

$\equiv_c$ and $<_c$. That is A$\equiv_c$B if A$\leq_c$B$\leq_c$A, and A$<_c$B if A$\leq_c$B **but** B$\not\leq_c$A. Now it is easy to see in the conciliatory Wadge hierarchy, every conciliating set A is non self-dual:$A \not\equiv_c {}^{-}A$. i.e. player *I* has a winning strategy in $\mathbf{C}(A, {}^{-}A)$. Namely, he passes its first move and later on, he always does what *II* did the move before: *I* passes if *II* passed, and *I* chooses the letter $a \in \Sigma_A$ if *II* chose this letter before. So they both produce the same sequence $\alpha \in \Sigma_A^{\leq\omega}$, which means *I* wins.

The conciliatory Wadge hierarchy joins the previous one when sets are crammed with a specific letter "b" (for blanks) so as to turn them into sets of infinite sequences. That is, given a conciliating set $A \subseteq \Sigma_A^{\leq\omega}$, let "b" denote a letter not in Σ_A, and set $A^b = \{\alpha \in (\Sigma_A \cup \{b\})^\omega / \alpha \textit{ is a variant of some } x \in A\}$; where α is a variant of x if it is, in this precise case, an infinite word reducing to x when all "blanks" are removed.
Given A, B two conciliating sets, it is easy to see that a player wins $\mathbf{C}(A, B)$ iff the same player wins W(A^b,B^b); so, A$\leq_c$B iff $A^b \leq_w B^b$; and in particular, $A^b \not\equiv_w {}^{-}A^b$. In fact, there is an obvious isomorphism between strategies σ for $\mathbf{C}(A, B)$ and σ^b for W(A^b,B^b): σ passes iff σ^b chooses a "blank".
One of the final outcomes of the whole present study will be the proof of the next theorem.

Theorem 7
For every Borel set $A \subseteq \Sigma_A^\omega$,

$$A \textit{ is non self-dual}$$
$$\Longleftrightarrow$$
$$\textit{there is a set } s(A) \textit{ of finite sequences from } \Sigma_A, \textit{ such that } d_c^\circ(A \cup s(A)) = d_w^\circ A$$

(hence $A \equiv_w (A \cup s(A))^b$).

Remarks 8 Thus the conciliating hierarchy is almost the same as the Wadge hierarchy restricted to non self dual sets

Now there is a trivial procedure which derives the structure of the self-dual classes from the non self-dual ones. So in the sequel, we will concentrate on the conciliatory hierarchy and let self-dual classes take care of themselves.

The introduction sketched Thm.1 in imprecise manner, since it did not define $A + B$, $A \bullet \nu$, $A^\sim$ and did not distinguish the two versions of these operations: the one for sets of infinite strings, and the one for conciliating sets. Here is the precise form.

Theorem 9

(a) for conciliating sets $A \subseteq \Sigma_A^{\leq\omega}$, $B \subseteq \Sigma_B^{\leq\omega}$, define $A + B$, $A \bullet \nu$, $A^\sim$ as done in Rem.2 and Def.3. Then

$$d_c^\circ A + B = d_c^\circ A + d_c^\circ B, \qquad d_c^\circ A \bullet \nu = d_c^\circ A.\nu, \qquad d_c^\circ A^\sim = \omega_1^{d_c^\circ A + \epsilon}$$
where ϵ=-1, 0, +1 depending on the ordinal $d_c^\circ A$

(b) For non conciliatory sets $A \subseteq \Sigma_A^\omega$, continue to define $A + B$ and $A \bullet \nu$ as in (a) but set

$$A^\sim = (A \cup s(A))^{\sim b}$$

where $s(A)$ comes from Thm.7 hence $A \cup s(A)$ becomes a conciliating set; then Thm.1 holds.

We next sketch the proof of Thm. 9(a) (clearly, (b) follows from (a) using Thm.7). Let A, B, C be conciliating sets with $B \leq_c C$; from a w.s. for *II* in $\mathbf{C}(B,C)$ using Rmk.6(a) and Rmk.2 it is easy to derive winning strategies for *II* in $\mathbf{C}(A+B, A+C)$, $\mathbf{C}(B^{\sim}, C^{\sim})$, and $\mathbf{C}(B \bullet \nu, C^{\sim})$ and to successively prove:

- $d_c^{\circ}(A+B) \geq d_c^{\circ}A + d_c^{\circ}B$
- $d_c^{\circ}(A \bullet \nu) \geq (d_c^{\circ}A).\nu \quad (0 < \nu < \omega_1)$
- $d_c^{\circ}(A^{\sim}) \geq d_c^{\circ}(A \bullet \nu)$ hence $\geq (d_c^{\circ}A).\omega_1$ (if $d_c^{\circ}A > 1$)
- $d_c^{\circ}(A^{\sim}) \geq \omega_1^{1+d_c^{\circ}A}$ ($>$ if $d_c^{\circ}A$ is of cofinality ω)

There remains to prove reverse inequalities:

- $d_c^{\circ}(A+B) \leq d_c^{\circ}A + d_c^{\circ}B$ is relatively easy using again Rmk.6(a).
- $d_c^{\circ}(A \bullet \nu) \leq (d_c^{\circ}A).\nu$ (hence $\leq (d_c^{\circ}A).\omega_1$) follows from above and Rem.2.
- And $d_c^{\circ}(A^{\sim}) \leq \omega_1^{(d_c^{\circ}A)+1}$ is proved essentially by providing an inverse operation: $A \longmapsto A^{\approx}$ which composed with: $A \longmapsto A^{\sim}$, is the identity: $d_c^{\circ}(A^{\sim\approx})$ is the smallest degree $d_c^{\circ}A$ of the form $\omega_1^{1+\alpha}$ with α successor or of cofinality ω_1.

  Thus the degrees $d_c^{\circ}A^{\sim}$ enumerate **all** degrees $\omega_1^{1+\alpha}$ ($\mathrm{cof}(\alpha) \neq \omega$); hence $d_c^{\circ}A^{\sim}$ cannot arrive before $\omega_1^{(d_c^{\circ}B)+1}$.
  Since $A \longmapsto A^{\sim}$ is the Borel counterpart of exponentiation, its inverse: $A \longmapsto A^{\approx}$ will be a kind of set theoretic logarithm - a rather new concept in Descriptive Set Theory !

3 The Set Theoretic Logarithm

Here self dual sets regain some usefulness; so in this § we return to the Wadge (non conciliatory) hierarchy. But by Thm.7, the results below which deal with non self dual sets immediately go over to the conciliatory hierarchy.
We introduce the property of being initializable. It is a property concerning both conciliatory sets and sets of infinite strings (only non self dual ones are concerned, because self dual sets do not share it). A set A is **strongly initializable** if every position for a player in charge of such a set in a (conciliatory) Wadge game is (conciliatory) Wadge-equivalent to the initial one. This means, in the case A is conciliatory, *II* has a w.s. in the game $\mathbf{C}(A,A)$ where I is allowed to erase, just once, all his previous moves; and in the case A is a set of infinite strings,*II* has a w.s. in the game $\mathbf{W}(A,A)$ where I has same extra possibility. And we say A is **initializable** if it is (conciliatory) Wadge-equivalent to a strongly initializable set.
Now, every set S which is not initializable is equivalent to a set of the form $A+B$ where A and B are both strictly below S, in the Wadge hierarchy; or of the form $\sup_{i\in I} A_i$ for a countable I and each A_i also striclty below S. In fact, we show that the $^{\sim}$ operation is 1-1 and onto the initializable sets. It is 1-1 because it preserves the conciliatory Wadge ordering,and it is onto due to the existence of an inverse operation $^{\approx}$ defined below.

Recall that, given an alphabet Σ, T is a **non empty puned (n.e.p.) tree** on Σ if it is:

- **a non empty tree** that is a non empty subset of $\Sigma^{<\omega}$ closed under subsequences: $v \in T$ and $u \subseteq v \Longrightarrow u \in T$,

- **with no leaves**: $\forall u \in T\ \exists v \in T\ u \subsetneq v$.

And the set of all infinite branches of such a tree is denoted $[T]$: $\alpha \in [T] \iff \forall n \in \omega\ \alpha \restriction n\ \in T$.

Definition 10 *Given $A \subseteq \Sigma_A^\omega$ and $(F_i)_{i\in\omega}$ a sequence of closed subsets of Σ_A^ω, we define $A^{(F_i)_{i\in\omega}} \subseteq [T_{A^{(F_i)_{i\in\omega}}}]$, where $T_{A^{(F_i)_{i\in\omega}}}$ is a non empty pruned (n.e.p.) tree on the alphabet $\Sigma_A \cup \{0\} \cup \{<1>{}^\frown u:\ u \in \Sigma_A\}$ and $A^{(F_i)_{i\in\omega}}$ is a set of infinite branches of this tree. In fact, $[T_{A^{(F_i)_{i\in\omega}}}]$ represents all the allowed positions for a player in charge of the set $A^{(F_i)_{i\in\omega}}$. So we describe the tree by explaining how goes a run with a set like $A^{(F_i)_{i\in\omega}}$ for a player in a Wadge game. In this kind of run, odd moves are auxiliary moves where the player answers questions about wether or not he will stay all along in the closed sets Fi's; even moves are letters from Σ_A. So assume the run is $\alpha \in [T_{A^{(F_i)_{i\in\omega}}}]$, for all integer n, one must have:*

- $\alpha(2n) \in \Sigma_A$
- $\alpha(2n+1) = 0$ *or* $\alpha(2n+1) =<1>{}^\frown u$ *with* $u \in \Sigma_A$*; depending on the option he takes:*

  $\alpha(2n+1) = 0$ *the player garantees the sequence $\frac{\alpha}{2} =< \alpha(2i)/i \in \omega >$, he is playing , will remain in F_n.*

  $\alpha(2n+1) =<1>{}^\frown u$ *the player chooses the option to get out of F_n (that's why he plays 1); and indicates with $u \in \Sigma_A^{<\omega}$ the way he wants to exit. This means α will satisfy $< \alpha(0), \alpha(2), \ldots, \alpha(2n) >{}^\frown u = v$ is an initial segment of $\frac{\alpha}{2}$ that leads out of F_n i.e. $v.\Sigma_A^\omega\ \cap\ F_n = \emptyset$. From now on, the next next $lh(u)$ odd moves are determined by this agreement.*

So, finally, $T_{A^{(F_i)_{i\in\omega}}}$ is the set of all legal moves in this game, and $A^{(F_i)_{i\in\omega}}$ is defined by

$$\alpha \in A^{(F_i)_{i\in\omega}} \iff \frac{\alpha}{2} \in A.$$

Borel Determinacy states that for any alphabet Σ_A, any n.e.p. tree T_A on such alphabet, and any $A \subseteq [T_A]$ which is Borel in $[T_A]$ for the topology induced by the usual topology on Σ_A^ω, the game $\mathbf{G}(A, T_A)$, is determined. This game is a two players game with perfect information, where players both take turn choosing a letter $\in \Sigma_A$, and restricting their moves to T_A, so that after ω moves, they have produced a string $(a_0, a_1, a_2, \ldots) \in [T_A]$. Player *I* starts, and wins iff $(a_0, a_1, a_2, \ldots) \in A$; player *II* wins in the other case.

Given non empty pruned trees T_A and T_B, define the (extended) Wadge game $\mathbf{W}(A, T_A, B, T_B)$ as the usual Wadge game $\mathbf{W}(A, B)$ with the difference that player *I*'s moves are restricted to T_A and player *II*'s move are restricted to T_B. So that at the end of a run *I* has produced $\alpha \in [T_A]$ and *II* has produced $\beta \in [T_B]$, the winning conditions remaining the same: *II* wins iff $\alpha \in A \longleftrightarrow \beta \in B$.
Now, if $A \cap [T_A]$ and $B \cap [T_B]$ are Borel, by Borel determinacy the extended Wadge game $\mathbf{W}(A, T_A, B, T_B)$ is determined; and furthermore, the Wadge ordering on Borel subsets of n.e.p. trees is well-founded.

Remarks 11 (a) For any sequence $(F_i)_{i\in\omega}$ of closed subsets of Σ_A^ω,

$$A^{(F_i)_{i\in\omega}} \leq_w A.$$

(b) The set of all $A^{(F_i)_{i\in\omega}}$, where $(F_i)_{i\in\omega}$ ranges over all possible sequences of closed subsets of Σ_A^ω, is well-founded; further more, any two minimal elements are Wadge equivalent.

So, we are now founded to set:

Definition 12 *Let $A \subseteq \Sigma_A^\omega$, A Borel,*

$$A^{\approx} \textit{ is a } \leq_w\textit{-minimal element of } \{A^{(F_i)_{i\in\omega}} : F_i \in \mathbf{\Pi}_1^0 \upharpoonright \mathbf{\Sigma}_A^\omega\}.$$

Nota Bene: AC is needed to guarantee that $A^{\approx}$ is chosen uniquely; but only its existence will be used below, not its uniqueness.

This operation satisfies:

Proposition 13 *Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$, A, B both Borel, and $C \subseteq \Sigma_c^{\leq\omega}$, C^b Borel*

(a) $A \leq_w B \Longrightarrow A^{\approx} \leq_w B^{\approx}$

(b) *A is initializable $\Longleftrightarrow$ $A^{\approx}$ is non self dual*

(c) $(C^{\sim})^{b^{\approx}} \equiv_w C^b$

4 Borel sets of finite rank

With the help of $^{\approx}$ which is a kind of operation allowing to ask questions about a countable number of closed sets (or from a topological point of view, to turn them into clopen sets) it is easy to change a $\mathbf{\Sigma}_{n+1}^\circ$ set into a $\mathbf{\Sigma}_n^\circ$ one (for $0 < n < \omega$). So, as $^{\approx}$ is the inverse of $^{\sim}$, this gives rise to:

Proposition 14 *Let A be a $\mathbf{\Sigma}_1^\circ$-complete set, $\forall n < \omega$, $(A^{\sim . n})^b$ is $\mathbf{\Sigma}_{n+1}^\circ$-complete: it is a $\mathbf{\Sigma}_{n+1}^\circ$ set, and every $\mathbf{\Sigma}_{n+1}^\circ$ set continuously reduces to it.*

Nota Bene: Our normal form of $\mathbf{\Sigma}_1^\circ$-complete sets is $\emptyset \bullet 2$, so that $(\emptyset \bullet 2^{\ \sim . n})^b$ is $\mathbf{\Sigma}_{n+1}^\circ$-complete.

With the help of operations $+$, sup and $^{\sim}$, one can generate, starting with the emptyset $\emptyset$, the initial segment of the Wadge hierarchy that corresponds to Borel sets of finite rank. In fact, we can reach this way all sets in $\mathbf{\Delta}_\omega^\circ = \mathbf{\Sigma}_\omega^\circ \cap \mathbf{\Pi}_\omega^\circ$; that is sets with Wadge degree below ϵ^{ω_1}, the ω_1^{th} fixpoint (the first which is not of cofinality ω) of exponentiation of base ω^1; while those in $\bigcup_{n\in\omega} \mathbf{\Sigma}_n^\circ$ have Wadge degree less than ϵ^1 the first fixpoint of exponentiation of base ω^1.

Definition 15 *Let $0 < \alpha < \epsilon^{\omega_1}$, α admits a Cantor normal form of base ω_1:*

$$\alpha = \omega_1^{\alpha_k}.\nu_k + \ldots + \omega_1^{\alpha_0}.\nu_0$$

$$\textit{with } \epsilon^{\omega_1} > \alpha \geq \alpha_k > \ldots > \alpha_0 \geq 0 \ \textit{ and } \ \forall i \leq k,\ 0 < \nu_i < \omega_1.$$

Define:
$\Omega(\alpha) = \Omega(\omega_1^{\alpha_k}) \bullet \nu_k \pm \ldots\ldots \pm \Omega(\omega_1^{\alpha_0}) \bullet \nu_0$

where $\Omega(\omega_1^\beta)$ *with* $0 \leq \beta < \epsilon^{\omega_1}$ *is defined by:*

If $\beta = 0$:	$\Omega(\omega_1^\beta) = \Omega(1) = \emptyset$
If $\beta = n \in \omega$ *and* $n \neq 0$:	$\Omega(\omega_1^\beta) = \Omega(\beta+1)^\sim$
If $\beta = \lambda + n$, $cof(\lambda) = \omega_1$:	$\Omega(\omega_1^\beta) = \Omega(\beta)^\sim$
If $\beta = \lambda + n$, $cof(\lambda) = \omega$ *and* $0 \neq n \in \omega$:	$\Omega(\omega_1^\beta) = \Omega(\beta - 1)^\sim$
If $\beta = \lambda$, $cof(\lambda) = \omega$:	$\Omega(\omega_1^\beta) = \sum_{\xi<\lambda} \Omega(\omega_1^\xi)$

It is a consequence of theorem 1 that for such an ordinal α, the conciliatory degree of $\Omega(\alpha)$ is precisely α. So, assuming theorem 7 we obtain:

Theorem 16 *Let* $A \subseteq \Sigma_A^\omega$, *A non self-dual.*

$$A \in \bigcup_{n<\omega} \mathbf{\Sigma}_n^\circ \iff \exists \alpha,\ 0 < \alpha < \epsilon^1 \textit{ such that } A \textit{ or } {}^{-}A \ \equiv_w \ \Omega(\alpha^{(b)})$$

Recall that if A is self-dual, either $\exists B$ *non self* $-$ *dual* $A \equiv_w 0.B \cup 1.^{-}B$
or $\exists\ < B_i / i < \omega >$ *each* $B_i\ <_c A$ *and non self* $-$ *dual and* $A \equiv_w \bigcup_{i<\omega} i.B_i$.

Thus theorem 16 provides for every set A of finite Borel rank a normal form $\Omega \equiv_w A$.

5 Borel sets of transfinite rank

Now, if we want to go further, that is to generate the whole Borel hierarchy, we need to iterate the $^\sim$ operation and unfold games. This means, with the help of an "iteration" of $^\sim$ called $^{\sim 1}$, we get all the $\mathbf{\Delta}^\circ_{\omega^2}$ sets, and more generally, with all operations $,^{\sim\xi}$ for $\xi \leq \alpha$, we get the whole $\mathbf{\Delta}^\circ_{\omega^{1+\alpha}}$ hierarchy.
To begin with, we describe the first new one: $^{\sim 1}$ by explaining when a sequence α played by a player in charge of $A^{\sim 1}$ does belong to this set.
First of all, the sequence α has to be regarded (with the help of any fixed bijection between ω and ω^2) as a countable sequence of ω sequences $(\alpha_n)_{n\in\omega}$ with α_n a sequence of the kind played by someone in charge of the set $A\underbrace{^{\sim\sim\cdots\sim}}_n$; now this sequence α_n gives rise (after reducing α_n by taking limits n times) to the sequence $\beta_n \in \Sigma_A^{\leq\omega}$ which by definition of $A\underbrace{^{\sim\sim\cdots\sim}}_n$, satisfies $\alpha_n \in A\underbrace{^{\sim\sim\cdots\sim}}_n \iff \beta_n \in A$. Finally, we consider only the first letter of β_n in order to form the sequence β of all the first letters of the β_n's: $\beta = (\beta_n(0))_{n\in\omega}$; and we set

$$\alpha \in A^{\sim 1} \iff \beta \in A$$

We define the operation $^{\sim\xi+1}$ on the basis of $^{\sim\xi}$ the same way we did $^{\sim 1}$ on the basis of $^\sim$. So $\alpha \in A^{\sim\xi+1} \iff \beta \in A$ where $\beta = (\beta_n(0))_{n\in\omega}$ each β_n defined by taking "ξ limits n times" of the sequence α_n: so that $\alpha_n \in A^{\underbrace{\sim\xi\sim\xi\cdots\sim\xi}_{n}} \iff \beta_n \in A$ holds by definition. And we set

$$\alpha \in A^{\sim\xi+1} \iff \beta \in A$$

For any countable limit ordinal $\lambda = \sup_{n\in\omega} \lambda_n$, $^{\sim\lambda}$ is constructed in a similar way, replacing the condition on β_n by: $\alpha_n \in A^{\sim\lambda_n} \iff \beta_n \in A$.

And for any countable ordinal ξ, we define $^{\not\sim\xi}$ a kind of "inverse" operation for $,^{\sim\xi}$ iterating what has been done to create $^{\not\sim}$. Of course, the tree of all legal positions for a player in charge of a set of the form $A^{\not\sim\xi}$ becomes very messy as ξ grows up, but the idea remains simple: the player must answer questions about $\boldsymbol{\Sigma}^{\circ}_{<\omega^{\xi}}$ sets.
These operations are the Borel counterpart of ordinal operations growing far much quicker than the exponential of base ω_1. They were first introduced by Veblen in the first decade of this century, and consists in functions, from ordinals to ordinals, enumerating the fixpoints of previous ones.

Definition 17 *Given a cardinal κ, the Veblen hierachy consists of the functions $(V_{\xi}^{\kappa^+})_{\xi\in On}$, $V_{\xi}^{\kappa^+} : On \longmapsto On$ defined by the fact $V_{\xi}^{\kappa^+}(0) = 1$ and:*

- *$V_0^{\kappa^+}$ is the exponentiation of base κ^+: $\alpha \longmapsto {\kappa^+}^{\alpha}$*
- *$V_{\xi+1}^{\kappa^+}$ is the functions that enumerates the fixpoints of the function $V_{\xi}^{\kappa^+}$ with cofinality κ^+. i.e. $V_{\xi+1}^{\kappa^+}(\mu)$ is the μ^{th} fixpoint of cofinality κ^+ of $V_{\xi}^{\kappa^+}$*
- *$V_{\lambda}^{\kappa^+}$, for λ limit is the functions that enumerates the ordinals of cofinality κ^+ which are fixpoints of each function $V_{\xi}^{\kappa^+}$ for $\xi < \lambda$. i.e. $V_{\lambda}^{\kappa^+}(\mu)$ is the μ^{th} fixpoint of cofinality κ^+ of every $V_{\xi}^{\kappa^+}$ for $\xi < \lambda$.*

So, what we called ϵ^{ω_1} takes place in the Veblen hierarchy as the ordinal $V_1^{\omega_1}(1)$.

Now the operations $^{\sim\xi}$ satisfy, given any "Borel" conciliatory set A and any countable ordinal ξ,

$$d_c^{\circ} A^{\sim\xi} = V_{\xi}^{\omega_1}(-1 + d_c^{\circ} A)$$

As with the simple $^{\sim}$ operation, and with the help of the "inverse operations", we obtain the right ordinal corresponding to a $\boldsymbol{\Sigma}^{\circ}_{\xi}$ set as we did for the finite rank in prop. 14 and assuming that $^{\sim 0}$ is $^{\sim}$:

Proposition 18 *Given $A \subseteq \Sigma_A^{\leq\omega}$, A $\boldsymbol{\Sigma}^{\circ}_1$-complete and $0 \leq \nu_0 \leq \nu_1 \leq \ldots \leq \nu_k < \omega_1$*

$$(((A^{\sim\nu_0})^{\sim\nu_1})^{\cdots})^{\sim\nu_k} \textit{ is } \boldsymbol{\Sigma}^{\circ}_{1+\omega^{\nu_k}+\ldots+\omega^{\nu_1}+\omega^{\nu_0}}\textit{-complete}$$

More generally, given $B \subseteq \Sigma_B^{\leq\omega}$, B $\boldsymbol{\Sigma}^{\circ}_{\xi}$-complete and $0 \leq \nu_0 \leq \nu_1 \leq \ldots \leq \nu_k < \omega_1$

$$(((B^{\sim\nu_0})^{\sim\nu_1})^{\cdots})^{\sim\nu_k} \textit{ is } \boldsymbol{\Sigma}^{\circ}_{\xi+\omega^{\nu_k}+\ldots+\omega^{\nu_1}+\omega^{\nu_0}}\textit{-complete}$$

So, this leads to the following way to define a Borel set of degree precisely α and be sure the whole Borel hierarchy of non self dual sets is covered (every Borel non self dual set being Wadge equivalent to a set Ω^b or to its complement, with Ω as below):

Definition 19 *Let* $0 < \alpha < V_{\omega_1}^{\omega_1}(1)$*,* α *admits a Cantor Normal Form of base* ω_1*:*

$$\alpha = \omega_1^{\alpha_k}.\nu_k + \ldots + \omega_1^{\alpha_0}.\nu_0$$

with

$$V_{\omega_1}^{\omega_1}(1) > \alpha_k > \ldots > \alpha_0 \geq 0 \text{ et } \forall i \leq k,\ 0 < \nu_i < \omega_1$$

Define:

$$\Omega(\alpha) = \Omega(\omega_1^{\alpha_k}) \bullet \nu_k \pm \ldots\ldots \pm \Omega(\omega_1^{\alpha_0}) \bullet \nu_0$$

where $\Omega(\omega_1^{\beta})$ *with* $0 \leq \beta < V_1^{\omega_1}$ *defined by:*

$\beta = 0:$	$\Omega(\omega_1^{\beta}) = \emptyset$
$\beta = n,\ \ 0 \neq n \in \omega:$	$\Omega(\omega_1^{\beta}) = \Omega(1+\beta)^{\sim}$
$\beta = \lambda + n, \lambda$ *of cofinality* ω_1 *and* $0 \neq n \in \omega:$	$\Omega(\omega_1^{\beta}) = \Omega(\beta)^{\sim}$
$\beta = \lambda + n, \lambda$ *of cofinality* ω *and* $0 \neq n \in \omega:$	$\Omega(\omega_1^{\beta}) = \Omega(\beta - 1)^{\sim}$
β *limit of cofinality* $\omega:$	$\Omega(\omega_1^{\beta}) = \sup_{\xi<\beta} \Omega(\omega_1^{\xi})$
β *limit of cofinality* ω_1	$\Omega(\omega_1^{\beta}) = \Omega(1+\gamma)^{\sim\nu}$

with ν *least ordinal* $\exists\gamma\ \beta = V_{\nu}^{\omega_1}(\gamma)$ *and* γ *is not a fixpoint of cofinality* ω_1 *of:* $x \longmapsto V_{\nu}^{\omega_1}(x)$*.*

6 The Generalized Wadge hierarchy

The study of the Wadge hierarchy rested on Martin's Borel determinacy which states that every game $\mathbf{G}(A, T_A)$ is determined provided A is Borel in T_A. This statement is not only true for T_A a n.e.p. tree built on a countable alphabet Σ_A, but for T_A a n.e.p. tree on an alphabet of any cardinality. As Borel determinacy induces Borel Wadge determinacy, every game $\mathbf{W}(A, B)$ where A and B are Borel subsets of respectively Σ_A^{ω} and Σ_B^{ω} is determined independently of the cardinality of Σ_A and Σ_B. So, this gives right to the study of the Wadge hierarchy on Borel subsets of Σ^{ω} where Σ is of cardinality $\kappa > \omega$.
Borel Wadge determinacy makes $\leq_w$ be a prewellordering of Borel subsets of Σ^{ω}; one also still has the fact two sets have same degree iff they are Wadge equivalent or Wadge equivalent to the complement of the other; and the structure of self dual sets derives from the structure of non self dual ones. The only difference is in the heigth of the hierarchy; more precisely, to get from floor $\mathbf{\Sigma}_{\xi}^{\circ}$ to floor $\mathbf{\Sigma}_{\xi+1}^{\circ}$ one needs to go κ^+ steps whereas one only did ω_1 of them with a countable alphabet. So we might say every thing is the same as in the countable situation, turning ω_1 into κ^+, except of course the fact that there remains only ω_1 Borel ranks: any Borel set is still $\mathbf{\Sigma}_{\xi}^{\circ}$ with $\xi < \omega_1$, not $\xi < \kappa^+$.

Now, considering an alphabet Σ of cardinality $\kappa > \omega$, with the help of the sup operation ($\sup_I$ where $card(I) < \kappa^+$ - cf Def.3), one can extend multiplication to any cardinal less than κ^+. And when looking at all Borel subsets of Σ^{ω} in terms of Wadge class generators, these simple extensions of the sup-operation suffice to derive the whole hierarchy, with the help of the other ones. This means that every non self dual Borel set is Wadge equivalent to a set Ω^b, or to its complement, where Ω is a conciliatory set of the form below.

Definition 20 *Let $0 < \alpha < V_{\omega_1}^{\kappa^+}(1)$, α admits a Cantor Normal Form of base κ^+:*

$$\alpha = \kappa^{+^{\alpha_k}}.\nu_k + \ldots + \kappa^{+^{\alpha_0}}.\nu_0$$

with

$$V_{\omega_1}^{\kappa^+}(1) > \alpha_k > \ldots > \alpha_0 \geq 0 \ et\ \forall i \leq k,\ 0 < \nu_i < \kappa^+$$

Define:

$$\Omega(\alpha) = \Omega(\kappa^{+^{\alpha_k}}) \bullet \nu_k \pm \ldots\ldots \pm \Omega(\kappa^{+^{\alpha_0}}) \bullet \nu_0$$

where $\Omega(\kappa^{+^{\beta}})$ with $0 \leq \beta < V_1^{\kappa^+}$ defined by:

$\beta = 0:$ $\qquad \Omega(\kappa^{+^{\beta}}) = \emptyset$

$\beta = n, \quad 0 \neq n \in \omega:$ $\qquad \Omega(\kappa^{+^{\beta}}) = \Omega(1+\beta)^{\sim}$

$\beta = \lambda + n, \lambda$ *of cofinality* κ^+ *and* $0 \neq n \in \omega:$ $\qquad \Omega(\kappa^{+^{\beta}}) = \Omega(\beta)^{\sim}$

$\beta = \lambda + n, \lambda$ *of cofinality* $< \kappa^+$ *and* $0 \neq n \in \omega:$ $\qquad \Omega(\kappa^{+^{\beta}}) = \Omega(\beta - 1)^{\sim}$

β *limit of cofinality less than* κ^+ : $\qquad \Omega(\kappa^{+^{\beta}}) = \sup_{\xi<\beta} \Omega(\kappa^{+^{\xi}})$

β *limit of cofinality* κ^+ $\qquad \Omega(\omega_1^{\beta}) = \Omega(1+\gamma)^{\sim^{\nu}}$
with ν least ordinal $\exists \gamma\ \beta = V_\nu^{\kappa^+}(\gamma)$ and γ is not a fixpoint of cofinality κ^+ of: $x \longmapsto V_\nu^{\kappa^+}(x)$.

The proofs are very close to the ones needed in the countable alphabet context, even when considering inverse operations for the $^{\sim\xi}$'s, the $^{\nsim\xi}$'s. The fact the topology is not polish at all, because a basis contains at least κ open sets, is not a problem any more, for our methods do not really use usual methods of Descriptive Set Theory. What helps us is only that games we play are in ω moves, i.e. the fact we consider ω-sequences makes the whole thing work. Countability that is needed is the one of the length of the games, not the one of the cardinality of the basis of the toplogy.

References

[1] **Becker H., van Engelen F., van Mill J.** [1994] *Disjoint Embeddings of compacta.* Mathematika, vol 41, Part 2 No. 82 (1994) 221-232.

[2] **Friedman H.** [1971] *Higher Set theory and mathematical practice.* Ann. Math. Logic vol. 2 (1971)

[3] **Gale D., Stewart F.M.** [1953] *Infinite games with perfect information.* Ann. Math. Studies, vol. 28 (1953) 245-266

[4] **Kechris A.S** [1994] *Classical descriptive set theory.* Graduate texts in mathematics; vol 156. Springer Verlag (1994)

[5] **Kuratowski K.** [1958] *Topologie,* vol.1 PWN, Polish. Sci. Publ., Warsawa (1958)

[6] **Louveau A.** [1983] *Some Results in the Wadge Hierarchy of Borel Sets.* Cabal Sem 79-81, Lecture Notes in Mathematics (1019) 28-55.

[7] **Louveau A.,St.Raymond J.** [1985] *The strength of Borel Wadge determinacy.* Cabal Sem 81-85, Lecture Notes in Mathematics (1333) 1-30.

[8] **Louveau A.,St.Raymond J.** [1988] *Les propriétés de réduction et de norme pour les classes de Boréliens.* Fund. Maths. vol. 131 (1988) 223-243

[9] **Martin D.A.** [1968] *the axiom of determinateness and reduction principles in the analitycal hierarchy*, Bull. Amer. Math. Soc. 74 (1968) 687-689

[10] **Martin D.A.** [1975] *Borel Determinacy.* Ann. Math. vol.102 (1975) 363-371.

[11] **Martin D.A.** [1985] *A purely inductive proof of Borel Determinacy.* Rec. Theory, Proc. Symp. Pure Math. 42. (1985) 303-308

[12] **Moschovakis Y.N.** [1980] *Descriptive Set Theory.* North-Holland (1980).

[13] **St.Raymond J.** [1989] *Classification de Wadge.* Notes d'exposé , février 89; séminaire d'initiation à l'analyse. G. Choquet, G. Godefroy, M. Rogalski, J. St.Raymond; 28ème année, 1988-1989

[14] **Steel J.R.** [1981] *Determinateness and the separation property.* Journal of Symbolic Logic, vol 46 (1981) 417-44.

[15] **Veblen O.** [1908] *Continuous increasing fonctions of finite and transfinite ordinals.* Transactions of the American Mathematical Society vol 9 (1908) 280-292.

[16] **van Engelen F.** [1994] *On Borel ideals.* Annals of Pure and Applied Logic, vol 70 (1994) 177-203.

[17] **van Wesep R.** [1978] *Separation Principles and the axiom of determinateness.* Journal of Symbolic Logic, vol 43 (1978) 77-81.

[18] **Wadge W.W.** [1972] *Degrees of complexity of subsets of the Baire space.* Notice A.M.S. (1972), A-714

[19] **Wadge W.W.** [1984] *Ph.D. Thesis*, Berkeley.