A hierarchy of Deterministic Context-free ω-languages *

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Abstract

Twenty years ago, Klaus. W. Wagner came up with a hierarchy of ω-regular sets that actually bears his name. It turned out to be exactly the Wadge hierarchy of the sets of ω-words recognized by Deterministic Finite Automata. We describe the Wadge hierarchy of context-free ω-languages, which stands as an extension of Wagner's work from Automata to Pushdown Automata.

Keywords: deterministic context-free ω-languages, pushdown automata, Wadge hierarchy.

1 Introduction

We recall that ω stands for the set of integers ($\mathbb{N}$). It is also, from the ordinal point of view, the first ordinal that is not finite. And ω_1 denotes the first uncountable ordinal.
Given a set Σ, called the alphabet, we write: Σ^* for the set of finite words from Σ (the set of sequences of finite length of elements from Σ); Σ^ω for the set of ω-words (the set of functions from $\mathbb{N}$ to Σ). The empty word is denoted ε and the restriction of a word x to its n first letters is denoted $x \upharpoonright n$.
The concatenation of words x and y is written $x \cdot y$. Given A some set of finite or infinite words and u any finite word, $u \cdot A$ stands for the set of all words $u \cdot y$ for $y \in A$. More generally, given A and B two sets of words, $A \cdot B$ denotes the sets of all words $x \cdot y$ for $x \in A$ and $y \in B$. Anyways, when we use the usual mathematical notation for sequences: $\langle a_0, a_1, \ldots, a_k\rangle$, we write $\langle a_0, a_1, \ldots, a_k\rangle^\frown\langle a_{k+1}, a_{k+2}, \ldots, a_{k+n}\rangle$ to denote the concatenation of the two sequences, namely $\langle a_0, a_1, \ldots, a_k, a_{k+1}, a_{k+2}, \ldots, a_{k+n}\rangle$. At last, given $B \subseteq \Sigma_B^\omega$ and $u \in \Sigma_B^*$, we write $B_{(u)}$ to denote the set $(u \cdot \Sigma_B^\omega) \cap B$.

In 1979 Klaus W. Wagner introduced a hierarchy on ω-regular sets, that is sets of ω words recognized by Deterministic Automata (DA) [27]. These ω-languages were crucial in the famous works on the decidability of the monadic second order logic with 1 successor (S1S) by Büchi and Landweber [1] [2].
This paper provides the extension of this hierarchy to Deterministic PushDown Automata ($DPDA$). We recall the definition of a $DPDA$ reading ω-words [23]:

Definition 1 $\mathcal{A} = (K, \Sigma,\ \Gamma_\perp,\ q_{init}, \perp,\ \delta,\ \mathfrak{F})$ *is some Deterministic PushDown Automaton if*

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- *K is the set of its states,*
- *Σ is its alphabet (necessarily finite). The case $|\Sigma| = 1$ being straightforward and meaningless, we assume that Σ contains at least two letters, among which are $\{0, 1\}$.*
- *$\Gamma_\perp$ is a finite pushdown alphabet. It stands for $\Gamma \cup \{\perp\}$, where $\perp$ is a special letter (the start symbol) that indicates when read, that the bottom of the stack has been reached,*
- *q_{init} is the initial state,*
- *$\perp$ is the bottom symbol, it cannot be erased: the content of the stack always takes the form $\perp \cdot \gamma$ for some $\gamma \in {\Gamma_\perp}^*$.*
- *δ is the transition function from $K \times (\Sigma \cup \{\varepsilon\}) \times \Gamma_\perp$ to $K \times {\Gamma_\perp}^*$ with the following restrictions*
  - *The transition function commands to read a letter iff it does not command any ε-move: that means for all $q \in K$ and $Z \in \Gamma_\perp$*
    - *either $\delta(q, \varepsilon, Z)$ does not exist and $\forall a \in \Sigma \quad \exists q' \in K \quad \exists \gamma \in {\Gamma_\perp}^*$ such that $\delta(q, a, Z) = (q', \gamma)$*
    - *or $\exists q' \in K \quad \exists \gamma \in {\Gamma_\perp}^* \quad \delta(q, \varepsilon, Z) = (q', \gamma)$ and $\forall a \in \Sigma \quad \delta(q, a, Z)$ does not exist.*
  - *The bottom symbol cannot be erased: for any $q \in K$, $a \in \Sigma \cup \{\varepsilon\}$, if $\delta(q, a, \perp)$ is defined, then for some $q' \in K$ and some $\gamma \in {\Gamma_\perp}^*$ such that $\delta(q, a, \perp) = (q', \perp \cdot \gamma)$.*
- *$\mathfrak{F} \subseteq \mathcal{P}(K)$ is the set of accepting conditions.*

Notation 2

(a) We recall that the DPDA *$\mathcal{A}$ is said to be without ε-moves if for any state q and any letter from the pushdown alphabet $Z \in \Gamma_\perp$ $\delta(q, \varepsilon, Z)$ does not exist.*

(b) An ω-word x is accepted by $\mathcal{A}$ iff the set of states the DPDA *runs through infinitely many times belongs to $\mathfrak{F}$. Formally: we write $a : \ (q, \gamma \cdot Z) \underset{\mathcal{A}}{\vdash\!\!\!-\!\!-} (q', \gamma \cdot \gamma')$ if*

- $a \in \Sigma \cup \{\varepsilon\}$,
- $q, q' \in K$,
- $\gamma, \gamma' \in {\Gamma_\perp}^*$,
- $z \in \Gamma_\perp$,
- $\delta(q, a, z) = (q', \gamma')$.

Given any infinite word $x = \prod\limits_{n \in \mathbb{N}} a_n \in \Sigma^\omega$; for any $n \in \mathbb{N}$ there exists

(i) a unique letter $b_n \in \Sigma \cup \{\varepsilon\}$

(ii) a unique state $q_n \in K$, with q_0 being the initial state of $\mathcal{A}$.

(iii) a unique word $\gamma_n \in {\Gamma_\perp}^*$

such that

$$\begin{cases} b_n : \ (q_n, \gamma_n) \underset{\mathcal{A}}{\vdash\!\!\!-\!\!-} (q_{n+1}, \gamma_{n+1}) \\ \qquad\qquad \textit{and} \\ \prod_{n \in \mathbb{N}} b_n = \prod_{n \in \mathbb{N}} a_n = x \end{cases}$$

The word $x = \prod_{n\in\mathbb{N}} a_n$ *is accepted by* $\mathcal{A}$ *iff* $\{q \in K : \forall n \exists m > n \quad q_m = q\}$ *belongs to* $\mathfrak{F}$.

(c) *We write* $\ulcorner \mathcal{A} \urcorner$ *the set of all* ω*-words from* Σ^ω *that are accepted by* $\mathcal{A}$.

(d) *In case* $\mathcal{A}$ *is* ε*-free, for any finite word* $u \in \Sigma^*$*, we write* $\mathcal{A}[u] = (q, \gamma)$ *the "configuration" of the automaton after reading* u*: if* $\overset{*}{\underset{\mathcal{A}}{\longmapsto}}$ *stands for the transitive closure of* $\underset{\mathcal{A}}{\longmapsto}$*, then* $\mathcal{A}[u] = (q, \gamma)$ *iff* $u : \ (q_{init}, \perp) \overset{*}{\underset{\mathcal{A}}{\longmapsto}} (q, \gamma)$.

A *DPDA* $\mathcal{A}$ has the Continuity Property iff for every word $x \in \Sigma^\omega$, the run of the machine consists in applying infinitely many transitions that are not of the form $\delta(\_, \varepsilon, \_) = (\_, \_)$. In other words, if the machine never enters an infinite ε-loop. It is known that for every ω-*DPDA* $\mathcal{A}$ there exists an ω-*DPDA* $\mathcal{A}'$ which has the Continuity Property and accepts exactly the same set of ω-words as $\mathcal{A}$ (see [4] and [13]).
A Deterministic Automaton is just the restriction of a *DPDA* once everything that concerns the stack is forgotten. In other words the notion of *DPDA* extends the notion of *DA*, so a natural problem is to describe the complexity of *DPDA* compared to the complexity of *DA*. This can be done either by looking directly at the very definition of the machines, or by analyzing the topological complexity of the languages they recognize. Our work takes the second option, therefore in next section we introduce a very fine topological hierarchy which is the battlefield of this work.
For more information on ω-languages see Ludwig Staiger [22].

We particularly want to thank P. Simonnet who first introduced us to the Wagner and the Wadge hierarchy, J-P Ressayre who encouraged us with stimulating remarks and advices and O. Finkel who suggested the problem and showed us they were still a lot of things to do with ω-languages (see [10][11][12]).

2 Introduction to the Wadge hierarchy

In the sixties W. W. Wadge [25][26] started to study a hierarchy of subsets of the Baire space (ω^ω) and more generally of subsets of Σ^ω where Σ is some at most countable alphabet. That particular topological space (ω^ω) is essential in Analysis since it is homeomorphic to $\mathbb{R} \smallsetminus \mathbb{Q}$ (isomorphic to the Irrational space with respect to the topological structure). More generally the elements of Σ^ω are the functions from $\mathbb{N}$ to Σ, that is the ω-words on the alphabet Σ. We consider such spaces equipped with the usual topology, that is the product of the discrete topology on Σ. So, basic non empty open sets have the following form:

$$s \cdot \Sigma^\omega \text{ for some finite sequence } s \in \Sigma^*.$$

In the sequel we assume that Σ contains at least the two different letters 0 and 1.
Wadge was mainly interested in the class of Borel subsets of Σ^ω. Namely the smallest one that contains the Open sets and is closed under countable union and complementation. Borel sets stand as the core of Analysis. They bear essential properties like measurability, the perfect set property and the Baire Property. As soon as they were introduced, Baire set the Borel sets down in a hierarchy that relies on enumerating the number of operations of countable union and complementation required to obtain them. Thus the so called Borel hierarchy looks like:

$$\underset{\sim}{\mathbf{\Sigma}}^\circ_1 = \{Open\ sets\}$$

$$(\text{for } \nu \geq 1)\quad \underset{\sim}{\mathbf{\Pi}}^\circ_\nu = \left\{\Sigma^\omega \setminus A : \quad A \in \underset{\sim}{\mathbf{\Sigma}}^\circ_\nu\right\}$$

$$(\text{for } \nu > 1)\quad \underset{\sim}{\mathbf{\Sigma}}^\circ_\nu = \left\{\bigcup_{n\in\omega} A_n : \ \forall n \in \omega \quad A_n \in \bigcup_{\xi<\nu} \underset{\sim}{\mathbf{\Pi}}^\circ_\xi\right\}$$

The class $\underset{\sim}{\mathbf{\Pi}}^\circ_1$ is the collection of closed sets, $\underset{\sim}{\mathbf{\Sigma}}^\circ_2$ the collection of countable unions of closed sets, etc... We warn the reader to be cautious not to confuse the Borel class $\underset{\sim}{\mathbf{\Sigma}}^\circ_\omega$ and the space of ω-words Σ^ω.
This hierarchy of classes looks like a ladder with two classes at each step ($\underset{\sim}{\mathbf{\Sigma}}^\circ_\nu$ and $\underset{\sim}{\mathbf{\Pi}}^\circ_\nu$). Each class contains all classes below it ($\underset{\sim}{\mathbf{\Sigma}}^\circ_\nu$ contains all sets in all $\underset{\sim}{\mathbf{\Pi}}^\circ_\xi$ and $\underset{\sim}{\mathbf{\Sigma}}^\circ_\xi$ for any $\xi < \nu$, same thing for $\underset{\sim}{\mathbf{\Pi}}^\circ_\nu$). And the length of the hierarchy is ω_1:

$$\underset{\sim}{\mathbf{\Sigma}}^\circ_1 \subsetneq \underset{\sim}{\mathbf{\Sigma}}^\circ_2 \subsetneq \dots \underset{\sim}{\mathbf{\Sigma}}^\circ_n \subsetneq \underset{\sim}{\mathbf{\Sigma}}^\circ_{n+1} \subsetneq \dots \subsetneq \underset{\sim}{\mathbf{\Sigma}}^\circ_\omega \subsetneq \underset{\sim}{\mathbf{\Sigma}}^\circ_{\omega+1} \subsetneq \dots \underset{\sim}{\mathbf{\Sigma}}^\circ_\nu \subsetneq \underset{\sim}{\mathbf{\Sigma}}^\circ_{\nu+1} \subsetneq \dots\dots$$

$$\underset{\sim}{\mathbf{\Pi}}^\circ_1 \subsetneq \underset{\sim}{\mathbf{\Pi}}^\circ_2 \subsetneq \dots \underset{\sim}{\mathbf{\Pi}}^\circ_n \subsetneq \underset{\sim}{\mathbf{\Pi}}^\circ_{n+1} \subsetneq \dots \subsetneq \underset{\sim}{\mathbf{\Pi}}^\circ_\omega \subsetneq \underset{\sim}{\mathbf{\Pi}}^\circ_{\omega+1} \subsetneq \dots \underset{\sim}{\mathbf{\Pi}}^\circ_\nu \subsetneq \underset{\sim}{\mathbf{\Pi}}^\circ_{\nu+1} \subsetneq \dots\dots$$

Different improvements of the Borel hierarchy were introduced during the century. But by far, the most refined is the one due to Wadge that is based on the very simple topological notion of inverse image by a continuous function.
We recall that $f : \Sigma_A^\omega \longmapsto \Sigma_B^\omega$ is continuous if the inverse image of any open set is open. Or in other words, if it commutes on direct limits:

$$f(\lim_{i\in\mathbb{N}} x_i) = \lim_{i\in\mathbb{N}} f(x_i).$$

(where $x = \lim_{i\in\mathbb{N}} x_i$ holds iff $\forall n \in \mathbb{N}\ \ \exists i \in \mathbb{N}\ \ \ \forall j \geq i \quad x \upharpoonright n = x_j \upharpoonright n$).
Wadge introduced a very natural relation $\leq_W$ between sets $A \subseteq \Sigma_A^\omega$ and $B \subseteq \Sigma_B^\omega$:

$$A \leq_W B \iff_{def} \exists\ f : \Sigma_A^\omega \longmapsto \Sigma_B^\omega \ \ continuous\ \ s.t.\ \ f^{-1}B = A$$

In other words, x belongs to A iff $f(x)$ belongs to B. Intuitively $A \leq_W B$ means that A is less complicated than B with regard to the topological structure. Whenever $A \leq_W B$ we say that A continuously reduces to B because the problem to know whether any word x belongs to A reduces to the problem of the continuous image of x belonging to B.
Directly related to the Wadge ordering ($\leq_W$) are the strict ordering ($<_W$) and the equivalence relation ($\equiv_W$):

$$A <_W B \iff_{def} A \leq_W B \ \ but \ \ B \not\leq_W A$$

$$A \equiv_W B \iff_{def} A \leq_W B \leq_W A$$

when $A \equiv_W B$ we say that A and B are Wadge equivalent.

For example, any clopen (closed and open) set $A \subsetneq \Sigma^\omega$ that is neither the empty set nor the whole space verifies

- $A \equiv_W u \cdot \Sigma^\omega$ for any non empty finite word $u \in \Sigma^*$.
- $\emptyset <_W A$ and $\Sigma^\omega <_W A$

- $A \equiv_W \Sigma^\omega \setminus A$
- $A <_W \{0^\omega\}$ and $A <_W \Sigma^\omega \setminus \{0^\omega\}$

To give an idea of the Wadge ordering, let ${}^{-}A$ denote the complement of A: (${}^{-}A = \Sigma^\omega \setminus A$) and given $\{\Sigma_+, \Sigma_-\}$ any partition of Σ into two **non empty** sets, we define ${}^{\pm}A$ as the set $(\Sigma_+ \cdot A) \cup (\Sigma_- \cdot {}^{-}A)$. There are two radically different kinds of sets: the ones that reduce to their complement (${}^{-}A \leq_W A$) - for example any clopen set that is neither the empty set nor the whole space satisfies this property) such sets are called **self dual**; they verify ${}^{-}A \equiv_W A$. And the other ones that do not reduce to their complement are called **non self dual**. For example, the empty set is non self dual since no function satisfies $f^{-1}\Sigma^\omega = \emptyset$. Same remark holds with the whole space Σ^ω. Any closed set that is not open (e.g. $\{0^\omega\}$) and also any open but not closed set (e.g. $\Sigma^\omega \setminus \{0^\omega\}$) is non self dual.
Moreover, if A is self dual, that is $A \equiv_W {}^{-}A$, then $A \equiv_W {}^{\pm}A$ holds. On the contrary, if A is non self dual, then $A \equiv_W {}^{\pm}A$ fails. In that case, one has $A <_W {}^{\pm}A$ and there is no B such that $A <_W B <_W {}^{\pm}A$. This means that with regards to the Wadge ordering, ${}^{\pm}A$ is (up to $\equiv_W$) the least set strictly above A in case A is non self dual.
In a similar way, in order to catch the immediate "successor" of A in the Wadge hierarchy, in case A is self dual, we just have to consider the set $(\Sigma \setminus \{0\})^* \cdot 0 \cdot A$. It satisfies $A <_W (\Sigma \setminus \{0\})^* \cdot 0 \cdot A$ and there is no B such that $A <_W B <_W (\Sigma \setminus \{0\})^* \cdot 0 \cdot A$.

Now $\leq_W$ could have been just any weird ordering between sets, but amazingly when restricted to Borel sets it turns out to be a pre-wellordering: it is not a total ordering ($\emptyset$ and ${}^{-}\emptyset$ are incomparable) but antichains have length at most 2 (there are no three elements incomparable with each other). Much more important is that it is wellfounded (there is no infinite descending chain $A_0 >_W A_1 >_W \ldots A_n >_W A_{n+1} >_W \ldots$). To be precise, the class $\bigcup_{\nu<\omega_1} \underset{\sim}{\mathbf{\Sigma}}^\circ_\nu$ of all Borel subsets of Σ^ω quotiented by the following equivalence relation

$$A \sim B \iff (A \equiv_W B \ \vee \ A \equiv_W \Sigma^\omega \setminus B)$$

is well ordered by $<_W$: it is a total ordering on $\bigcup_{\nu<\omega_1} \underset{\sim}{\mathbf{\Sigma}}^\circ_\nu / \sim$ with no infinite descending chain. In other words the structure $\left(\bigcup_{\nu<\omega_1} \underset{\sim}{\mathbf{\Sigma}}^\circ_\nu / \sim, <_W\right)$ is isomorphic to $(|WH|, <)$ where $|WH|$ is some ordinal that represents the height of the Wadge hierarchy and $<$ is the usual ordering on ordinals (for more details see [7][8]).

Since $<_W$ is wellfounded on Borel sets, it induces a notion of height on these sets. Clearly there is no set A such that $A <_W \emptyset$ holds; there is also no other set than the whole space which is incomparable with the empty set. Therefore $\emptyset$ and ${}^{-}\emptyset$ are located at the lowest level of the pre-wellordering. We set $ht(\emptyset) = ht({}^{-}\emptyset) = 0$ and inductively, for any set A that is neither the empty set nor the whole space, we set $ht(A) = sup\{ht(B) + 1 : \ B <_W A\}$.
The Wadge ordering on Borel subsets of Σ^ω verifies:

- $ht(\emptyset) = ht({}^{-}\emptyset) = 0$,
- $ht(A) = 1$ iff A is closed and open but different from the empty set and the whole space,
- $ht(A) = 2$ iff A is closed but not open (e.g. $\{0^\omega\}$) or if it is open but not closed (e.g. ${}^{-}\{0^\omega\}$),

- $ht(A)$ is odd iff A is self dual ($^{-}A \equiv_W A$),
- $ht(A)$ is even iff A is non self dual (limit ordinals are even),
- A and B are incomparable iff $ht(A) = ht(B)$ is even and $^{-}A \equiv_W B$,
- $ht((\Sigma^* \cdot 0)^\omega) = \omega_1$.

In order to catch up with the Borel hierarchy we introduce the Wadge class of a set A denoted $[A]_w$:

$$[A]_w =_{def} \quad \{B \subseteq \Sigma^\omega : \quad B \leq_W A\}$$

So a Wadge class is any class that is closed under continuous preimage and admits a **generator** (namely the set A). For example every Borel class $\underset{\sim}{\mathbf{\Sigma}}^\circ_\nu$ is a Wadge class since any set $A \in \underset{\sim}{\mathbf{\Sigma}}^\circ_\nu \setminus \underset{\sim}{\mathbf{\Pi}}^\circ_\nu$ is a generator for $\underset{\sim}{\mathbf{\Sigma}}^\circ_\nu$. Same remark holds with $\underset{\sim}{\mathbf{\Pi}}^\circ_\nu$, but the class of all Borel sets ($\bigcup_{\nu<\omega_1} \underset{\sim}{\mathbf{\Sigma}}^\circ_\nu$) although it is closed under continuous pre-image does not form a Wadge class for it does not have any generator: any potential generator would be in some class $\underset{\sim}{\mathbf{\Sigma}}^\circ_\nu$ for some ν large enough; but then all its continuous preimages would be in the same class $\underset{\sim}{\mathbf{\Sigma}}^\circ_\nu$, too small to contain all Borel sets.

Since we are dealing with finite machines we are limited to finite alphabets, therefore the spaces of the form Σ^ω where Σ is finite are compact and in such spaces self dual sets can easily be described by use of non self dual ones: for any self dual set $A \subseteq \Sigma_A^\omega$ there exists some non self dual one $B \subseteq A$ and a partition of Σ_A in two non empty sets $\{\Sigma_+, \Sigma_-\}$ such that

$$A \equiv_W (\Sigma_+ \cdot B) \ \cup \ (\Sigma_- \cdot {}^{-}B)$$

So we can get rid of the self dual sets and concentrate on the non self dual ones. To each **non self dual** set A we attribute an ordinal ($d_W^\circ(A)$) representing its level in the Wadge hierarchy. This is done by induction starting at 1 instead of 0 for reasons that will be clear later.

Definition 3 *For any Borel n.s.d. subset A of Σ^ω,*

$$\begin{aligned} d_W^\circ A &= 1 \quad \textit{iff} \quad (A = \emptyset \ \vee \ A = \Sigma^\omega) \\ d_W^\circ A &= \sup\{(d_W^\circ B) + 1 : \quad B \ \textit{n.s.d.} \ \wedge \ B <_W A\} \quad (\textit{any } A >_W \emptyset) \end{aligned}$$

To give an idea of the power of the Wadge hierarchy let us indicate how high the Wadge degrees of Borel sets go. The Wadge degree of an open set that is not closed is 2, the degree of a set in $\underset{\sim}{\mathbf{\Sigma}}^\circ_2 \setminus \underset{\sim}{\mathbf{\Pi}}^\circ_2$ is ω_1, and more generally the degree of a set belonging to $\underset{\sim}{\mathbf{\Sigma}}^\circ_{n+1} \setminus \underset{\sim}{\mathbf{\Pi}}^\circ_{n+1}$ for some non null integer n is $\underbrace{\omega_1^{(\omega_1^{(\ldots\omega_1))}}}_{n \ times}$. This is rather impressive but things get really much worse when it comes to infinite Borel ranks [8].

Fortunately this paper is not concerned with the whole Wadge hierarchy. Sets recognized by *DPDA* are simple, like the ones accepted by Deterministic Turing Machines, or more generally by any deterministic finite-state machine with a condition of acceptance of ω-words stated in terms of going through a certain subset of states infinitely many times. All these sets are all boolean combinations of $\underset{\sim}{\mathbf{\Sigma}}^\circ_2$ sets. Hence their Wadge degree is strictly less than ω_1^ω.

3 The basic notions about the Wadge hierarchy

The main device in working with the Wadge hierarchy consists in replacing the notion of the inverse image by a continuous function by the one of a winning strategy for a player in a suitable game: the Wadge Game; so that the whole artillery developed in game theory becomes available.

Definition 4 (The Wadge game) *Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$ the Wadge game $\mathbf{W}(A\ ,\ B)$ is an infinite two player game where players take turn: I playing letters in Σ_A and II playing letters in Σ_B. Player I begins the run. Player II is allowed to skip provided he plays infinitely many letters; but player I is not. So that at the end of a run (in ω moves), I has produced an ω-word $x \in \Sigma_A^\omega$ and II has produced $y \in \Sigma_B^\omega$. We say*

$$\textit{II} \text{ wins } \mathbf{W}(A\ ,\ B) \quad \Longleftrightarrow \quad (x \in A \longleftrightarrow y \in B)$$

A strategy for player *II* is a mapping $\sigma :\ \Sigma_A^+ \longmapsto \Sigma_B \cup \{s\}$ where s is a letter that is not in the alphabet Σ_B (s stands for "skip"). Player *II* applies this strategy in the above game if the sequence y he plays verifies $y = y'$ once every occurrence of s is erased where $y'(n) = \sigma(x \restriction (n+1))$. A strategy for I is defined similarly; and we say that σ is a winning strategy (abbreviated w.s.) for a player if that player always wins when using σ. As usual in game theory, we write $\sigma_\star\ :\ \Sigma_A^\omega \longmapsto \Sigma_B^\omega$ the mapping induced by the strategy σ
Wadge designed the game $\mathbf{W}(A\ ,\ B)$ so that there is a strategy σ that is winning for *II* in this game if and only if there exists a continuous mapping $f :\ \Sigma_A^\omega \longmapsto \Sigma_B^\omega$ such that $f^{-1}B = A$ ($\sigma_\star$ is that continuous function). In other words:

$$\textit{II} \text{ has a w.s. in } \mathbf{W}(A\ ,\ B) \quad \Longleftrightarrow \quad A \leq_W B$$

Remarks 5 By Borel Determinacy ([19]), given A and B both Borel, the game $\mathbf{W}(A\ ,\ B)$ is determined. this means one of the two players has a winning strategy. As an immediate consequence, it is easy to see that if I has a w.s. in $\mathbf{W}(A\ ,\ B)$ then *II* has a w.s. in $\mathbf{W}(B\ ,\ {}^-A)$. This leads to the Wadge lemma: if A and B are Borel then one of the two following possibilities holds:

$$A \leq_W B \textit{ or } B \leq_W {}^-A$$

Moreover, by the same kind of argument that shows that $<_W$ is wellfounded on Borel sets, Martin and Wadge obtained that

$$A \text{ is non self dual} \quad \Longleftrightarrow \quad \exists x \in \Sigma_A^\omega \ \ \forall n \in \mathbb{N} \quad A \leq_W A_{(x \restriction n)}$$

From a playful point of view, this means that a player who is in charge of a self dual set B will sooner or later reach a position $u \in \Sigma^*$ such that $B_{(u)}$ is some set topologically strictly less complicated than B. In fact, there exists some non self dual set $A <_W B$ such that ${}^\pm A \equiv_W B$.

As a warm-up, we give a proof of the following "canonical" successors for Borel sets which was stated in the previous section.

Proposition 6 *Let $A \subseteq \Sigma_A^\omega$ be Borel,*

(a) If A is non self dual, then $A <_W {}^\pm A$ and, for all Borel set B, $A <_W B \longrightarrow {}^\pm A \leq_W B$

(b) If A is self dual, then $A <_W (\Sigma_A \smallsetminus \{0\})^* \cdot 0 \cdot A$ and, for all Borel set B,

$$A <_W B \longrightarrow \Big((\Sigma_A \smallsetminus \{0\})^* \cdot 0 \cdot A \leq_W B \ \vee\ (\Sigma_A \smallsetminus \{0\})^* \cdot 0 \cdot A \leq_W {}^-B \Big).$$

(c) If A is self dual, then there exists B non self dual such that $A \equiv_W {}^{\pm}B$.

Proof of proposition 6:

(a) Clearly both $A \leq_W {}^{\pm}A$ and ${}^{-}A \leq_W {}^{\pm}A$ hold. Since A is non self dual, $A \not\leq_W {}^{-}A$ holds too. Hence $A <_W {}^{\pm}A$. To show that $A <_W B \longrightarrow {}^{\pm}A \leq_W B$, it is enough to remark that $A <_W B$ implies A and ${}^{-}A \leq_W B$. i.e. there are strategies σ_+ and σ_- that are winning for *II* in respectively $\mathbf{W}(A\ ,\ B)$ and $\mathbf{W}({}^{-}A\ ,\ B)$. From σ_+ and σ_- it is straightforward to derive a w.s. for *II* in $\mathbf{W}({}^{\pm}A\ ,\ B)$.

(b) Set $C = (\Sigma_A \smallsetminus \{0\})^* \cdot 0 \cdot A$. The fact $A \leq_W C$ is obvious since there is an immediate w.s. for *II* in $\mathbf{W}(A\ ,\ C)$ (play 0 at first move and then keep on copying *I*'s run until the end). To prove that $C \not\leq_W A$ it is enough to remark that for all integer n, $C_{(1^n)} \leq_W C$ holds, proving that there exists an ω-word $x = 1^\omega$ that satisfies

$$\forall n \in \mathbb{N}\ C_{(x\restriction n)} \leq_W C.$$

Hence C is non self dual. Now, assuming $C \leq_W A$ would lead to $C \leq_W {}^{-}A$ since A is self dual, that is ${}^{-}C \leq_W A$ and finally ${}^{-}C \leq_W C$ contradicting the fact C is non self dual.

Towards a contradiction, we assume that $A <_W B$, $C \not\leq_W B$ and $C \not\leq_W {}^{-}B$. This is the same as assuming $A <_W B$, $B \leq_W C$ and ${}^{-}B \leq_W C$. Therefore there exist strategies σ_+ and σ_- that are winning for *II* in respectively $\mathbf{W}(B\ ,\ C)$ and $\mathbf{W}({}^{-}B\ ,\ C)$. From σ_+ and σ_- it is easy to derive a w.s. for *II* in $\mathbf{W}(B\ ,\ C)$ that is restricted to $(\Sigma_A \cup \{0\})^* \cdot 0 \cdot \Sigma_A^\omega$. i.e. it always plays the letter "0" at some point. But from that strategy one gets a w.s. for *II* in $\mathbf{W}(B\ ,\ A)$, proving that $B \leq_W A$, a contradiction.

(c) We define a set $S \subseteq \Sigma^*$ by

$$u \in S \quad \longleftrightarrow \quad B_{(u)} <_W B \leq_W B_{(u\restriction(lh(u)-1))}$$

Since A is self dual, S satisfies $\{u \cdot \Sigma^\omega :\ u \in S\}$ is a partition of Σ^ω in basic open sets. However, since Σ is finite, the topological space Σ^ω is compact. Therefore, S is some finite set, hence there exists some "maximal" sets $B_{(v_+)}$ and $B_{(v_-)}$ for some v_+ and v_- both belonging to S, such that

$$B_{(v_+)} \equiv_W {}^{-}B_{(v_-)} \text{ and } B_{(u)} \leq_W B_{(v_+)} \ \vee\ B_{(u)} \leq_W {}^{-}B_{(v_-)}$$

holds for any $u \in S$. Clearly $B_{(v_+)}$ is non self dual and

$$B \equiv_W 0 \cdot B_{(v_+)} \ \cup\ (\Sigma \smallsetminus 0) \cdot B_{(v_-)} \equiv_W {}^{\pm}B_{(v_+)} \equiv_W {}^{\pm}B_{(v_-)}.$$

$\dashv$ 6

In [7] we gave a description of the Wadge hierarchy restricted to Borel sets of finite ranks (the ones that are in some $\underset{\sim}{\mathbf{\Sigma}}^\circ_n$ for some finite n) that we extended to all ranks in [8]. This description was based on the introduction of set theoretic operations that are the counterpart of the ordinal operations needed to compute the Wadge degrees. This allowed to give a canonical definition of Borel sets of every Wadge degree, simply by replacing in the canonical description of its ordinal (its Cantor Normal Form of base ω_1) the ordinal 1 by the empty set and the arithmetical functions by their set theoretical counterpart. We did this by introducing a new hierarchy that is isomorphic to the Wadge hierarchy of non self dual Borel sets. Anyways, Sets accepted by

DPDA are all Boolean combinations of $\underset{\sim}{\mathbf{\Sigma}}^\circ_2$-sets; hence their Wadge degree is located strictly below than ω_1^ω. Moreover, any ordinal $\alpha < \omega_1^\omega$ can be described by use of the following ordinal operations: sum, countable multiplication and multiplication by ω_1. This means that there exists some unique integer k, a unique strictly increasing sequence of integers $0 < n_0 < \ldots < n_k$ and some unique $0 < \nu_i < \omega_1$ (any $0 \leq i \leq k$) such that

$$\begin{aligned} \alpha &= (1 \cdot \underbrace{\omega_1 \cdot \cdots \cdot \omega_1}_{n_k}) \cdot \nu_k + \ldots + (1 \cdot \underbrace{\omega_1 \cdot \cdots \cdot \omega_1}_{n_0}) \cdot \nu_0 \\ &= \omega_1^{n_k} \cdot \nu_k + \ldots + \omega_1^{n_0} \cdot \nu_0. \end{aligned}$$

Our purpose is to define operations on non self dual sets that be the counterparts of the aforementioned ordinal operations (in particular, only producing non self dual sets) and preserve the Wadge ordering ($\leq_W$). This means to define:

- A set theoretical sum: $(A, B) \longmapsto A{+}B$ that satisfies

$$d^\circ_W(A{+}B) = (d^\circ_W A) + (d^\circ_W B).$$

- A set theoretical countable multiplication: $(A, \xi) \longmapsto A \bullet \xi$ that satisfies

$$d^\circ_W(A \bullet \xi) = (d^\circ_W A) \cdot \xi \quad (\textit{for } 0 < \xi < \omega_1).$$

- A set theoretical multiplication by ω_1: $A \longmapsto A^\natural$ that satisfies

$$d^\circ_W(A^\natural) = (d^\circ_W A) \cdot \omega_1.$$

4 The set theoretical sum

We first remark that given any set $A \subseteq \Sigma_A^\omega$ and alphabet $\Sigma \supsetneq \Sigma_A$ one may find a set $B \subseteq \Sigma^\omega$ such that $B \equiv_W A$ (e.g. define B as the inverse image of A by the homomorphism $h(a) = 0$ for $a \in \Sigma \setminus \Sigma_A$ and $h(a) = a$ otherwise). So in the sequel we always assume that the alphabets involved are rich enough to allow the partitioning operations we need, or are simply enriched to fit the goal.

Definition 7 *Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$, assume $\Sigma_A \subsetneq \Sigma_B$ and $\{\Sigma_+, \Sigma_-\}$ is a partition of $\Sigma_A \setminus \Sigma_B$ in two* **non empty** *sets,*

$$B{+}A = A \ \cup\ \Sigma_A^* \cdot \Sigma_+ \cdot B \ \cup\ \Sigma_A^* \cdot \Sigma_- \cdot {}^-B$$

From the point of view of a player in charge of the set $B{+}A$ in a Wadge game, everything goes as if that player was starting the game being in charge of A. So provided he only plays letters in Σ_A the question whether the resulting sequence x he will have produced at the end of the run belongs to $B{+}A$ or not reduces to knowing whether x belongs to A or not. But at any moment of the run, he can play a letter "A" which is not in Σ_A. Then everything looks like the whole sequence played since the beginning of the game is erased and he is now starting from scratch but this time not being in charge of A anymore but in charge of: B if $a \in \Sigma_+$ and ${}^-B$ if $a \in \Sigma_-$. When a player is in charge of $B{+}A$, we say that the player stays inside the tail part of $B{+}A$ if that player always plays letters in Σ_A. We say that he exits the tail part of $B{+}A$ when he plays a letter in $\Sigma_B \setminus \Sigma_A$ for the first time. And finally, we say that this player chooses to be in charge of B (respectively ${}^-B$) if that letter belongs to Σ_+ (respectively Σ_-).

Remarks 8 Given $A,\ A',\ B,\ B'$ non self dual Borel sets,

(a) ${}^{-}(A{+}B) \equiv_W A{+}({}^{-}B)$

(b) The operation $+$ preserves the Wadge ordering:

$$(A' \leq_W A \quad \wedge \quad B' \leq_W B) \quad \longrightarrow \quad A'{+}B' \leq_W A{+}B.$$

Proposition 9 *Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$, A and B both non self dual Borel sets,*

$$d_W^\circ(A{+}B) = d_W^\circ(A) + d_W^\circ(B)$$

Proof of proposition 9: By induction on $d_W^\circ(B)$.

(a) if $d_W^\circ(B) = 1$ then $B \equiv_W \emptyset$ or $B \equiv_W {}^{-}\emptyset$. Since for any set C $d_W^\circ(C) = d_W^\circ({}^{-}C)$ it is enough to prove the result when $B = \emptyset$. The result relies on:

Claim 10 *Let $C \subseteq \Sigma_C^\omega$, C non self dual,*

$$({}^{-}C \leq_W A{+}\emptyset \quad \wedge \quad C \leq_W A{+}\emptyset) \quad \longleftrightarrow \quad ({}^{-}C \leq_W A \quad \vee \quad C \leq_W A)$$

Proof of claim 10:

($\Longleftarrow$) is obvious since A and ${}^{-}A\ \leq_W {}^{\pm}A \leq_W A{+}\emptyset$.

($\Longrightarrow$) Since C is non self dual, there exists $x \in \Sigma_C^\omega$ such that

$$\forall n \in \omega \quad C_{(x \upharpoonright n)} \equiv_W C.$$

(i) Assume $x \in C$. Then consider the Wadge game $\mathbf{W}(C\ ,\ A{+}\emptyset)$. Player *II* has a winning strategy σ in that game. If *I* plays x and *II* replies applying σ, then at some move n_0, σ requires *II* to play a first letter $a \in \Sigma_A \setminus \Sigma_\emptyset$ (otherwise the question "$\sigma\star x \in (A{+}\emptyset)$?" would be that same as "$\sigma\star x \in \emptyset$?" which admits a negative answer, contradicting the fact σ is winning. Now if that first letter $a \in \Sigma_A \setminus \Sigma_\emptyset$ leads *II* to start the run again being in charge of A we obtain $C \leq_W C_{(x\upharpoonright n)} \leq_W A$. And if on the contrary *II* finds himself being in charge of ${}^{-}A$ then one gets $C \leq_W C_{(x\upharpoonright n)} \leq_W {}^{-}A$.

(ii) Assume $x \notin C$. Then consider the Wadge game $\mathbf{W}({}^{-}C\ ,\ A{+}\emptyset)$ and proceed as in (a)(i).

$\dashv$ 10

So on one side $A, {}^{-}A \leq_W A{+}\emptyset$ implies $d_W^\circ(A{+}\emptyset) \geq d_W^\circ A + 1$.

On the other side:

$$\begin{array}{rcl} d_W^\circ(A{+}\emptyset) & = & \sup\{d_W^\circ(C)+1 :\ C\ n.s.d. \wedge\ C <_W A{+}\emptyset\} \\ & = & \sup\{d_W^\circ(C)+1 :\ C\ n.s.d. \wedge\ (C \leq_W A\ \vee\ C \leq_W {}^{-}A)\} \\ & = & \sup\{d_W^\circ(C)+1 :\ C\ n.s.d. \wedge\ d_W^\circ(C) \leq d_W^\circ(A)\} \\ & \leq & d_W^\circ(A)+1 \end{array}$$

(b) If $d_W^\circ(B) > 1$ then the result relies on

Claim 11 *Let $C \subseteq \Sigma_C^\omega$, C non self dual,*

$$({}^{-}C \text{ and } C \leq_W A{+}B) \quad \longleftrightarrow \quad \exists D <_W B \ \ s.t. \ \ (C \text{ or } {}^{-}C \leq_W A{+}D)$$

Proof of claim 11:

($\Longleftarrow$) Since both D *and* ${}^{-}D \leq_W B$ and $+$ preserves $\leq_W$, one gets $A{+}D \leq_W A{+}B$ and $A{+}{}^{-}D \leq_W A{+}B$. And since $A{+}{}^{-}D \equiv_W {}^{-}A{+}D$ holds, one finally obtains $A{+}D$ *and* $A{+}{}^{-}D \leq_W A{+}B$.

($\Longleftarrow$) Assume σ_+ is a winning strategy for $I\!I$ in $\mathbf{W}(C \ , \ A{+}B)$ and σ_- is a winning strategy for $I\!I$ in $\mathbf{W}(C \ , \ A{+}{}^{-}B)$. Set

$$F = \{x \in \Sigma_C^\omega : \ \sigma_+ \star \ x \textbf{ and } \sigma_- \star \ x \text{ does not contain a letter in } \Sigma_A \setminus \Sigma_B\}$$

If $F = \emptyset$ then $C \leq_W A{+}\emptyset$ holds and gives the result. Otherwise, we remark that F is closed. So let T_F be the underlying tree:

$$T_F = \{x \restriction n : \ x \in F \ \wedge \ n \in \omega\}$$

and

$$E = (C \cap F) \cup \{u \cdot a \cdot x \in \Sigma_C^\omega : \ u \in T_F \wedge a \in \Sigma_C \wedge u \cdot a \notin T_F \wedge u \cdot \Sigma_C^\omega \cap (C \cap F) \neq \emptyset\}.$$

The set E is defined this way so to provide, with the help of σ_+, a w.s. for $I\!I$ in $\mathbf{W}(E \ , \ B)$; and, with the help of σ_-, a w.s. for $I\!I$ in $\mathbf{W}(E \ , \ {}^{-}B)$. In other words, one gets $E \leq_W B$ and $E \leq_W {}^{-}B$.

(i) If E is non self dual, by hypothesis there exists $D \subseteq \Sigma_D^\omega$ such that $D <_W B$ and $E \leq_W D$. So $I\!I$ has a w.s. τ in $\mathbf{W}(E \ , \ D)$. From τ, σ_+ and σ_- we build a w.s. for $I\!I$ in $\mathbf{W}(C \ , \ A{+}D)$: as long as I remains in F, $I\!I$ applies τ (thus remaining in charge of D). If I exits F then $I\!I$ applies σ_+ or σ_- depending on the one that chooses not to "be in charge of B anymore".

(ii) If E is self dual then there exists $U \subseteq \Sigma_C^*$ such that

$$\{u \cdot \Sigma_C^\omega : \ u \in U\} \text{ is a partition of } \Sigma_C^\omega$$

and

$$\forall u \in U \quad E_{(u)} \text{ is non self dual and } E_{(u)} <_W E.$$

Since C is non self dual there exists $x \in \Sigma_C^\omega$ such that, for all integer n, $C_{(x \restriction n)} \equiv_W C$. Then, for some unique integer n_0, the finite word $x \restriction n_0$ belongs to U. Now

$$E_{(x \restriction n_0)} \text{ is non self dual and } E_{(x \restriction n_0)} \leq_W B \textit{ and } {}^{-}B.$$

Hence by hypothesis there exists $D <_W B$ such that $E_{(x \restriction n_0)} \leq_W D$ holds. i.e. by same argument as in (b)(i), we show there exists τ a w.s. for $I\!I$ in $\mathbf{W}\left(C_{(x \restriction n_0)} \ , \ A{+}D\right)$. This shows that $C_{(x \restriction n_0)} \leq_W A{+}D$ holds, hence $C \leq_W A{+}D$ is satisfied.

$\dashv$ 11

Clearly, if $D <_W B$ then $D, {}^-D \leq_W B$ so that $A{+}D$, $A{+}{}^-D \leq_W A{+}B$ which means that $A{+}D$, ${}^-(A{+}D) \leq_W A{+}B$ i.e. $A{+}D <_W A{+}B$. So we get:

$$\begin{aligned} d^\circ_W A{+}B &= \sup\{(d^\circ_W C) + 1 :\ C\ n.s.d.\ \wedge\ C <_W A{+}B\} \\ &= \sup\{(d^\circ_W C) + 1 :\ C\ n.s.d.\ \wedge\ \exists D <_W B \quad C \leq_W A{+}D\} \\ &= \sup\{d^\circ_W (A{+}D) + 1 :\ D <_W B\} \\ &= \sup\{d^\circ_W (A) + d^\circ_W (D) + 1 :\ D <_W B\} \\ &= d^\circ_W (A) + \sup\{d^\circ_W (D) + 1 :\ D <_W B\} \\ &= (d^\circ_W A) + (d^\circ_W B) \end{aligned}$$

$\dashv$ 9

5 The set theoretical supremum of countable cofinality

Given a countable set of Borel sets we define the set theoretic counterpart of the ordinal supremum:

Definition 12 *Let $\Phi : \mathbb{N} \longmapsto I$ be some mapping 1-1 and onto some countable set I. And, for all $i \in I$, $A_i \subseteq \Sigma^\omega$ be some non self dual Borel set,*

$$\mathbf{sup}_{i \in I} A_i = \bigcup_{n \in \mathbb{N}} (\Sigma \setminus \{0\})^n \cdot 0 \cdot A_{\Phi(n)}$$

From a player point of view, being in charge of $\mathbf{sup}_{i\in I} A_i$ means "choosing" the set that player wants to be in charge of among all the A_i's. This can be done by playing "0" for the first time in the run and indicated by the length (modulo Φ) of the word preceding that letter.
This operation preserves the Wadge ordering, that is, if for all $i \in I$ $A_i \leq_W B_i$ holds, then $\mathbf{sup}_{i\in I} A_i \leq_W \mathbf{sup}_{i\in I} B_i$ holds too.

Proposition 13 *Let I be countable, and for all $i \in I$, $A_i \subseteq \Sigma^\omega$ be non self dual Borel sets. Assume also that $\forall i \in I \quad \exists j \in I \quad A_i <_W A_j$, then*

$$d^\circ_W(\mathbf{sup}_{i\in I} A_i) = \sup\{d^\circ_W A_i :\ i \in I\}.$$

Proof of proposition 13: Clearly, $\mathbf{sup}_{i\in I} A_i \leq_W A_j$ holds for any $j \in I$. Hence one has

$$d^\circ_W(\mathbf{sup}_{i\in I} A_i) \geq \sup\{d^\circ_W A_i :\ i \in I\}.$$

The fact that $d^\circ_W(\mathbf{sup}_{i\in I} A_i \leq \sup\{d^\circ_W A_i :\ i \in I\}$ relies on

Claim 14 *Let $A \subseteq \Sigma^\omega_A$ be non self dual,*

$${}^-A, A \leq_W (\mathbf{sup}_{i\in I} A_i) \implies \exists i \in I\ A \leq_W A_i$$

Proof of claim 14: Since A is non self dual:

$$\exists x \in \Sigma^\omega_A \quad \forall n \in \omega \quad A_{(x\restriction n)} \geq_W A.$$

(a) If $x \in A$, then consider the game $\mathbf{W}(A\ ,\ \mathbf{sup}_{i\in I}A_i)$ where *I* plays x and *II* replies applying a winning strategy. Necessarily, at some move n, *II* plays a first letter "0" (otherwise his run would surely not belong to $\mathbf{sup}_{i\in I}A_i$ while x would belong to A). Then one gets $A_{(x\lceil n)} \leq_W A_i$, hence $A \leq_W A_i$.

(b) If $x \notin A$, taking ${}^-A$ into account instead of A, the same argument as above leads to ${}^-A \leq_W A_i$. Then by hypothesis $\exists j \in I \ \ A_i \leq_W A_j$ hence ${}^-A, A \leq_W A_j$.

$\dashv$ 14

Granted with the claim one obtains:

$$\begin{array}{rcl} d^\circ_W(\mathbf{sup}_{i\in I}A_i) & = & \sup\{(d^\circ_W A)+1:\ A\ n.s.d.\ \wedge\ A \leq_W \mathbf{sup}_{i\in I}A_i\} \\ & = & \sup\{(d^\circ_W A)+1:\ A\ n.s.d.\ \wedge\ \exists i \in I\ \ A \leq_W A_i\} \\ & \leq & \sup\{(d^\circ_W A_i)+1:\ i \in I\} \\ & \leq & \sup\{(d^\circ_W A_i):\ i \in I\} \end{array}$$

$\dashv$ 13

6 The set theoretical countable multiplication

With the help of ordinal sum and countable supremum we easily define the set theoretic counterpart of the countable multiplication.

Definition 15 *Let $A \subseteq \Sigma_A^\omega$, inductively define:*

- $A \bullet 1 = A$
- $A \bullet (\nu+1) = (A \bullet \nu) + A$
- $A \bullet \lambda = \mathbf{sup}_{\xi\in\lambda} A \bullet \xi$ *when λ is some limit countable ordinal.*

By convention, just like when dealing with the arithmetical sum and multiplication, the operation $\bullet$ takes priority over $+$. i.e. the notations $A+B\bullet\nu$ and $B\bullet\nu+A$ stand for respectively $A+(B\bullet\nu)$ and $(B\bullet\nu)+A$. We also write $A_0+A_1+A_2+\ldots+A_k$ for

$$(\ldots((A_0+A_1)+A_2)+\ldots\ldots\ldots)+A_{k-1})+A_k).$$

So, in general, $A_0\bullet\nu_0+A_1\bullet\nu_1+A_2\bullet\nu_2+\ldots+A_k\bullet\nu_k$ stands for

$$(\ldots((A_0(\bullet\nu_0)+(A_1\bullet\nu_1))+(A_2\bullet\nu_2))+\ldots\ldots\ldots(+A_k\bullet\nu_k)).$$

We remark that the operation $\bullet$ preserves $\leq_W$. i.e. for any countable ξ:

$$A \leq_W B \ \longrightarrow\ A\bullet\xi \leq_W B\bullet\xi.$$

And also

Proposition 16 *Let $A \subseteq \Sigma_A^\omega$ and $0 < \xi < \omega_1$,*

$$d^\circ_W(A\bullet\xi) = (d^\circ_W A)\cdot\xi$$

Proof of proposition 16: By induction on ξ,

- **assume $\boldsymbol{\xi = 1}$**: $d_W^\circ(A \bullet 1) = d_W^\circ A = (d_W^\circ A) \cdot 1$
- **assume $\boldsymbol{1 < \xi = \nu + 1}$**: $d_W^\circ(A \bullet (\nu+1)) = d_W^\circ((A \bullet \nu) + A) = d_W^\circ(A \bullet \nu) + d_W^\circ(A) = d_W^\circ(A) \cdot \nu + d_W^\circ A = d_W^\circ(A) \cdot (\nu+1)$.
- **assume $\boldsymbol{\xi}$** is limit: $d_W^\circ(A \bullet \xi) = d_W^\circ(\mathbf{sup}_{\zeta \in \xi} A \bullet \zeta) = \sup_{\zeta \in \xi} d_W^\circ(A \bullet \zeta) = \sup_{\zeta \in \xi} d_W^\circ(A) \cdot \zeta = d_W^\circ(A) \cdot \xi$.

$\dashv$ 16

From the player point of view, being in charge of a set of the form $A \bullet \xi$ is like being in charge of A with the additional option to restart the run at any moment being in charge of ${}^{-}A$ instead of A and start again and again replacing ${}^{-}A$ by A then A by ${}^{-}A$ etc... But **provided** that at every such changing the player decreases the ordinal ξ. Therefore, during the run, this trick will produce a decreasing sequence of ordinals (hence necessarily finite), preventing him from initializing the game indefinitely. We reach next operation by allowing such a process to be infinite. The ω_1-multiplication is this process going on for ever without any ordinal decreasing limitation; simply considering that a run that changes one's mind ω times is rejected.

7 The set theoretical ω_1-multiplication

Definition 17 *Let $A \subseteq \Sigma_A^\omega$ and O_+, O_- be two different letters not in Σ_A; set $\Sigma = \Sigma_A \cup \{O_+, O_-\}$,*

$$A^\natural \;=\; (\Sigma^* \cdot O_+ \cdot A) \;\cup\; (\Sigma^* \cdot O_- \cdot {}^{-}A).$$

Remarks 18 (a) This operation preserves the Wadge ordering since for any A and B both Borel:

$$A \leq_W B \longrightarrow A^\natural \leq_W B^\natural$$

(b) Clearly, for any countable ordinal $\nu \neq 0$, the following holds:

$$A \bullet \nu \leq_W A^\natural$$

(c) Unlike the sets obtained by use of the previously defined operations $+$, $\mathbf{sup}$, $\bullet\nu$, the set $A^\natural$ satisfies the property that, from the point of view of a player in charge of such a set in a Wadge game, **every position** is equivalent to the initial one. Just like when dealing with the emptyset or the whole space. More generally:

Definition 19 *Let $A \subseteq \Sigma_A^\omega$, O_+ be a letter not in Σ_A and $A' \subseteq (\Sigma_A \cup \{O_+\})^\omega$ be the inverse image of A by the homomorphism $h(O_+) = 0$ and $h(a) = a$ (for any $a \in \Sigma_A$). We say that A is* **initializable** *if*

$$A \equiv_W A \;\cup\; (\Sigma_A^* \cdot O_+ \cdot A').$$

In other words, A is initializable iff there exists a winning strategy for *II* in $\mathbf{W}(A\ ,\ A)$ such that every position u reached by *II* when applying this strategy satisfies $A_{(u)} \equiv_W A$. Amazingly there is a simple relation between initializability and Wadge degrees of Borel sets: a set A is initializable iff there exists some ordinal ξ which is not of countable cofinality such that $d_W^\circ A = \omega_1^\xi$ (see [7] and [8]). These peculiar sets will be very useful in the main proofs of this paper.

Proposition 20 *Let $A \subseteq \Sigma_A^\omega$ be Borel and non self dual,*

$$d_W^\circ(A^\natural) = (d_W^\circ A) \cdot \omega_1$$

Proof of proposition 20: Since $A \bullet \nu \leq_W A^\natural$ holds for any countable ν, the inequality $d_W^\circ(A^\natural) \geq d_W^\circ(A \bullet \nu)$ is immediate. Therefore:

$$\begin{array}{rcl} d_W^\circ(A^\natural) & \geq & \sup\{d_W^\circ(A \bullet \nu) : \ \nu < \omega_1\} \\ & \geq & \sup\{d_W^\circ(A) \cdot \nu) : \ \nu < \omega_1\} \ . \\ & \geq & d_W^\circ(A) \cdot \omega_1 \end{array}$$

The other inequality relies on:

Claim 21 *let $C \subseteq \Sigma_C^\omega$, C non self dual,*

$$({}^{-}C \leq_W A^\natural \ \wedge \ C \leq_W A^\natural) \quad \longrightarrow \quad \exists \nu < \omega_1 \quad C \leq_W A \bullet \nu.$$

This claim leads to

$$\begin{array}{rcl} d_W^\circ(A^\natural) & = & \sup\{(d_W^\circ C) + 1 : \ C \ n.s.d. \ \wedge \ C <_W (A^\natural)\} \\ & = & \sup\{(d_W^\circ C) + 1 : \ C \ n.s.d. \ \wedge \ \exists \nu < \omega_1 \ \ C \leq_W (A \bullet \nu)\} \\ & \leq & \sup\{(d_W^\circ(A \bullet \nu)) + 1 : \ \nu < \omega_1\} \\ & \leq & \sup\{((d_W^\circ A) \cdot \nu) + 1 : \ \nu < \omega_1\} \\ & \leq & \sup\{((d_W^\circ A) \cdot \nu) : \ \nu < \omega_1\} \\ & \leq & (d_W^\circ A) \cdot \omega_1 \end{array} \quad .$$

Proof of claim 21: Set $B = A^\natural \ \cup \ \{O_+, \ O_-\}^\omega$. It is easy to see that ${}^{-}(A^\natural) \equiv_W B$. By hypothesis, there are strategies σ_+ and σ_- that are winning for *II* in respectively $\mathbf{W}\left(C \ , \ A^\natural\right)$ and $\mathbf{W}(C \ , \ B)$. From σ_+ and σ_-, one can define a winning strategy for *II* in $\mathbf{W}\left(C \ , \ A^\natural\right)$ with the property that it never compels player *II* to play infinitely many letters among $\{O_+, \ O_-\}$. Essentially σ follows the strategy (σ_+ or σ_-) that plays the least number of O_+ and O_-. The fact that $\{O_+, \ O_-\}^\omega \subseteq B$ and $\{O_+, \ O_-\}^\omega \cap A^\natural = \emptyset$ guarantees that switching strategies will only occur finitely many times. Now set:

$$T = \left\{\left(x \restriction k \ , \ \sigma\!\!\star (x \restriction k)\right) : \ k \in \mathbb{N} \ \wedge \ \left(\sigma\!\!\star (x \restriction (k)) = O_+ \ \vee \ \sigma\!\!\star (x \restriction (k)) = O_-\right)\right\}$$

and the ordering $\leq_T$ on T by

$$(u,v) \leq_T (u',v') \quad \Longleftrightarrow_{\text{df}} \ \left(lh(u) \leq lh(u') \ \wedge \ lh(v) \leq lh(v') \ \wedge \ u' \restriction lh(u) = u \ \wedge \ v' \restriction lh(v) = v\right).$$

As usual, the strict ordering $c <_T c'$ stands for $c \leq_T c' \not\leq_T c$. We remark that there is no infinite sequence $(u_0,v_0) <_T (u_1,v_1) <_T \ldots <_T (u_n,v_n) <_T (u_{n+1},v_{n+1}) <_T \ldots$; because otherwise, let (x,y) be the unique couple such that,

$$\forall i \in \mathbb{N} \quad x \restriction lh(u_i) = u_i \quad \wedge \quad y \restriction lh(v_i) = v_i.$$

By definition, $\sigma\!\!\star \ x = y$ contains infinitely many O_+ or O_-. A contradiction.
Inductively define a mapping $Ord_T \ : \ T \longmapsto On$ by

$$\begin{array}{l} Ord_T(u,v) = 1 \iff \left((u,v) \leq_T (u',v') \longrightarrow (u,v) = (u',v')\right) \\ Ord_T(u,v) = \sup\{Ord_T(u',v') + 1 : \supsetneq (u,v) <_T (u',v')\} \text{ otherwise.} \end{array}$$

Finally, we set $\nu = \sup\{Ord_T(u,v) \ : \ (u,v) \in T\} + 1$. Clearly $\nu < \omega_1$ holds because T is countable. From σ and T it is then really tedious but straightforward to derive, by induction on ν, a strategy that is winning for *II* in $\mathbf{W}\,(C\ ,\ A \bullet \nu)$.

$\dashv$ 21

$\dashv$ 20

For now, the Wadge-degree-related properties of the set theoretical operations $+$, $\bullet\,\nu$ and $\natural$ allow us to define "canonical" non self dual sets for any Wadge degree below ω_1^ω, the very same sets that are subject to be recognized by the machines we are interesting in in that paper.

8 canonical non self dual sets

Definition 22 *Let $\alpha < \omega_1^\omega$, then α admits a Cantor Normal Form of base ω_1:*

$$\alpha = \omega_1^{n_k} \cdot \nu_k + \ldots + \omega_1^{n_0} \cdot \nu_0$$

$$\textit{where} \ \ 0 \leq n_0 < \ldots < n_k < \omega \quad \textit{and} \ \ \forall i \leq k \ \ 0 < \nu_i < \omega_1$$

Set

$$\Omega(\alpha) = \left((\emptyset^{\natural n_k}) \bullet \nu_n\right) + \ldots\ldots + \left((\emptyset^{\natural n_0}) \bullet \nu_0\right).$$

where $A^{\natural n}$ is inductively defined by: $A^{\natural 0} = A$ and $A^{\natural(n+1)} = \left(A^{\natural n}\right)^{\natural}$.

Propositions 9, 16 and 20 guarantee that $d^\circ_W \Omega(\alpha) = \alpha$. So any non self dual set of degree $< \omega_1^\omega$ is Wadge equivalent to some $\Omega(\alpha)$ or ${}^{-}\Omega(\alpha)$ for some ordinal α.

Wagner showed in [27] that any ω-regular set is Wadge equivalent to some $\Omega(\alpha)$, ${}^{-}\Omega(\alpha)$ or ${}^{\pm}\Omega(\alpha)$ with all the parameters of the Cantor Normal Form of base ω_1 of α being integers:

$$\alpha = \omega_1^{n_k} \cdot m_k + \ldots + \omega_1^{n_0} \cdot m_0 \quad (n_i, m_i \in \mathbb{N})$$

So the order type of Wagner hierarchy is ω^ω. We intend to show that any deterministic context free ω-language is Wadge equivalent to some $\Omega(\alpha)$, ${}^{-}\Omega(\alpha)$ or ${}^{\pm}\Omega(\alpha)$ for some α with a Cantor Normal Form of base ω_1 with parameters less than ω^ω:

$$\alpha = \omega_1^{n_k} \cdot \nu_k + \ldots + \omega_1^{n_0} \cdot \nu_0 \quad (n_i \in \mathbb{N},\ \nu_i < \omega^\omega).$$

Hence while the height of Wagner's hierarchy is ω^ω, its extension from *DA* to *DPDA* (the Wadge Hierarchy of Context-free ω-languages) reaches $(\omega^\omega)^\omega = \omega^{(\omega^2)}$.

Let $\mathfrak{W}_{\mathrm{DA}} :\ \omega^\omega \longmapsto \omega_1^\omega$ be the Wagner function defined by: assuming the Cantor Normal Form of base ω of α is $\alpha = \omega^{n_k} \cdot m_k + \ldots + \omega^{n_0} \cdot m_0$ then

$$\mathfrak{W}_{\mathrm{DA}}(\alpha) = \omega_1^{n_k} \cdot m_k + \ldots + \omega_1^{n_0} \cdot m_0$$

For each ordinal $0 < \alpha < \omega^\omega$ the Wagner construction gives a Deterministic Automaton $\mathbb{A}(\alpha)$ which Wadge degree is precisely $\mathfrak{W}_{\mathrm{DA}}(\alpha)$.

In this paper we do, *mutatis mutandis*, the same kind of thing but with *DPDA* instead of *DA*. Set $\mathfrak{W}_{\mathrm{DPDA}} :\ \omega^{(\omega^2)} \longmapsto \omega_1^\omega$ be the Wagner function defined by: assuming the Cantor Normal Form of base ω^ω of α is $\alpha = (\omega^\omega)^{n_k} \cdot m_k + \ldots + (\omega^\omega)^{n_0} \cdot m_0$ then

$$\mathfrak{W}_{\mathrm{DPDA}}(\alpha) = \omega_1^{n_k} \cdot m_k + \ldots + \omega_1^{n_0} \cdot m_0$$

For each ordinal $0 < \alpha < \omega^{(\omega^2)}$ we produce $\mathbb{A}(\alpha)$, a *DPDA* whose Wadge degree is precisely $\mathfrak{W}_{\mathrm{DPDA}}(\alpha)$. We do so by defining operations on *DPDA* ($+_{\mathbb{A}}$, $\bullet_{\mathbb{A}}\nu$, $^{\natural_{\mathbb{A}}}$) that are the counterpart of the ordinal operations when it comes to compute the Wadge degree of the ω-languages involved. In a second time, we prove that, for any *DPDA* $\mathcal{A}$, there exists some ordinal $\alpha < \omega^{(\omega^2)}$ such that

$$\ulcorner\mathcal{A}\urcorner \equiv_W \ulcorner\mathbb{A}(\alpha)\urcorner \quad \vee \quad \ulcorner\mathcal{A}\urcorner \equiv_W {}^{-}\ulcorner\mathbb{A}(\alpha)\urcorner \quad \vee \quad \ulcorner\mathcal{A}\urcorner \equiv_W {}^{\pm}\ulcorner\mathbb{A}(\alpha)\urcorner.$$

We do so by assuming that α admits a Cantor Normal Form of base ω_1:

$$\alpha = \omega_1^{n_k} \cdot \nu_k + \ldots + \omega_1^{n_0} \cdot \nu_0$$

with at least one $\nu_i \geq \omega^\omega$ and then showing that the set $\Omega(\alpha)$ cannot be Wadge equivalent to some set recognized by a *DPDA*.

9 "Arithmetical" operations on Deterministic Pushdown Automata

In that section we describe $\mathbb{A}(\alpha)$, a canonical *DPDA*, for each α admitting a Cantor Normal Form of base ω_1:

$$\alpha = \omega_1^{n_k} \cdot \nu_k + \ldots + \omega_1^{n_0} \cdot \nu_0 \quad \text{(each } \nu_i < \omega^\omega\text{)}$$

We do so by defining operations on Deterministic PushDown Automata:

(a) $\mathcal{A}, \mathcal{B} \longmapsto \mathcal{B} +_{\mathbb{A}} \mathcal{A}$,

(b) $\mathcal{A} \longmapsto \mathcal{A} \bullet_{\mathbb{A}} \omega$,

(c) $\mathcal{A} \longmapsto \mathcal{A}^{\natural_{\mathbb{A}}}$

that verify:
If $\ulcorner\mathcal{A}\urcorner$ and $\ulcorner\mathcal{B}\urcorner$ are the sets of infinite words accepted respectively by $\mathcal{A}$ and $\mathcal{B}$, then

(a) $\ulcorner\mathcal{B} +_{\mathbb{A}} \mathcal{A}\urcorner \equiv_W \ulcorner\mathcal{B}\urcorner + \ulcorner\mathcal{A}\urcorner$,

(b) $\ulcorner\mathcal{A} \bullet_{\mathbb{A}} \omega\urcorner \equiv_W \ulcorner\mathcal{A}\urcorner \bullet \omega$

(c) $\ulcorner\mathcal{A}^{\natural_{\mathbb{A}}}\urcorner \equiv_W \ulcorner\mathcal{A}^{\natural}\urcorner$

The operations $\mathcal{A}, \mathcal{B} \longmapsto \mathcal{B} +_{\mathbb{A}} \mathcal{A}$ and $\mathcal{A} \longmapsto \mathcal{A}^{\natural_{\mathbb{A}}}$ are not only defined on *DPDA* but also on Deterministic Automata, since formally they don't require a stack. But this remark fails when it comes to the third operation: $\mathcal{A} \longmapsto \mathcal{A} \bullet_{\mathbb{A}} \omega$. Here the stack is absolutely necessary to count indefinitely throughout all integers.

Definition 23 *Let* $\mathcal{A} = (K, \Sigma, \Gamma_\perp, q_{init}, \perp, \delta, \mathfrak{F})$ *be some* DPDA, *we define* ${}^{-}\mathcal{A}$ *by*

$${}^{-}\mathcal{A} = (K, \Sigma, \Gamma_\perp, q_{init}, \perp, \delta, \mathcal{P}(K) \smallsetminus \mathfrak{F}).$$

By its very definition, ${}^{-}\mathcal{A}$ recognizes exactly the set $\Sigma^\omega \smallsetminus \ulcorner\mathcal{A}\urcorner$.

We now define a *DPDA* ${}^{\pm}\mathcal{A}$ which in first move allows to choose between $\mathcal{A}$ and ${}^{-}\mathcal{A}$. So the set recognized by ${}^{\pm}\mathcal{A}$ is Wadge equivalent to ${}^{\pm}\ulcorner\mathcal{A}\urcorner$:

Definition 24 *Let* $\mathcal{A} = (K, \Sigma,\ \Gamma_\perp,\ q_{init}, \perp,\ \delta,\ \mathfrak{F})$ *be some* DPDA, *Assume* $\{\Sigma_+, \Sigma_-\}$ *is some partition of* Σ *into two* **non empty** *sets. We build a* DPDA *that accepts the following set:*

$$(\Sigma_+ \cdot \ulcorner \mathcal{A} \urcorner) \ \cup\ (\Sigma_- \cdot {}^{-}\ulcorner \mathcal{A} \urcorner).$$

Assume K_+ *and* K_- *are two different copies of* $K = \{q_0, \ldots, q_k\}$*:* $K_+ = \{q_0^+, \ldots, q_k^+\}$ *and* $K_- = \{q_0^-, \ldots, q_k^-\}$ *with* $K_- \cap K_+ = \emptyset$*. Assume also that* $q' \notin K_+ \cup K_-$*. We define*

$${}^{\pm}\mathcal{A} = (K_+ \cup K_- \cup \{q'\}, \Sigma,\ \Gamma_\perp,\ q', \perp,\ \delta',\ \mathfrak{F}').$$

With δ' *and* $\mathfrak{F}'$ *defined by:*
$\forall n \leq k \quad \forall \varrho \in \{+, -\} \quad \forall a \in \Sigma \quad \forall Z \in \Gamma_\perp$

$$\delta(q_n, a, Z) = (q_m, w) \implies \delta'(q_n^\varrho, a, Z) = (q_m^\varrho, w)$$

$$\forall a \in \Sigma \quad \forall Z \in \Gamma_\perp \quad \delta'(q', a, Z) = \begin{cases} (q_i^+, Z) \ \textit{iff}\ a \in \Sigma_+ \\ \qquad \textit{and} \\ (q_i^-, Z) \ \textit{iff}\ a \in \Sigma_- \end{cases}$$

(remember that q_{init} *was the initial state of* $\mathcal{A}$*). Finally*

$$\mathfrak{F}' = \begin{array}{c} \left\{\{q_{j_0}^+, \ldots, q_{j_n}^+\} \subseteq K_+ : \quad \{q_{j_0}, \ldots, q_{j_n}\} \in \mathfrak{F}\right\} \\ \bigcup \\ \left\{\{q_{j_0}^-, \ldots, q_{j_n}^-\} \subseteq K_- : \ \{q_{j_0}, \ldots, q_{j_n}\} \notin \mathfrak{F}\right\} \end{array}.$$

So from a player's point of view, the *DPDA* ${}^{\pm}\mathcal{A}$ is just two copies of $\mathcal{A}$, one being the exact inverse of the other one, joined by a state q' that requires on first move to choose between the copy of $\mathcal{A}$ and its complement.

Also from the playful point of view, the next *DPDA* $(\mathcal{B} +_{\mathbb{A}} \mathcal{A})$ can be described as the one that allows a player to start with $\mathcal{A}$ and at any moment of the game to get rid of $\mathcal{A}$ and to start again from scratch, with $\mathcal{B}$ or ${}^{-}\mathcal{B}$ instead of $\mathcal{A}$.

Definition 25
Let $\mathcal{A} = (K_A, \Sigma_A,\ \Gamma^A_{\perp_A},\ q_i^A, \perp_A,\ \delta_A,\ \mathfrak{F}_A)$ *and* $\mathcal{B} = (K_B, \Sigma_B,\ \Gamma^B_{\perp_B},\ q_i^B, \perp_B,\ \delta_B,\ \mathfrak{F}_B)$ *be two* DPDA *with* $\Sigma_A \subsetneq \Sigma_B$ *and* $\{\Sigma_+, \Sigma_-\}$ *be a partition of* $\Sigma_B \setminus \Sigma_A$ *in two* **non empty** *sets. We build* $\mathcal{B} +_{\mathbb{A}} \mathcal{A}$ *as a* DPDA *that accepts:*

$$\ulcorner \mathcal{A} \urcorner \ \cup\ \Sigma_A^* \cdot \Sigma_+ \cdot \ulcorner \mathcal{B} \urcorner \ \cup\ \Sigma_A^* \cdot \Sigma_- \cdot {}^{-}\ulcorner \mathcal{B} \urcorner.$$

Let $\mathcal{B}^- = (K_B^-, \Sigma_B,\ \Gamma^B_{\perp_B},\ q_i^{B^-}, \perp_B,\ \delta_B^-,\ \mathcal{P}(K_B^-) \setminus \mathfrak{F}_B^-)$ *be a copy of* ${}^{-}\mathcal{B}$ *with completely different states. i.e.* $q_i^{B^-}$, δ_B^-, $\mathfrak{F}_B^-$ *are just the exact images of* q_i^B, δ_B, $\mathfrak{F}_B$ *induced by the isomorphism between* K_B *and* K_B^-*.*
Assume $K_A \cap K_B \cap K_B^- = \emptyset$, $\Gamma^A_{\perp_A} \cap \Gamma^B_{\perp_B} = \emptyset$, $\Sigma_A \subsetneq \Sigma_B$

$$\mathcal{B} +_{\mathbb{A}} \mathcal{A} = (K_A \cup K_B \cup K_B^-, \Sigma_B,\ \Gamma^A_{\perp_A} \cup \Gamma^B_{\perp_B},\ q_i^A, \perp_A,\ \delta,\ \mathfrak{F})$$

With $\mathfrak{F}$ defined by $\mathfrak{F} = \mathfrak{F}_A \cup \mathfrak{F}_B \cup (\mathcal{P}(K_B^-) \smallsetminus \mathfrak{F}_B^-)$ and δ defined by

$$\forall q \in K_A \quad \forall a \in \Sigma_A \quad \forall Z \in \Gamma_{\perp_A}^A \qquad \begin{cases} \delta(q,a,Z) = \delta_A(q,a,Z), \text{ if defined} \\ \delta(q,\varepsilon,Z) = \delta_A(q,\varepsilon,Z), \text{ otherwise} \end{cases}$$

$$\forall q \in K_A \quad \forall a \in \Sigma_+ \quad \forall Z \in \Gamma_{\perp_A}^A \qquad \delta(q,a,Z) = (q_i^B, \perp_B)$$

$$\forall q \in K_A \quad \forall a \in \Sigma_- \quad \forall Z \in \Gamma_{\perp_A}^A \qquad \delta(q,a,Z) = (q_i^{B^-}, \perp_B)$$

$$\forall q \in K_B \quad \forall a \in \Sigma_B \quad \forall Z \in \Gamma_{\perp_B}^B \qquad \begin{cases} \delta(q,a,Z) = \delta_B(q,a,Z), \text{ if defined} \\ \delta(q,\varepsilon,Z) = \delta_B(q,\varepsilon,Z), \text{ otherwise} \end{cases}$$

$$\forall q \in K_B^- \quad \forall a \in \Sigma_B \quad \forall Z \in \Gamma_{\perp_B}^B \qquad \begin{cases} \delta(q,a,Z) = \delta_{B^-}(q,a,Z), \text{ if defined} \\ \delta(q,\varepsilon,Z) = \delta_{B^-}(q,\varepsilon,Z), \text{ otherwise} \end{cases}$$

The definition of δ in all other cases doesn't matter because the five conditions above prevent them from ever happening.

It is almost immediate to see that the set $\ulcorner \mathcal{B} +_{\mathbb{A}} \mathcal{A} \urcorner$ is Wadge equivalent to the set $\ulcorner \mathcal{B} \urcorner + \ulcorner \mathcal{A} \urcorner$.

The main idea to define the $\bullet_{\mathbb{A}} \omega$ operation is that, from the point of view of a player in charge of a set recognized by such a *DPDA*, the first moves consist in filling up the stack with a special letter. Each occurrence of that letter being then used to count and limit the number of times the player may decide to start anew being in charge of $\mathcal{A}$ or of its complement. So writing n occurrences of this special letter in the stack will limit the reset process to at most n times. Ensuring this will make almost immediate to see that the set $\ulcorner \mathcal{A} \bullet_{\mathbb{A}} \omega \urcorner$ is Wadge equivalent to the set $\ulcorner \mathcal{A} \urcorner \bullet \omega$.

Definition 26 *Let $\mathcal{A} = (K, \Sigma_A,\ \Gamma_\perp,\ q_{init}, \perp,\ \delta,\ \mathfrak{F})$ be some* DPDA*. Let O_+ and O_- be two different letters not in Σ_A; we build a* DPDA *that recognizes the following set:*

$$\bigcup_{n \in \mathbb{N}} \quad \Sigma_A^n \cdot (\{O_+, O_-\} \cdot \Sigma_A^*)^{\leq n} \cdot \left(O_+ \cdot \ulcorner \mathcal{A} \urcorner \quad \cup \quad O_- \cdot {}^{-}\ulcorner \mathcal{A} \urcorner\right)$$

Assume K_+ and K_- are two different copies of $K = \{q_0, \ldots, q_k\}$: $K_+ = \{q_0^+, \ldots, q_k^+\}$ and $K_- = \{q_0^-, \ldots, q_k^-\}$ with $K_- \cap K_+ = \emptyset$.
Assume also that q', q_+, q_- are three different states not in $K_+ \cup K_-$; and pick two different letters $⫫$ and $\top$ that are not in $\Gamma_\perp$; And also choose two different letters O_+ and O_- that do not belong to Σ_A and define

$$\mathcal{A} \bullet_{\mathbb{A}} \omega = (K_+ \cup K_- \cup \{q',\ q_+,\ q_-\}, \Sigma_A \cup \{O_+, O_-\},\ \Gamma \cup \{\perp, ⫫, \top\},\ q', ⫫,\ \delta',\ \mathfrak{F}')$$

where δ' is defined by

(a) when involving the initial state q':
$\forall a \in \Sigma_A \quad \delta'(q', a, ⫫) = (q', ⫫ \cdot \top)$ *and* $\delta'(q', a, \top) = (q', \top \cdot \top)$,
$\delta'(q', O_+, ⫫) = (q_{init}^+, ⫫ \cdot \perp)$ *and* $\delta'(q', O_+, \top) = (q_{init}^+, \top \cdot \perp)$,
$\delta'(q', O_-, ⫫) = (q_{init}^-, ⫫ \cdot \perp)$ *and* $\delta'(q', O_-, \top) = (q_{init}^-, \top \cdot \perp)$.

(b) when involving a state q from $K_+ \cup K_-$:

$$\forall a \in \Sigma_A \quad \forall Z \in \Gamma_\perp \quad \begin{cases} \textit{if } \delta(q,a,Z) \textit{ is defined, then } \delta'(q,a,Z) = \delta(q,a,Z) \\ \textit{otherwise, } \delta'(q,\varepsilon,Z) = \delta(q,\varepsilon,Z). \end{cases}$$

$$\forall Z \in \Gamma_\perp \quad \delta'(q,O_+,Z) = (q_+,\varepsilon) \textit{ and } \delta'(q,O_-Z) = (q_-,\varepsilon)$$

(c) when involving q_+ or q_-:

$$\forall Z \in \Gamma_\perp \quad \begin{cases} \delta'(q_+,\varepsilon,Z) = (q_+,\varepsilon) \\ \delta'(q_-,\varepsilon,Z) = (q_-,\varepsilon) \end{cases}$$ *(this is to erase the stack until the letter $\top$ or $\perp\!\!\!\perp$ is reached).*

$$\forall a \in \Sigma_A \cup \{O_+,\ O_-\} \quad \begin{cases} \delta'(q_+,a,\top) = (q^+_{init},\perp) \textit{ and } \delta'(q_+,a,\perp\!\!\!\perp) = (q_+,\perp\!\!\!\perp) \\ \delta'(q_-,a,\top) = (q^-_{init},\perp) \textit{ and } \delta'(q_-,a,\perp\!\!\!\perp) = (q_-,\perp\!\!\!\perp) \end{cases}$$

Defining all other alternatives is useless since everything is designed so to prevent them from ever happening.

Finally $\mathfrak{F}'$ is defined by:

$$\mathfrak{F}' = \left\{\{q^+_{j_0},\dots,q^+_{j_n}\} \subseteq K_+ :\ \{q_{j_0},\dots,q_{j_n}\} \in \mathfrak{F}\right\} \ \cup\ \left\{\{q^-_{j_0},\dots,q^-_{j_n}\} \subseteq K_- :\ \{q_{j_0},\dots,q_{j_n}\} \notin \mathfrak{F}\right\}$$

Last but not least, we define the operation that can be regarded as allowing to "jump" indefinitely from $\mathcal{A}$ to ${}^-\mathcal{A}$, repeating this reset process as much as desired. Therefore $\ulcorner \mathcal{A}^{\natural_{\mathbb{A}}} \urcorner$ is Wadge equivalent to $(\ulcorner \mathcal{A} \urcorner)^\natural$

Definition 27 *Let $\mathcal{A} = (K,\Sigma_A,\ \Gamma_\perp,\ q_{init},\perp,\ \delta,\ \mathfrak{F})$ be some* DPDA. *Assume K_+ and K_- are two different copies of $K = \{q_0,\dots,q_k\}$: $K_+ = \{q^+_0,\dots,q^+_k\}$ and $K_- = \{q^-_0,\dots,q^-_k\}$ with $K_- \cap K_+ = \emptyset$. Choose two different states q_+ and q_- that are not in $K_+ \cup K_-$ and two different letters $O_, - O_+ \notin \Sigma_A$ and define $\mathcal{A}^{\natural_{\mathbb{A}}}$ as a* DPDA *that accepts:*

$$\ulcorner \mathcal{A} \urcorner \ \cup\ (\Sigma_A \cup \{O_-O_+\})^* \cdot O_+ \cdot \ulcorner \mathcal{A} \urcorner \ \cup\ (\Sigma_A \cup \{O_-O_+\})^* \cdot O_- \cdot {}^-\ulcorner \mathcal{A} \urcorner.$$

$\mathcal{A}^{\natural_{\mathbb{A}}} = (K_+ \cup K_- \cup \{q_+,q_-\},\Sigma_A \cup \{O_+,O_-\},\ \Gamma_\perp,\ q_+,\perp,\ \delta',\ \mathfrak{F}')$ where δ' is defined by:

(a) $\forall Z \in \Gamma_\perp \quad \delta'(q_+,\varepsilon,Z) = (q^+_i,Z)$ *and* $\delta'(q_-,\varepsilon,Z) = (q^-_i,Z)$

(b) $\forall a \in \Sigma_A \quad \forall j \le k \quad \forall \varrho \in \{+,-\} \quad \forall Z \in \Gamma_\perp \quad \begin{cases} \delta'(q^\varrho_j,a,Z) = \delta(q_j,a,Z) \textit{ if defined} \\ \delta'(q^\varrho_j,\varepsilon,Z) = \delta(q_j,\varepsilon,Z) \textit{ otherwise} \end{cases}$

(c) $\forall j \le k \quad \forall \varrho \in \{+,-\} \quad \forall Z \in \Gamma_\perp \quad \delta'(q^\varrho_j,O_+,Z) = (q^+,\perp)$

(d) $\forall j \le k \quad \forall \varrho \in \{+,-\} \quad \forall Z \in \Gamma_\perp \quad \delta'(q^\varrho_j,O_-Z) = (q^-,\perp).$

And $\mathfrak{F}'$ is defined by

$$\mathfrak{F}' = \left\{\{q^+_{j_0},\dots,q^+_{j_n}\} \subseteq K_+ :\ \{q_{j_0},\dots,q_{j_n}\} \in \mathfrak{F}\right\} \ \cup\ \left\{\{q^-_{j_0},\dots,q^-_{j_n}\} \subseteq K_- :\ \{q_{j_0},\dots,q_{j_n}\} \notin \mathfrak{F}\right\}$$

We iterate the $+_{\mathbb{A}}$ operation to get $\bullet_{\mathbb{A}} n$ (the analog of the arithmetical multiplication) and we mix it with the $\bullet_{\mathbb{A}}\omega$ operation to get the analog of multiplication by any ordinal below ω^ω. We also iterate $\natural_{\mathbb{A}}$ a finite number of times to get $\natural_{\mathbb{A}} n$.

Definition 28 *Let $\mathcal{A}$ be some* DPDA, *(a) For any non null integer n we inductively define $\mathcal{A}\bullet_{\mathbb{A}} n$ by*

$$\mathcal{A}\bullet_{\mathbb{A}} 1 = \mathcal{A} \quad \text{and} \quad \mathcal{A}\bullet_{\mathbb{A}}(n+1) = (\mathcal{A}\bullet_{\mathbb{A}} n) +_{\mathbb{A}} \mathcal{A}.$$

and $\mathcal{A}\bullet_{\mathbb{A}}\omega^n$ by

$$\mathcal{A}\bullet_{\mathbb{A}}\omega^1 = \mathcal{A}\bullet_{\mathbb{A}}\omega \quad \text{and} \quad \mathcal{A}\bullet_{\mathbb{A}}\omega^{(n+1)} = (\mathcal{A}\bullet_{\mathbb{A}}\omega^n)\bullet_{\mathbb{A}}\omega.$$

(b) So for any ordinal $0 < \nu < \omega^\omega$, let $\omega^{n_k} \cdot m_k + \ldots + \omega^{n_0} \cdot m_0$ be the Cantor Normal Form of base ω of ν, we define $\mathcal{A}\bullet_{\mathbb{A}}\nu$ by

$$\mathcal{A}\bullet_{\mathbb{A}}\nu = (\mathcal{A}\bullet_{\mathbb{A}}\omega^{n_k})\bullet_{\mathbb{A}} m_k +_{\mathbb{A}} \ldots +_{\mathbb{A}} (\mathcal{A}\bullet_{\mathbb{A}}\omega^{n_0})\bullet_{\mathbb{A}} m_0$$

(c) For any integer n we inductively define $\mathcal{A}^{\natural_{\mathbb{A}} n}$ by

$$\mathcal{A}^{\natural_{\mathbb{A}} 0} = \ominus \quad \text{and} \quad \mathcal{A}^{\natural_{\mathbb{A}}(n+1)} = (\mathcal{A}^{\natural_{\mathbb{A}} n})^{\natural_{\mathbb{A}}}.$$

Where $\ominus$ stands for any DPDA *that rejects every word, hence accepts the emptyset.*

We finally put everything together and define canonical DPDA $\mathbb{A}(\alpha)$ *from the normal form of the ordinal α that indicates its level in this hierarchy:*

Definition 29 *Let $0 < \alpha < \omega^{(\omega^2)} = (\omega^\omega)^\omega$, α admits a Cantor Normal Form of base ω^ω:*

$$\alpha = (\omega^\omega)^{n_k} \cdot \nu_k + \ldots + (\omega^\omega)^{n_0} \cdot \nu_0$$

with $0 \le n_0 < \ldots < n_k < \omega$ and $0 < \nu_i < \omega^\omega$ (for each $i \le k$).
Define $\mathbb{A}(\alpha)$ by

$$\mathbb{A}(\alpha) = \mathbb{A}((\omega^\omega)^{n_k})\bullet_{\mathbb{A}}\nu_k \;+_{\mathbb{A}}\; \ldots \;+_{\mathbb{A}}\; \mathbb{A}((\omega^\omega)^{n_0})\bullet_{\mathbb{A}}\nu_0$$

where $\mathbb{A}((\omega^\omega)^{n_i})$ is defined by $\mathbb{A}((\omega^\omega)^{n_i}) = \ominus^{\natural_{\mathbb{A}} n_i}$. Hence

$$\mathbb{A}(\alpha) = (\ominus^{\natural_{\mathbb{A}} n_k})\bullet_{\mathbb{A}}\nu_k \;+_{\mathbb{A}}\; \ldots \;+_{\mathbb{A}}\; (\ominus^{\natural_{\mathbb{A}} n_0})\bullet_{\mathbb{A}}\nu_0$$

Following our convention, $\ulcorner\mathbb{A}(\alpha)\urcorner$ denotes the set of infinite words accepted by $\mathbb{A}(\alpha)$.

Remark 30 If the Cantor Normal Form of base ω^ω of $\alpha < \omega^{(\omega^2)}$ is

$$\alpha = (\omega^\omega)^{n_k} \cdot \nu_k + \ldots + (\omega^\omega)^{n_0} \cdot \nu_0$$

then $\ulcorner\mathbb{A}(\alpha)\urcorner \equiv_W \Omega(\omega_1^{n_k} \cdot \nu_k + \ldots + \omega_1^{n_0} \cdot \nu_0) = \ominus^{\natural n_k} \bullet \nu_k + \ldots + \ominus^{\natural n_0} \cdot \nu_0$.

Everything is now ready to state the main result of this paper.

Theorem 31 *Let $\mathcal{A}$ be any* DPDA, *there exists some unique $\alpha < \omega^{(\omega^2)}$ such that one and only of the following possibilities holds:*

$$\ulcorner\mathcal{A}\urcorner \equiv_W \ulcorner\mathbb{A}(\alpha)\urcorner \quad \text{or} \quad \ulcorner\mathcal{A}\urcorner \equiv_W \ulcorner{}^{-}\mathbb{A}(\alpha)\urcorner \quad \text{or} \quad \ulcorner\mathcal{A}\urcorner \equiv_W \ulcorner{}^{\pm}\mathbb{A}(\alpha)\urcorner.$$

From now on, the fore-coming sections are entirely devoted to the proof of theorem 31. So assuming that ${}^{+}\mathcal{A}$ stands for $\mathcal{A}$, theorem 31 states that (up to $\equiv_W$) the following set

$$\{\ulcorner{}^{\varrho}\mathbb{A}(\alpha)\urcorner : \quad \varrho \in \{-,+,\pm\}, \quad 0 < \alpha < \omega^{(\omega^2)}\}$$

is the set of all possible sets of ω-words accepted by *DPDA*. In fact, we will only prove the restriction of it to ε-free *DPDA* mainly because they are more simple machines to work with than the *DPDA* with ε-moves (see [4]). However, it is well-known that there exists sets accepted by *DPDA* that cannot be recognized by any ε-free *DPDA*. But such a phenomenon cannot occur when equality is replaced by Wadge equivalence, as shown in proposition below. Therefore, proving theorem 31 for *ew*-free *DPDA* comes to proving it for any *DPDA*.

Proposition 32 *If* $\mathcal{A} = (K, \Sigma, \Gamma_\perp, q_{init}, \perp, \delta_A, \mathfrak{F})$ *is some* DPDA *which has the Continuity Property, then there exists* $\mathcal{B} = (K, \Sigma, \Gamma_\perp, q_{init}, \perp, \delta_B, \mathfrak{F})$, *some* ε**-free** DPDA *such that*

$$\ulcorner \mathcal{A} \urcorner \equiv_W \ulcorner \mathcal{B} \urcorner$$

Proof of proposition 32: We simply set:

$$\forall q \in K \quad \forall a \in \Sigma \quad \forall Z \in \Gamma_\perp \qquad \begin{array}{rcl} \delta_B(q,a,Z) & = & \delta_A(q,a,Z) \text{ if defined} \\ & = & \delta_A(q,\varepsilon,Z) \text{ otherwise.} \end{array}$$

- We prove first that $\ulcorner \mathcal{A} \urcorner \leq_W \ulcorner \mathcal{B} \urcorner$ holds. For any word $x \in \Sigma^\omega$, there exists some **unique** word $x_\xi \in (\{\xi\}^* \cdot \Sigma)^\omega$ such that x_ξ is the **exact reading** of x by $\mathcal{A}$ when every occurrence of ξ is interpreted as an ε-move. Moreover, since $\mathcal{A}$ has the Continuity Property, the mapping $\phi: \ x \longmapsto x_\xi$ from Σ^ω into $(\{\xi\} \cup \Sigma)^\omega$ is continuous. It is then easy to describe a w.s. for *II* in the Wadge game $\mathbf{W}(\ulcorner A \urcorner, \ulcorner B \urcorner)$ which consists in replacing *I*'s run x by x_ξ (this is done continuously) and play exactly the same ω-word as x_ξ once every occurrence of the letter ξ is replaced by any letter (say 0 for instance).
- To prove that $\ulcorner \mathcal{B} \urcorner \leq_W \ulcorner \mathcal{A} \urcorner$ holds, we consider the following strategy for *II* in the Wadge game $\mathbf{W}(\ulcorner B \urcorner, \ulcorner A \urcorner)$: we assume *I* was in position $u \in \Sigma^*$ and as just played the letter $a \in \Sigma$, so that he is now in position $u \cdot a$; and *II* is in position v and has the choice either to play some letter in Σ or to skip. We assume $\mathcal{B}[u] = (q, \gamma.Z)$. There are two cases:
  - if $\delta_A(q,a,Z)$ is defined, then *II* plays the letter "A",
  - if $\delta_A(q,a,Z)$ is not defined, then *II* skips.

The Continuity Property of $\mathcal{A}$ guarantees that *II* will not skip indefinitely.

$\dashv$ 32

From now on, we assume, unless differently mentioned, that all *DPDA* we work with are ε-free. Also, if $\ulcorner \mathcal{A} \urcorner \subseteq \Sigma^\omega$ is self dual then there exists some $u \in \Sigma^*$ such that the set

$$\ulcorner \mathcal{A} \urcorner_{(u)} = \{x \in \Sigma^\omega : \quad u \cdot x \in \ulcorner \mathcal{A} \urcorner\}$$

is non self dual and $^\pm(\ulcorner \mathcal{A} \urcorner_{(u)}) \equiv_W \ulcorner \mathcal{A} \urcorner$; meaning that $\ulcorner \mathcal{A} \urcorner_{(u)}$ is the $\leq_W$-greatest (up to $\equiv_W$ and complementation) non self dual set below $\ulcorner \mathcal{A} \urcorner$. We already saw this and know that it is true for any Borel set and doesn't have anything special to do with context-free ω-languages. We notice that, by definition, $\ulcorner \mathcal{A} \urcorner_{(u)}$ is precisely the set recognized by the Deterministic PushDown Automaton $\mathcal{A}$ starting in configuration $\mathcal{A}[u]$. That is $\ulcorner \mathcal{A}[u] \urcorner = \ulcorner \mathcal{A} \urcorner_{(u)}$.

This last remark shows that in order to prove theorem 31 we just need to concentrate on non self dual *DPDA* and prove for any *DPDA* $\mathcal{A}$ that

$$\ulcorner \mathcal{A} \urcorner \text{ is n.s.d.} \quad \Longleftrightarrow \quad \exists \varrho \in \{-,+\} \quad \exists \alpha < \omega^{(\omega^2)} \qquad \ulcorner \mathcal{A} \urcorner \equiv_W {}^{\varrho}\ulcorner \mathbb{A}(\alpha) \urcorner$$

because the remark on non self dual *DPDA* implies that self dual *DPDA* are necessarily of the form $^\pm\mathbb{A}(\alpha)$ $(0 < \alpha < \omega^{(\omega^2)})$. Indeed, if some self dual *DPDA* would not be of this form then one non self dual *DPDA* would not be of the form $\mathbb{A}(\alpha)$ or $^-\mathbb{A}(\alpha)$.

Our proof needs two steps. First we prove that if $\ulcorner \mathcal{A} \urcorner \equiv_W E \bullet \nu$ holds for some initializable set E, then $\nu < \omega^\omega$ holds too. Secondly we prove that if $\ulcorner \mathcal{A} \urcorner \equiv_W (E^{\natural_n} \bullet \nu_n + \ldots + E^{\natural_1} \bullet \nu_1 + E \bullet \nu)$

holds for some initializable E then it implies that ν is strictly below ω^ω. Combining these two results gives a proof of theorem 31.

The first proof requires a preliminary result about product of initializable sets, that is why the first part of next section is devoted to strict descriptive set theory.

10 Product of initializable sets

We introduce the notion of **combination of Wadge games**. In the sequel we will talk about Wadge games with n players: I, II, III, *etc...* ($n > 1$). This stands for a game where each player plays infinitely many moves. Player I is not allowed to skip while other players are; but all players must produce infinite sequences. Each player's alphabet is easily understood by the sets each of them is in charge of. Such games are just like extended Wadge games where each player produces an infinite word. By saying that player P, being in charge of A_p, reduces player P', who is in charge of $A_{p'}$, we mean that P applies a w.s. against player P' such that at the end of a run if P has played x_p and P' has played $x_{p'}$ then $x_p \in A_p$ iff $x_{p'} \in A_{p'}$.

Lemma 33 *Let $A \subseteq \Sigma_A^\omega$ be an initializable set; λ, θ be some countable ordinals and $B \subseteq \Sigma_B^\omega$,*

$$A \bullet (\theta+1) \leq_W B \leq_W A \bullet \lambda \implies \exists u \in \Sigma_B^* \begin{cases} B_{(u)} \equiv_W A \bullet (\theta+1) \\ \quad\quad or \\ B_{(u)} \equiv_W {}^-(A \bullet (\theta+1)) \end{cases}$$

Proof of lemma 33: The case $\theta + 1 = \lambda$ is obvious since $u = \varepsilon$ works. So in the sequel we assume $\theta + 1 < \lambda$. The proof goes by induction on λ.

Assume λ is limit : there are two different cases.

(a) If $B \equiv_W A \bullet \lambda$, then the set

$$\{d_W^\circ B_{(u)} : u \in \Sigma_B^* \text{ and } B_{(u)} <_W B\}$$

is unbounded in $d_W^\circ(A \bullet \lambda) = d_W^\circ(A) \cdot \lambda$ which is a limit ordinal of countable cofinality. To see that last point, assume towards a contradiction that

$$\{d_W^\circ B_{(u)} : u \in \Sigma_B^* \text{ and } B_{(u)} <_W B\}$$

is bounded in $d_W^\circ(A) \cdot \lambda$. Consider the tree

$$T = \{u \in \Sigma_B^* : \ B_{(u)} \equiv_W B\}$$

and the set

$$S = \{x \in \Sigma_B^\omega : \ \exists y \in \Sigma_B^\omega \ \ \exists n \in \mathbb{N} \ \ (x \restriction n) \in T \ \wedge \ (x \restriction (n+1)) \notin T \ \wedge \ (x \restriction n) \cdot y \in [T]\}$$

where $[T] = \{x \in \Sigma_B^\omega : \ \forall n \in \mathbb{N} \ \ (x \restriction n) \in T\}$ is the set of infinite branches of T.
Set $C = (B \cap [T]) \ \cup \ S$. Clearly C is initializable and $C \leq_W B$ holds. It is easy to see that both $A \leq_W C$ and ${}^-A \leq_W C$ are true. Therefore $A \bullet \mu \leq_W C$ holds for any countable ordinal μ. Thus $A \bullet \mu \leq_W B$ is verified for any countable μ; a contradiction.

Hence there exists some ordinal $\xi < \lambda$ and some $v \in \Sigma_B^*$ that both satisfy

$$A \bullet (\theta+1) \leq_W B_{(v)} \leq_W A \bullet \xi <_W A \bullet \lambda.$$

Then by induction hypothesis one gets some $u' \in \Sigma_B^*$ such that

$$\begin{cases} {B_{(v)}}_{(u')} \equiv_W A \bullet (\theta+1) \text{ or} \\ {B_{(v)}}_{(u')} \equiv_W {}^{-}(A \bullet (\theta+1)). \end{cases}$$

hence $u = v \cdot u'$ works.

(b) If $B <_W A \bullet \lambda$, then $d_W^\circ(B) < d_W^\circ(A \bullet \lambda) = d_W^\circ(A) \cdot \lambda$. So, for some $\xi < \lambda$,

$$A \bullet (\theta+1) \leq_W B \leq_W A \bullet \xi$$

holds; which gives the result using the induction hypothesis on ξ.

Assume $\boldsymbol{\lambda}$ is successor :

(a) **Assume $\boldsymbol{\lambda = \zeta + 2}$**:

(i) **Assume $\boldsymbol{{}^{-}(A \bullet (\zeta+1)) \leq_W B \leq_W A \bullet (\zeta+2)}$**:
consider the following combination of Wadge games with 3 players : I, II and III:

- I is in charge of the set $(A \bullet \zeta) + {}^{-}A$, which is Wadge equivalent to ${}^{-}(A \bullet (\zeta+1))$.
- II is in charge of B and
- III is in charge of $A \bullet (\zeta+2)$

II applies a w.s. that reduces I and III applies a w.s. that reduces II. This means that if I plays x_1, II plays x_2 and III plays x_3 then

$$II \text{ reduces } I \text{ if } x_1 \in (A \bullet \zeta) + {}^{-}A \longleftrightarrow x_2 \in B$$

and

$$III \text{ reduces } II \text{ if } x_3 \in A \bullet (\zeta+2) \longleftrightarrow x_2 \in B.$$

Assume now that player I remaining in the right tail ${}^{-}A$ of $(A \bullet \zeta) + {}^{-}A$ (i.e. without going into the part $(A \bullet \zeta)$ or ${}^{-}(A \bullet \zeta)$, or in other words, without playing a letter not in Σ_A) copies III's run as long as III stays in the tail part A of $(A \bullet (\zeta+1)) + A$. Necessarily after a finite number of moves player III exits the right most A and chooses ${}^{-}(A \bullet (\zeta+1))$ - the other choice $A \bullet (\zeta+1)$ would be a losing one. Let v be the position of player II at that point, one has:

$${}^{-}(A \bullet (\zeta+1)) \leq_W B_{(v)} \leq_W {}^{-}(A \bullet (\zeta+1))$$

therefore

$$B_{(v)} \equiv_W {}^{-}(A \bullet (\zeta+1)).$$

If $\theta + 1 = \zeta + 1$ holds, then we are done. Otherwise we have

$${}^{-}(A \bullet (\theta+1)) \leq_W {}^{-}B_{(v)} \leq_W (A \bullet (\zeta+1))$$

By induction hypothesis there exists u extending v such that

$${}^{-}B_{(u)} \equiv_W {}^{\varrho}(A \bullet (\theta+1)) \text{ which is the same as } B_{(u)} \equiv_W {}^{-\varrho}(A \bullet (\theta+1))$$

holds for some ϱ belonging to $\{+,-\}$.

(ii) **Assume $\boldsymbol{B \leq_W A \bullet (\zeta+1)}$**:
the result relies on the induction hypothesis since $\zeta + 1 < \lambda$ holds together with

$$(A \bullet (\theta+1)) \leq_W B \leq_W (A \bullet (\zeta+1)).$$

(b) **Assume $\lambda = \zeta + 1$, with ζ limit**:

If $B \leq_W A \bullet \zeta$ holds, then the result follows from the induction hypothesis. Thus we assume that

$$^{-}(A \bullet \zeta) \leq_W B \leq_W ((A \bullet \zeta) + A)$$

Play the following combination of Wadge games with 3 players where:

- I is in charge of the set $(((A \bullet \theta) + A) + A) + {}^{-}A)$, which is Wadge equivalent to $^{-}(A \bullet (\theta + 3))$.
- II is in charge of B and
- III is in charge of $(A \bullet \zeta) + A$

II applies a w.s. that reduces I and III applies a w.s. that reduces II. As long as III's run remains in the tail part A of $(A \bullet \zeta) + A$, player I, while remaining in the tail part ^{-}A of $(((A \bullet \theta) + A) + A) + {}^{-}A)$ applies a winning strategy for player I in $\mathbf{W}\,({}^{-}A\ ,\ A)$; forcing this way player III to exit the tail part of $(A \bullet \zeta) + A$. Necessarily after a finite number of moves player II exits the first A and chooses $^{-}(A \bullet \zeta)$ or $(A \bullet \zeta)$. Since A is non self dual, there exists some $x \in \Sigma_A^\omega$ such that for all integer n, $A \equiv_W A_{(x \restriction n)}$ holds. We pick one such x and depending on if it belongs to A or not:

- if $x \in A$, player I chooses to go into $^{-}((A \bullet \zeta + A) + A)$ if III chooses $^{-}(A \bullet \zeta)$ and into $((A \bullet \zeta + A) + A)$ if III chooses $(A \bullet \zeta)$.
- If $x \notin A$, player I chooses to go into $((A \bullet \zeta + A) + A)$ if III chooses $^{-}(A \bullet \zeta)$ and into $^{-}((A \bullet \zeta + A) + A)$ if III chooses $(A \bullet \zeta)$.

Then I plays x until III chooses to "enter" $A \bullet \gamma$ for some $\gamma < \zeta$ (recall that by definition $A \bullet \zeta = \mathbf{sup}_{\gamma < \zeta} A \bullet \gamma$). After III makes that choice, II is in a position v that satisfies for some $\varrho \in \{+, -\}$:

$$^{\varrho}(A \bullet (\theta + 2)) \leq_W B_{(v)} \leq_W (A \bullet \gamma)$$

thus

$$A \bullet (\theta + 1) \leq_W B_{(v)} \leq_W (A \bullet \gamma)$$

The fact that $\gamma < \lambda$ allows to apply the induction hypothesis and gives the result.

$\dashv$ 33

After this little ride inside Descriptive Set Theory, we come back to our machines. Given any *DPDA*, we focus on the language that it accepts when it starts in a certain state different from the initial one and never encounters the bottom symbol at the bottom of its stack.

Definition 34 *Assume $\mathcal{A} = (K, \Sigma,\ \Gamma_\perp,\ q_{init},\ \ \perp,\ \delta,\ \mathfrak{F})$ is some deterministic pushdown automaton. Given $q \in K$ and $Z \in \Gamma_\perp$ let $\mathcal{A}_q^Z$ define the following automaton:*

- *if $Z \neq \perp$:*
  - *$\mathcal{A}_q^Z = (K \cup \{q_q^Z\}, \Sigma,\ \Gamma_\perp,\ q_q^Z,\ \ \perp,\ \delta_q^Z,\ \mathfrak{F})$. with q_q^Z a new state not in K and δ_q^Z been defined by:*

    (a) $\forall q \in K \quad \forall a \in \Sigma \quad \forall Z \in \Gamma_\perp \qquad \begin{cases} \delta_q^Z(q, a, Z) = \delta(q, a, Z), \text{ if defined} \\ \delta_q^Z(q, \varepsilon, Z) = \delta(q, \varepsilon, Z), \text{ otherwise} \end{cases}$

    (b) $\delta_q^Z(q_q^Z, \varepsilon, \perp) = (q, \perp \cdot Z)$.

Defining all other alternatives is useless since they never happen.
Thus one may also look at $\mathcal{A}_q^Z$ as exactly the same DPDA *as $\mathcal{A}$ except that the initial state is now q and the automaton starts with the two letter word $\perp \cdot Z$ written in the stack instead of the simple $\perp$.*

- *We also define $\ddot{\ulcorner}\mathcal{A}_q^Z\ddot{\urcorner}$ as the subset of Σ^ω which is accepted by $\mathcal{A}_q^Z$* **without ever reading $\perp$ at the bottom of the stack***:*

$$\ddot{\ulcorner}\mathcal{A}_q^Z\ddot{\urcorner} = \{x \in \ulcorner\mathcal{A}_q^Z\urcorner : \ \forall q' \in K \ \forall n \in N \setminus \{0\} \ \ \mathcal{A}_q^Z[x \upharpoonright n] \neq (q', \perp)\}$$

◦ *if $Z = \perp$:*

- *$\mathcal{A}_q^\perp = (K, \Sigma, \ \Gamma_\perp, \ q, \ \ \perp, \ \delta, \ \mathfrak{F})$. In this case the only difference between $\mathcal{A}$ and $\mathcal{A}_q^\perp$ is that the initial state q_{init} has been turned into q.*
- *Here $\ddot{\ulcorner}\mathcal{A}_q^\perp\ddot{\urcorner}$ denotes simply the subset of Σ^ω which is accepted by $\mathcal{A}_q^\perp$. There is no mention here of reading or not reading the bottom symbol: $\ddot{\ulcorner}\mathcal{A}_q^\perp\ddot{\urcorner} = \ulcorner\mathcal{A}_q^\perp\urcorner$.*

We use the definition of the $\mathcal{A}_q^Z$ to prove next lemma which is exactly what is required to show that any *DPDA* $\mathcal{A}$ that satisfies $\ulcorner\mathcal{A}\urcorner \equiv_W E \bullet \nu$ where E is initializable, also verifies $\nu < \omega^\omega$.

Lemma 35 *Given any* DPDA $\mathcal{A} = (K, \Sigma, \ \Gamma_\perp, \ q_{init}, \ \ \perp, \ \delta, \ \mathfrak{F})$,
Assume*:*

(a) $B \subseteq \Sigma_B^\omega$ satisfies $B \leq_W \ulcorner\mathcal{A}\urcorner$ and the Wadge degree of B is an ordinal of cofinality ω.

(b) For each $Z \in \Gamma_\perp$, the set $C_Z \subseteq \Sigma_c^\omega$ verifies

$$(C_Z <_W B) \quad \textit{and} \quad \forall q \in K \quad (\ddot{\ulcorner}\mathcal{A}_q^Z\ddot{\urcorner} <_W B \Longrightarrow \ddot{\ulcorner}\mathcal{A}_q^Z\ddot{\urcorner} \leq_W C_Z)$$

For instance any set C_Z that verifies

$$d_W^\circ C_Z = \sup\left(\{d_W^\circ \ddot{\ulcorner}\mathcal{A}_q^Z\ddot{\urcorner} : \ q \in K \ \wedge \ \ddot{\ulcorner}\mathcal{A}_q^Z\ddot{\urcorner} <_W B\} \cup \{1\}\right)$$

works.

(c) $u \in \Sigma^*$ satisfies $\ulcorner\mathcal{A}\urcorner_{(u)} <_W B$ and $\mathcal{A}[u] = (q, \perp \cdot Z_0 \cdot \cdots \cdot Z_{n-1})$, i.e. after reading u the automaton is in state q with the word $\perp \cdot Z_0 \ldots Z_{n-1}$ filling up its stack.

then

$$\ulcorner\mathcal{A}\urcorner_{(u)} \leq_W C_\perp + C_{Z_0} + \ldots + C_{Z_{n-1}}$$

Proof of lemma 35: The proof goes by induction on n.

$\boldsymbol{n = 0}$: there is just $\perp$ written in the stack. Necessarily the state q that has been reached verifies $\ddot{\ulcorner}\mathcal{A}_q^\perp\ddot{\urcorner} <_W B$, therefore $\ddot{\ulcorner}\mathcal{A}_q^\perp\ddot{\urcorner} \leq_W C_\perp$ holds and the result relies on:

$$\ulcorner\mathcal{A}\urcorner_{(u)} \equiv_W \ddot{\ulcorner}\mathcal{A}_q^\perp\ddot{\urcorner} \leq_W C_\perp$$

$n > 0$: remark first that $\ddot{\ulcorner}\mathcal{A}_q^{Z_{n-1}}\ddot{\urcorner} \leq_W \ulcorner\mathcal{A}\urcorner_{(u)} <_W B$ holds which implies that $\ddot{\ulcorner}\mathcal{A}_q^{Z_{n-1}}\ddot{\urcorner} \leq_W C_{Z_{n-1}}$. Secondly we need to prove that player *II* has a w.s. in the Wadge game where *I* is in charge of $\ulcorner\mathcal{A}\urcorner_{(u)}$ and *II* is in charge of $C_\perp + C_{Z_0} + \ldots + C_{Z_{n-1}}$). Clearly *II* has a w.s. against *I* as long as the automaton $\mathcal{A}_q^{Z_{n-1}}$ reading *I*'s run does not read the bottom symbol on the stack. If it happens that it reads it, then let $v \in \Sigma^*$ satisfy

$$\mathcal{A}_q^{Z_{n-1}}[v] = (q', \perp \cdot Z_0 \cdot \cdots \cdot Z_{n-2}) \text{ and } \forall i < lh(v) \;\; \forall w \in \Gamma_\perp{}^* \quad \mathcal{A}_q^{Z_{n-1}}[v \restriction i] \neq (q', v)$$

It immediately implies that

$$\mathcal{A}[u \cdot v] = (q', \perp \cdot Z_0 \ldots Z_{n-2}).$$

Now, by induction hypothesis, one has

$$\ulcorner\mathcal{A}\urcorner_{(u \cdot v)} \leq_W (C_\perp + C_{Z_0} + \ldots + C_{Z_{n-2}})$$

which gives player *II* a winning strategy against player *I*.

$\dashv$ 35

Remark 36 With same hypothesis as in lemma 35:
if $\lambda = \sup\{d_W^\circ C_Z : \; Z \in \Gamma_\perp\}$, then $d_W^\circ \ulcorner\mathcal{A}\urcorner_{(u)} \;\leq\; \lambda \cdot (n+1)$. Where the integer $n+1$ denotes the height of the stack after $\mathcal{A}$ reads the word u. Thus in particular $d_W^\circ \ulcorner\mathcal{A}\urcorner_{(u)} \;<\; \lambda \cdot \omega$.

Proof of remark 36: By lemma 35, the following holds:

$$d_W^\circ \ulcorner\mathcal{A}\urcorner_{(u)} \;\leq\; d_W^\circ(((C_\perp + C_{Z_0}) + \ldots) + C_{Z_{n-1}}) \;=\; (d_W^\circ C_\perp) + (d_W^\circ C_{Z_0}) + \ldots + d_W^\circ(C_{Z_{n-1}})$$

which leads to

$$d_W^\circ \ulcorner\mathcal{A}\urcorner_{(u)} \;\leq\; \lambda \cdot (n+1)$$

$\dashv$ 36

Corollary 37 Assume $\mathcal{A}$ is some *DPDA* and $B \subseteq \Sigma_B^\omega$ is a set that satisfies both $B \leq_W \ulcorner\mathcal{A}\urcorner$ and $cof(d_W^\circ B) = \omega$ then

$$\exists\, \lambda < d_W^\circ B \;\; \forall u \in \Sigma^* \;\; \big(\ulcorner\mathcal{A}\urcorner_{(u)} <_W B \;\; \longrightarrow \;\; d_W^\circ \ulcorner\mathcal{A}\urcorner_{(u)} < \lambda \cdot \omega\big)$$

Proof of corollary 37: Straightforward from lemma 35 and remark 36, once noticed that the ordinal λ satisfies $\lambda < d_W^\circ B$ since $d_W^\circ B$ is some limit ordinal of cofinality ω and that each $d_W^\circ C_Z$ is strictly less than $d_W^\circ B$ by condition (b) of lemma 35.

$\dashv$ 37

Theorem 38 *Let $\mathcal{A}$ be some* DPDA, *$E \subseteq \Sigma_E^\omega$ be an initializable set and $\nu > 0$ some countable ordinal,*

$$\ulcorner\mathcal{A}\urcorner \equiv_W (E \bullet \nu) \quad \Longrightarrow \quad \nu < \omega^\omega$$

Proof of theorem 38: Towards a contradiction we assume $\nu \geq \omega^\omega$ holds and we apply corollary 37:

- set $B = E \bullet \omega^\omega$, notice that B satisfies both conditions required by corollary 37.

∘ By corollary 37 there exists some $\lambda < d_W^\circ B$ such that for any $u \in \Sigma^*$

$$\ulcorner\mathcal{A}\urcorner_{(u)} <_W B \implies d_W^\circ \ulcorner\mathcal{A}\urcorner_{(u)} < \lambda \cdot \omega$$

But $d_W^\circ B = (d_W^\circ E) \cdot \omega^\omega$, so

$$\lambda < d_W^\circ B \Longrightarrow \lambda < d_W^\circ(E) \cdot \omega^\omega \Longrightarrow \exists n \in \omega \quad \lambda < d_W^\circ(E) \cdot \omega^n$$

Hence $\lambda \cdot \omega < d_W^\circ(E) \cdot \omega^{n+1}$ holds. By lemma 33 there exists a word $u \in \Sigma^*$ and $\varrho \in \{-,+\}$ such that $\ulcorner\mathcal{A}\urcorner_{(u)} \equiv_W {}^\varrho(E \bullet (\omega^{n+1}+1))$. All this leads to

$$\lambda \cdot \omega < (d_W^\circ E) \cdot \omega^{n+1} < (d_W^\circ E) \cdot (\omega^{n+1}+1) = d_W^\circ \ulcorner\mathcal{A}\urcorner_{(u)} < \lambda \cdot \omega$$

which comes to $\lambda \cdot \omega > \lambda \cdot \omega$, a contradiction. ⊣ 38

11 Sum of Product of initializable sets

In the last section we showed that if a *DPDA* is Wadge equivalent to a set of the form $E \bullet \nu$ where E is initializable, then ν must be strictly less than ω^ω. In that section we generalize this fact to sums of sets of this form. That is: sets like $E_n \bullet \nu_n + \ldots + E_0 \bullet \nu_0$, each E_i being initializable and $E_0 <_W \ldots <_W E_n$.

Here also we need a set theoretical result which has nothing to do with machines and *DPDA*. It is almost a copy of lemma 33.
By convention given any sets A, B, the notation $A+(B \bullet 0)$ simply stands for the set A. So, in general, assuming that ν_k is not zero and $\langle \nu_{i_j}/j \leq n \rangle$ is the greatest subsequence of $\langle \nu_i/i \leq k \rangle$ such that each ν_{i_j} is different from 0, the set $B_k \bullet \nu_k + \ldots + B_0 \bullet \nu_0 = \sum_{i=k}^{0} B_i \bullet \nu_i$ denotes exactly the set $\sum_{j=n}^{0} B_{i_j} \bullet \nu_{i_j}$.

Lemma 39 *Let $A \subseteq \Sigma_A^\omega$ be an initializable set, λ, θ be countable ordinals;*
$D = ((A^{\natural_n} \bullet \nu_n)+ \ldots)+(A^{\natural_1} \bullet \nu_1)$, $n > 0$ with all parameters ν_i $(0 < i \leq n)$ being countable, not all of them being 0. Let also $B \subseteq \Sigma_B^\omega$,

$$D+A \bullet (\theta+1) \leq_W B \leq_W D+A \bullet \lambda \implies \exists u \in \Sigma_B^* \begin{cases} B_{(u)} \equiv_W D+A \bullet (\theta+1) \\ \text{or} \\ B_{(u)} \equiv_W {}^-(D+A \bullet (\theta+1)) \end{cases}$$

Proof of lemma 39: The case $\theta + 1 = \lambda$ is obvious since $u = \varepsilon$ works. So in the sequel we assume $\theta + 1 < \lambda$. The proof goes by induction on λ.

Assume λ is limit : there are two different cases.

(a) If $B \equiv_W D+A \bullet \lambda$, then, by the same kind of argument as the one developed in lemma 33 we clearly see that the set

$$\{d_W^\circ B_{(u)} : u \in \Sigma_B^* \text{ and } B_{(u)} <_W B\}$$

is unbounded in $d_W^\circ(D+A \bullet \lambda) = d_W^\circ(D) + (d_W^\circ(A) \cdot \lambda)$. Hence there exists some ordinal $\xi < \lambda$ and some $v \in \Sigma_B^*$ that both satisfy

$$D+A \bullet (\theta+1) \leq_W B_{(v)} \leq_W D+A \bullet \xi <_W D+A \bullet \lambda$$

Then by induction hypothesis one gets some $u' \in \Sigma_B^*$ such that

$$\begin{cases} B_{(v)_{(u')}} \equiv_W D{+}A \bullet (\theta+1) \\ \text{or} \\ B_{(v)_{(u')}} \equiv_W {}^-(D{+}A \bullet (\theta+1)) \end{cases}$$

Hence $u = v \cdot u'$ works.

(b) If $B <_W D{+}A \bullet \lambda$, then

$$d_W^\circ(B) < d_W^\circ(D{+}A \bullet \lambda) = d_W^\circ(D) + d_W^\circ(A) \cdot \lambda.$$

So, for some $\xi < \lambda$, the following holds

$$D{+}A \bullet (\theta+1) \leq_W B \leq_W D{+}A \bullet \xi$$

which gives the result once the induction hypothesis on ξ is taken into account.

Assume $\boldsymbol{\lambda}$ is successor :

(a) **Assume $\boldsymbol{\lambda = \zeta + 2}$**:

(i) **assume $\boldsymbol{{}^-(D{+}A \bullet (\zeta+1)) \leq_W B \leq_W D{+}A \bullet (\zeta+2)}$**:
consider the following combination of Wadge games with 3 players : I, II and III:
- I is in charge of $D{+}A{\bullet}\zeta{+}{}^-A$ - which is Wadge equivalent to ${}^-(D{+}A{\bullet}(\zeta{+}1))$.
- II is in charge of B and
- III is in charge of $D{+}A \bullet (\zeta+2)$.

II applies a w.s. that reduces I and III applies a w.s. that reduces II. Assume now that player I remaining in the right tail ${}^-A$ of $D{+}A \bullet \zeta{+}{}^-A$ (i.e. without ever playing a letter which is not in Σ_A) copies III's run as long as III stays in the tail part A of $(D{+}A \bullet (\zeta+1)){+}A$. Necessarily, after a finite number of moves, player III exits the right most A and chooses ${}^-(D{+}A \bullet (\zeta+1))$ - the other choice $D{+}A \bullet (\zeta+1)$ would be a losing one. Let v be the position of player II at that point, one has:

$${}^-(D{+}A \bullet (\zeta+1)) \leq_W B_{(v)} \leq_W {}^-(D{+}A \bullet (\zeta+1))$$

therefore

$$B_{(v)} \equiv_W {}^-(D{+}A \bullet (\zeta+1))$$

If $\theta+1 = \zeta+1$, then we are done. Otherwise we have

$${}^-(D{+}A \bullet (\theta+1)) \leq_W {}^-B_{(v)} \leq_W D{+}A \bullet (\zeta+1)$$

By induction hypothesis there exists u extending v such that

$${}^-B_{(u)} \equiv_W {}^\varrho(D{+}A \bullet (\theta+1)) \text{ i.e. } B_{(u)} \equiv_W {}^{-\varrho}(D{+}A \bullet (\theta+1))$$

for some ϱ belonging to $\{+,-\}$.

(ii) **assume $\boldsymbol{B \leq_W D{+}A \bullet (\zeta+1)}$**:
the result relies on the induction hypothesis since $\zeta+1 < \lambda$ holds together with

$$D{+}A \bullet (\theta+1) \leq_W B \leq_W D{+}A \bullet (\zeta+1).$$

(b) **Assume $\lambda = \zeta + 1$, ζ limit**:

If $B \leq_W D{+}A \bullet \zeta$ holds, then the result follows from the induction hypothesis. Thus we assume that

$$^{-}(D{+}A \bullet \zeta) \leq_W B \leq_W (D{+}A \bullet \zeta){+}A$$

Play the following combination of Wadge games with 3 players where:

- I is in charge of $D{+}(A{\bullet}\theta){+}A{+}A{+}^{-}A)$ - which is Wadge equivalent to $^{-}(D{+}A \bullet (\theta+3))$.
- $I\!I$ is in charge of B and
- $I\!I\!I$ is in charge of $D{+}A \bullet \zeta{+}A)$

$I\!I$ applies a w.s. that reduces I, and $I\!I\!I$ applies a w.s. that reduces $I\!I$. Player I copies $I\!I\!I$'s run as long as it remains in the tail part A of $D{+}(A \bullet \zeta){+}A$. Necessarily after a finite number of moves player $I\!I$ exits the first A and chooses $^{-}(D{+}A \bullet \zeta)$ or $D{+}A \bullet \zeta$. Since A is non self dual, there exists some $x \in \Sigma_A^\omega$ such that for all integer n, $A \equiv_W A_{(x \restriction n)}$ holds. We pick one such x and:

- if $x \in A$, player I chooses to go into $^{-}(D{+}A{\bullet}\zeta{+}A{+}A)$ if $I\!I\!I$ chooses $^{-}(D{+}A{\bullet}\zeta)$ and to go into $D{+}A \bullet \zeta{+}A{+}A$ if $I\!I\!I$ chooses $D{+}A \bullet \zeta$.
- If $x \notin A$, player I chooses to go into $D{+}A \bullet \zeta{+}A{+}A$ if $I\!I\!I$ chooses $^{-}(D{+}A \bullet \zeta)$ and to go into $^{-}(D{+}A \bullet \zeta{+}A{+}A)$ if $I\!I\!I$ chooses $D{+}A \bullet \zeta$.

Then I plays x until $I\!I\!I$ chooses to "enter" $D{+}A \bullet \gamma$ for some $\gamma < \zeta$ (recall that by definition $D{+}A \bullet \zeta = D{+}(\mathbf{sup}_{\gamma<\zeta} A \bullet \gamma) \equiv_W \mathbf{sup}_{\gamma<\zeta} D{+}A \bullet \gamma$). After $I\!I\!I$ makes that choice, $I\!I$ is in a position v that satisfies:

$$^{\varrho}(D{+}A \bullet (\theta+2)) \leq_W B_{(v)} \leq_W D{+}A \bullet \gamma$$

thus

$$D{+}A \bullet (\theta+1) \leq_W B_{(v)} \leq_W D{+}A \bullet \gamma$$

The fact that $\gamma < \lambda$ allows to apply the induction hypothesis and gives the result.

$\dashv$ 39

We come back to our machines and define a sort of analog, for the issue of this section, of what the *DPDA* $\mathcal{A}_q^Z$ was used for in the problem of last section.

Definition 40 *Let $\mathcal{A} = (K, \Sigma,\ \Gamma_\perp,\ q_{init}, \perp,\ \delta,\ \mathfrak{F})$ be some* DPDA.

(a) Given $Z \in \Gamma$ and $q \in K$, we define T_q^Z as the subtree of Σ^* made of all sequences u that verify: $\mathcal{A}_q^Z$ while reading u never encounters the bottom symbol at the bottom of its stack. Formally:

$$T_q^Z = \{u \in \Sigma^* :\ \forall i < lh(u)\ \forall q' \in K\ \mathcal{A}_q^Z[u \restriction i] \neq (q', \perp)\}.$$

(b) Let $\langle C_p / p \in K\rangle$ be any sequence of subsets of Σ^ω, Z be any letter in Γ (by definition Z is different from $\perp$) and q' be any state from K, we define $\ulcorner\!\!\ulcorner \mathcal{A}_{q'}^Z \langle C_p/p \in K\rangle \urcorner\!\!\urcorner$ as the following set:

$$\ulcorner\!\!\ulcorner \mathcal{A}_{q'}^Z \urcorner\!\!\urcorner \cup \left\{u \cdot x \in \Sigma^\omega :\ u \in T_{q'}^Z \wedge u \cdot (x \restriction 1) \notin T_{q'}^Z \wedge \mathcal{A}_{q'}^Z[u \cdot (x \restriction 1)] = (p, \perp) \wedge x \in C_p\right\}$$

(c) *Let w be any* **word** *belonging to Γ^* (by definition w does not contain any bottom symbol) and q be any state from K, define $\mathcal{A}_q^{\perp \cdot w}$ as the automaton that only differs from $\mathcal{A}$ by the fact it starts in state q with $\perp \cdot w$ written in the stack. Formally, $\mathcal{A}_q^{\perp \cdot w}$ stands for the following automaton: $\mathcal{A}_q^{\perp \cdot w} = (K \cup \{q'\}, \Sigma,\ \Gamma_\perp,\ q',\ \ \perp,\ \delta',\ \mathfrak{F})$, with q' a new state not in K and δ' been defined by:*

$$\text{(i)}\quad \forall q \in K \quad \forall a \in \Sigma \quad \forall Z \in \Gamma_\perp \qquad \begin{cases} \delta'(q,a,Z) = \delta(q,a,Z), \text{ if defined} \\ \delta'(q,\varepsilon,Z) = \delta(q,\varepsilon,Z), \text{ otherwise} \end{cases}$$

$$\text{(ii)}\quad \delta'(q',\varepsilon,\perp) = (q, \perp \cdot w).$$

Since all other alternatives never happen, it is useless to define them.

As usual $\ulcorner A_q^{\perp \cdot w} \urcorner$ stands for the set of ω-words accepted by the DPDA $\mathcal{A}_q^{\perp \cdot w}$.

In other words, the set $\ddot{\ulcorner} \mathcal{A}_{q'}^Z \langle C_p / p \in K \rangle \ddot{\urcorner}$ consists in all sequences that are accepted by the automaton $\mathcal{A}_{q'}^Z$ without ever reading the bottom sign at the bottom of the stack, plus all words that start with a finite word that involves a reading of $\perp$ by $\mathcal{A}_{q'}^Z$ and from that precise point ends by an infinite word belonging to C_p where p was the state in which $\perp$ was read.

From a playful point of view, a player who is in charge of $\ddot{\ulcorner} \mathcal{A}_{q'}^Z \langle C_p / p \in K \rangle \ddot{\urcorner}$ is like starting the game being in charge of $\ddot{\ulcorner} \mathcal{A}_{q'}^Z \ddot{\urcorner}$ (remind that this means that the winning set is made of all sequences read by the automaton $\mathcal{A}_{q'}^Z$ without ever reading $\perp$ on the stack) but he can also decide (if possible) to reach a position where, for the first time, it reads the bottom symbol: at that point the whole automaton is in configuration $(p, \gamma \cdot \perp)$ for some state $p \in K$ and $\gamma \in {\Gamma_\perp}^*$, everything then goes like the game is starting again, the player being in charge of C_p instead of $\ddot{\ulcorner} \mathcal{A}_{q'}^Z \ddot{\urcorner}$, and stays in charge of C_p until the end of the run.
In the sequel we define for any given countable ordinal a finite set of ordinals below it. Then we use that set of ordinals to determine, from a given canonical set, a finite set of canonical sets that all reduce to it.

Definition 41 *Assume $0 < \nu < \omega_1$, the Cantor Normal Form of base ω of ν is*

$$\nu = \sum_{i=k}^{0} \omega^{\nu_i} \cdot n_i \quad (= \omega^{\nu_k} \cdot n_k + \ldots + \omega^{\nu_1} \cdot n_1 + \omega^{\nu_0} \cdot n_0)$$

with $\omega_1 > \nu_k > \ldots > \nu_0$ and each n_i strictly positive integers

We define $\mathcal{O}(\nu)$ as a **finite** *set of ordinals, all of them being $\leq \nu$:*

$$\mathcal{O}(\nu) = \bigcup_{0<j\leq k} \left\{ \left(\sum_{i=k}^{j} \omega^{\nu_i} \cdot n_i \right) + \omega^{\nu_{(j-1)}} \cdot m :\ 0 \leq m \leq n_{(j-1)} \right\} \cup \{\omega^{\nu_k} \cdot m :\ 0 \leq m \leq n_k\} \cup \{1\}$$

Definition 42
Let $B_0 <_W \ldots <_W B_k$ be initializable sets and $0 < \nu_i < \omega_1$ (any $0 \leq i \leq k$), set

$$B = B_k \bullet \nu_k + \ldots + B_0 \bullet \nu_0.$$

We define $\mathfrak{S}(B)$ a **finite** *set of sets (that all reduce to ${}^{\pm}B$) as the closure of the following set $\mathfrak{s}(B)$ by operations ${}^{-}$, ${}^{+}$ and ${}^{\pm}$:*

$$\mathfrak{s}(B) = \bigcup_{0<j\leq k} \left\{ \left(\sum_{i=k}^{j} B_i \bullet \nu_i \right) + B_{(j-1)} \bullet \mu : \ \mu \in \mathcal{O}(\nu_{(j-1)}) \right\} \cup \{B_k \bullet \mu : \ \mu \in \mathcal{O}(\nu_k)\} \cup \{\emptyset\}$$

i.e. $C \in \mathfrak{S}(B) \iff \exists D \in \mathfrak{s}(B)$ *and* $\varrho \in \{{}^{-}, {}^{+}, {}^{\pm}\}$ *such that* $C = {}^{\varrho}D$

Definition 43
Let E be some initializable set, k an integer and for each $i < k$, B_i stands for $E^{\natural_{i+1}}$, we consider the following set

$$\underbrace{B_k \bullet \nu_k + \ldots + B_0 \bullet \nu_0}_{B} + E \bullet \nu$$

where $\nu_k \neq 0$, $0 \leq \nu_i < \omega_1$ (each $i < k$) and $0 < \nu < \omega_1$. We set $B = B_k \bullet \nu_k + \ldots + B_0 \bullet \nu_0$.

Given any set $C \subseteq \Sigma_C^{\omega}$ such that $C \leq_W B + E \bullet \nu$, we assign a set Υ_C satisfying both

$$\Upsilon_C \leq_W C \qquad \textit{and} \qquad \Upsilon_C \in \mathfrak{S}(B)$$

(a) If $C \geq_W {}^{\pm}B$ then $\Upsilon_C = {}^{\pm}B$

(b) If $C \equiv_W {}^{-}B$ then $\Upsilon_C = {}^{-}B$

(c) If $C \equiv_W B$ then $\Upsilon_C = B$

(d) If $C <_W B_k$ then $\Upsilon_C = \emptyset$

(e) If $C \geq_W B_k$ or ${}^{-}B_k$ and $C <_W B$ then

$$C \equiv_W {}^{\varrho}(B_k \bullet \mu_k + \ldots + B_0 \bullet \mu_0 + F) \text{ for some } F <_W B_0 \text{ and } \varrho \in \{{}^{-}, {}^{+}, {}^{\pm}\}$$

Necessarily $\mu_i < \nu_i$ holds otherwise $C <_W B$ would fail. Now we get the greatest i such that $\mu_i \neq \nu_i$. Two different cases appear:

(i) If $\mu_i = 0$, then

(1) if $C \equiv_W B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1}$ then

$$\Upsilon_C = B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1}$$

(2) if $C \equiv_W {}^{-}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1})$ then

$$\Upsilon_C = {}^{-}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1})$$

(3) if $C \geq_W {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1})$ then

$$\Upsilon_C = {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1})$$

(ii) If $\mu_i \neq 0$, then consider the Cantor Normal Forms of base ω of μ_i and ν_i:

$$\nu_i = \sum_{j=l}^{0} \omega^j \cdot n_j = \omega^l \cdot n_l + \ldots + \omega^0 \cdot n_0 \quad \textit{and} \quad \mu_i = \sum_{j=l}^{0} \omega^j \cdot m_j = \omega^l \cdot m_l + \ldots + \omega^0 \cdot m_0$$

with $n_l \neq 0$, the other n_j's being integers and all the μ_j's integers not all of them $= 0$. We get the greatest j such that $n_j \neq m_j$ (necessarily $n_j > m_j$ holds) and we set:

$$\theta = \omega^l \cdot n_l + \ldots + \omega^{j+1} \cdot n_{j+1} + \omega^j \cdot m_j$$

We remark that θ belongs to $\mathcal{O}(\nu_i)$. We may also well say that another way of defining θ is to describe it as the greatest ordinal in $\mathcal{O}(\nu_i)$ that is less or equal than μ_i. Finally we set:

(1) if

$$C \equiv_W B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1} + B_i \bullet \theta$$

then

$$\Upsilon_C = B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1} + B_i \bullet \theta$$

(2) if

$$C \equiv_W {}^{-}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1} + B_i \bullet \theta)$$

then

$$\Upsilon_C = {}^{-}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1} + B_i \bullet \theta)$$

(3) if

$$C \geq_W {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1} + B_i \bullet \theta)$$

then

$$\Upsilon_C = {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1} + B_i \bullet \theta)$$

By previous remark, θ belongs to $\mathcal{O}(\nu_i)$ therefore Υ_C belongs to $\mathfrak{S}(B)$

All preceding definitions find their justifications in next lemma which is the core of the main result of this section.

Lemma 44 *Let $\mathcal{A} = (K, \Sigma,\ \Gamma_\perp,\ q_{init}, \perp,\ \delta,\ \mathfrak{F})$ be some* DPDA*, $K = \{q_0, \ldots, q_h\}$, $\langle C_j / j \leq h\rangle$ be a sequence of subsets of Σ^ω. Assume there exists an integer n, an initializable set E, and for all $i < k$, there exists ordinals $0 \leq \nu_i < \omega_1$ with $\nu_k \neq 0$, such that, for all j, the following holds:*

$$C_j \leq_W \underbrace{\left(\sum_{i=k}^{0} B_i \bullet \nu_i\right)}_{B} + E \bullet \omega^n$$

where B_i stands for $E^{\natural_{i+1}}$ $(i < k)$.

We recall that the set Υ_{C_j} (introduced in definition 43) belongs to $\mathfrak{S}(B)$.
Assume also that given $Z \in \Gamma$ and $q \in K$

$$B + E \bullet (\omega^{n+1} + 1) \geq_W \ddot{\ulcorner}\mathcal{A}_q^Z \langle C_j / j \leq h\rangle \ddot{\urcorner} \geq_W B + E \bullet \omega^{n+1} \quad or \quad {}^{-}(B + E \bullet \omega^{n+1})$$

Then

$$\ddot{\ulcorner}\mathcal{A}_q^Z \langle C_j / j \leq h\rangle \ddot{\urcorner} \equiv_W \ddot{\ulcorner}\mathcal{A}_q^Z \langle \Upsilon_{C_j} / j \leq h\rangle \ddot{\urcorner}$$

Proof of lemma 44: We play both underlying Wadge games.

To show that $\ddot{\ulcorner}\mathcal{A}_q^Z \langle C_j / j \leq h\rangle \ddot{\urcorner} \geq_W \ddot{\ulcorner}\mathcal{A}_q^Z \langle \Upsilon_{C_j} / j \leq h\rangle \ddot{\urcorner}$ is obvious, since by the very definition of Υ_{C_j}, for any j, $\Upsilon_{C_j} \leq_W C_j$ holds.
So we concentrate on proving that $\ddot{\ulcorner}\mathcal{A}_q^Z \langle C_j / j \leq h\rangle \ddot{\urcorner} \leq_W \ddot{\ulcorner}\mathcal{A}_q^Z \langle \Upsilon_{C_j} / j \leq h\rangle \ddot{\urcorner}$. The result follows from

Claim 45 $\forall g < h+1$

$$\ddot{\ulcorner}\mathcal{A}_q^Z \langle C_j/0 \leq j \leq h\rangle \ddot{\urcorner} \quad \leq_W \quad \ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g\right\rangle ^\frown \langle C_j/g \leq j \leq h\rangle \ddot{\urcorner}$$

Proof of claim 45: The proof goes by induction on g. The case $g = 0$ is straightforward since both sets are the same. So we assume the result holds for $g < h+1$ and we show that it still holds for $g+1$. So the hypothesis is

$$\ddot{\ulcorner}\mathcal{A}_q^Z \langle C_j/0 \leq j \leq h\rangle \ddot{\urcorner} \quad \leq_W \quad \ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g\right\rangle ^\frown \langle C_j/g \leq j \leq h\rangle \ddot{\urcorner}$$

and we need to prove

$$\ddot{\ulcorner}\mathcal{A}_q^Z \langle C_j/0 \leq j \leq h\rangle \ddot{\urcorner} \quad \leq_W \quad \ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j \leq g\right\rangle ^\frown \langle C_j/g < j \leq h\rangle \ddot{\urcorner}$$

which will be done by proving:

$$\ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g\right\rangle ^\frown \langle C_j/g \leq j \leq h\rangle \ddot{\urcorner} \quad \leq_W \quad \ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j \leq g\right\rangle ^\frown \langle C_j/g < j \leq h\rangle \ddot{\urcorner}$$

We first remark that the case $\Upsilon_{C_g} = C_g$ is obvious since both sets

$$\ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g\right\rangle ^\frown \langle C_j/g \leq j \leq h\rangle \ddot{\urcorner} \text{ and } \ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j \leq g\right\rangle ^\frown \langle C_j/g < j \leq h\rangle \ddot{\urcorner}$$

are the same. So we concentrate on cases where $\Upsilon_{C_g} \neq C_g$, that is when $\Upsilon_{C_g} <_W C_g$ holds. There are 4 such cases:

(a) $C_g >_W {}^{\pm}B$ and $\Upsilon_{C_g} = {}^{\pm}B$.

(b) $C_g <_W B_k$ and $\Upsilon_{C_g} = \emptyset$.

(c) $C_g >_W B_k$ and $C_g <_W B$; i.e.

$$C_g \equiv_W {}^{\varrho}(B_k \bullet \mu_k + \ldots + B_0 \bullet \mu_0) \text{ for some } \varrho \in \{^-,\ ^+,\ ^\pm\}$$

or

$$C_g \equiv_W {}^{\varrho}(B_k \bullet \mu_k + \ldots + B_0 \bullet \mu_0 + F) \text{ for some } F <_W B_0 \text{ and } \varrho \in \{^-,\ ^+,\ ^\pm\}$$

In each case, get the greatest i such that $\mu_i \neq \nu_i$. Necessarily $\mu_i < \nu_i$ holds otherwise $C_g <_W B$ would fail. We distinguish between two different cases:

(i) $\mu_i = 0$:

$$C_g >_W {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1})$$

and

$$\Upsilon_{C_g} = {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1})$$

(ii) $\mu_i \neq 0$:

$$C_g >_W {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1} + B_i \bullet \theta)$$

and

$$\Upsilon_{C_g} = {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1} + B_i \bullet \theta)$$

where

$$\theta = \omega^l \cdot n_l + \ldots + \omega^{j+1} \cdot n_{j+1} + \omega^j \cdot m_j$$

With j being the greatest integer such that $n_j \neq m_j$ in the comparison of the following Cantor Normal Forms of base ω of ν_i and μ_i:

$$\nu_i = \omega^l \cdot n_l + \ldots + \omega^0 \cdot n_0 \quad \text{and} \quad \mu_i = \omega^l \cdot m_l + \ldots + \omega^0 \cdot m_0$$

We consider each and everyone of these cases and show that in any of them the following holds:

$$\ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g\right\rangle ^\frown \langle C_j/g \leq j \leq h\rangle \ddot{\urcorner} \quad \leq_W \quad \ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j \leq g\right\rangle ^\frown \langle C_j/g < j \leq h\rangle \ddot{\urcorner}.$$

(a) $\boldsymbol{C_g >_W {}^{\pm}B}$and $\boldsymbol{\Upsilon_{C_g} = {}^{\pm}B}$:

We consider the following combination of Wadge games where five different players interfere:

- Player I is in charge of $\ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g\right\rangle ^\frown \langle C_j/g \leq j \leq h\rangle \ddot{\urcorner}$ and plays anything he wants.
- Player $I\!I$ is in charge of a **canonical set** ($\mathfrak{CS}$) which is Wadge equivalent to

  $$\ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g\right\rangle ^\frown \langle C_j/g \leq j \leq h\rangle \ddot{\urcorner}$$

  and applies a w.s. against I. This means he reduces I's play.
  The canonical set ($\mathfrak{CS}$) mentioned above takes one of the following forms:
  (a) $\mathfrak{CS} = {}^{\varrho}\left((\sum_{i=k}^{0} B_i \bullet \nu_i + E \bullet \omega^{n+1} \right)$ for some $\varrho \in \{-,+,\pm\}$.
  (b) $\mathfrak{CS} = {}^{\varrho}\left(\sum_{i=k}^{0} B_i \bullet \nu_i + E \bullet \omega^{n+1} + G \right)$ for some $\varrho \in \{-,+,\pm\}$ and $G \leq_W E$.
  The reason why we want that player to be in charge of a canonical set is because looking at this player's run makes it much easier to understand what's going on in I's play. Things that are clear and obvious when playing with a canonical set (as for example to precisely know what the power of the actual position is) may be hidden in a set that does not have that canonical feature.
- Player $I\!I\!I$ is in charge of the same canonical set $\mathfrak{CS}$ and copies $I\!I$'s run except when mentioned
- Player $I\!V$ is in charge of $\ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g\right\rangle ^\frown \langle C_j/g \leq j \leq h\rangle \ddot{\urcorner}$ and reduces player $I\!I\!I$'s run (he applies a w.s. against $I\!I\!I$).
- Player V is in charge of $\ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g+1\right\rangle ^\frown \langle C_j/g+1 \leq j \leq h\rangle \ddot{\urcorner}$ and reduces $I\!V$'s run.

Player $I\!I\!I$, as $I\!I$ doesn't enter in the part $E \bullet \omega^{n+1}$ of $\mathfrak{CS}$ (i.e. as long as the positions u reached by $I\!I$ satisfy

$$\mathfrak{CS}_{(u)} \leq_W (B + E \bullet \omega^{n+1}) \text{ or } {}^{-}(B + E \bullet \omega^{n+1})$$

$I\!I\!I$ just copies $I\!I$'s run. If $I\!I$ reaches some position u such that

$$\mathfrak{CS}_{(u)} <_W B + E \bullet \omega^{n+1},$$

then for some $m \in \omega$:

$$\mathfrak{CS}_{(u)} \leq_W B + E \bullet (\omega^n \cdot m).$$

So $I\!I\!I$ reduces $I\!I$ always "keeping an $E \bullet \omega^n$" on the side" in advance as long as $I\!I$ does not reach a position u' that satisfies $\mathfrak{CS}_{(u')} \leq_W {}^{\pm}B$. Properly this means that if $I\!I$ has reached some position v such that

$${}^{-}(B + E \bullet \xi) \text{ or } B + E \bullet \xi \leq_W \mathfrak{CS}_{(v)} <_W B + E \bullet (\xi + 1)$$

for some **minimal** $\xi \leq \omega^n \cdot m$, then *III* must have reached a position v' such that

$$^{-}(B{+}E \bullet (\omega^n + \xi)) \text{ or } B{+}E \bullet \xi \leq_W \mathfrak{CS}_{(v')} <_W B{+}E \bullet (\omega^n + \xi + 1).$$

Of course, if *II* never "enters" $^{\pm}B$, then *III* never "enters" $B{+}E \bullet \omega^n$ nor $^{-}(B{+}E \bullet \omega^n)$, therefore *IV* **never** "uses" C_g", hence V's strategy that consists in copying *IV*'s run is clearly winning.

If *II* enters $^{\pm}B$, then at that precise moment *III* and *IV* are still in some position that reduces $B{+}E \bullet \omega^n$, so player V, who copies *IV*'s run, has not "chosen" yet to be in charge of C_g; which means:

(a) Either V is not playing with the $\ddot{\ulcorner}\mathcal{A}_q^Z\ddot{\urcorner}$ part of his set (i.e. he "got in charge" of some set C_i or Υ_{C_i} - for some $i \neq g$). In that case V is in a position that easily reduces *II*'s actual run because *II* is in some position $\leq_W {}^{\pm}B$. So V has clearly a winning strategy to reduce *II* which means reducing I.

(b) Or V is still playing with the $\ddot{\ulcorner}\mathcal{A}_q^Z\ddot{\urcorner}$: say V is in a position u such that for all u' initial segment of u, there exists some non empty word w' and $q \in K$ such that $\mathcal{A}_q^Z[u'] = (q, \perp \cdot w')$ holds. Then

   (i) Either there exists some minimal v extending u such that $\mathcal{A}_q^Z[v] = (q_g, \perp)$ with any strict initial segment of v' of v never satisfying $\mathcal{A}_q^Z[v] = (q, \perp)$ for any $q \in K$. The strategy for V consists then in reaching that position v so that everything goes now as if he would be starting a new run being in charge of Υ_{C_g}. But since $\Upsilon_{C_g} = {}^{\pm}B$, V has now a w.s. to beat *II* who is in a position $\leq_W {}^{\pm}B$. This is clearly winning.

   (ii) Or there is no such v extending u. In that case we just demand *III* to reduce *II* without any special requirement, *IV* to reduce *III* and V to copy *IV*'s run. This strategy is also clearly winning since *IV* will **never** "choose" to be in charge of C_g. So V will also never have to chose to be in charge of Υ_{C_g}.

(b) $\boldsymbol{C_g <_W B_k}$ and $\boldsymbol{\Upsilon_{C_g} = \emptyset}$:

This case is easier than the previous one. We consider the following combination of Wadge games with four different players:

- Player I is in charge of $\ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g \right\rangle ^\frown \langle C_j/g \leq j \leq h \rangle \ddot{\urcorner}$ and plays anything he wants.
- Player *II* is in charge of a **canonical set ($\mathfrak{CS}$)** Wadge equivalent to

$$\ddot{\ulcorner}\mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g \right\rangle ^\frown \langle C_j/g \leq j \leq h \rangle \ddot{\urcorner}$$

  and applies a w.s. against I:
  Here also the canonical set ($\mathfrak{CS}$) takes one of the following form:

  (a) $\mathfrak{CS} = {}^{\varrho}\left(\sum_{i=k}^{0} B_i \bullet \nu_i {+} E \bullet \omega^{n+1}\right)$ for some $\varrho \in \{-, +, \pm\}$.

  (b) $\mathfrak{CS} = {}^{\varrho}\left(\sum_{i=k}^{0} B_i \bullet \nu_i {+} E \bullet \omega^{n+1} {+} G\right)$ for some $\varrho \in \{-, +, \pm\}$ and $G \leq_W E$

- Player *III* is in charge of $\ulcorner\!\!\!\ddot{\ }\mathcal{A}_q^Z\left\langle \Upsilon_{C_j}/j<g\right\rangle^\frown\langle C_j/g\leq j\leq h\rangle\ddot{\urcorner}$ and reduces player *II*'s run (he applies a w.s. against *II*).
- Player *IV* is in charge of $\ddot{\ulcorner}\mathcal{A}_q^Z\left\langle \Upsilon_{C_j}/j<g+1\right\rangle^\frown\langle C_j/g+1\leq j\leq h\rangle\ddot{\urcorner}$ and reduces *III* by **always** copying *III*'s run: we will show that this is a winning strategy.

We remark that B_k being initializable, if I comes to some position u such that everything goes as if I is now in charge of C_g, then *II* is necessarily in some position u' such that $\mathfrak{CS}_{(u')}\geq_W B_k$ or ${}^-B_k$. Therefore *III* **cannot** have "chosen" to be in charge of C_g, because otherwise this would be losing. So *III* never "chooses" C_g, hence *IV* never "chooses" Υ_{C_g}. So the strategy is clearly winning.

(c) $\boldsymbol{C_g\geq_W B_k}$ **and** $\boldsymbol{C<_W B}$: one has

$$C_g\equiv_W {}^{\varrho}(B_k\bullet\mu_k+\ldots+B_0\bullet\mu_0)\text{ for some }\varrho\in\{^-,\ ^+,\ ^\pm\}$$

or

$$C_g\equiv_W {}^{\varrho}(B_k\bullet\mu_k+\ldots+B_0\bullet\mu_0+F)\text{ for some }F<_W B_0\text{ and }\varrho\in\{^-,\ ^+,\ ^\pm\}$$

because otherwise $C_g<_W B$ would fail. We distinguish two different cases:

(c)(i) $\boldsymbol{\mu_i=0}$ where i is the greatest integer such that $\mu_i\neq\nu_i$. So we have

$$C_g>_W {}^{\pm}(B_k\bullet\mu_k+\ldots+B_{i+1}\bullet\mu_{i+1})$$

and

$$\Upsilon_{C_g}={}^{\pm}(B_k\bullet\mu_k+\ldots+B_{i+1}\bullet\mu_{i+1})$$

Here also we consider a combination of Wadge games that involves four different players.

- Player I is in charge of $\ddot{\ulcorner}\mathcal{A}_q^Z\left\langle \Upsilon_{C_j}/j<g\right\rangle^\frown\langle C_j/g\leq j\leq h\rangle\ddot{\urcorner}$ and plays anything he wants.
- Player *II* is in charge of a **canonical set** **($\mathfrak{CS}$)** Wadge equivalent to

$$\ddot{\ulcorner}\mathcal{A}_q^Z\left\langle \Upsilon_{C_j}/j<g\right\rangle^\frown\langle C_j/g\leq j\leq h\rangle\ddot{\urcorner}$$

  and applies a w.s. against I:
  Here also the canonical set ($\mathfrak{CS}$) takes one of the following form:

  (a) $\mathfrak{CS}={}^{\varrho}\left(\sum_{i=k}^{0}B_i\bullet\nu_i+E\bullet\omega^{n+1}\right)$ for some $\varrho\in\{-,+,\pm\}$.

  (b) $\mathfrak{CS}={}^{\varrho}\left(\sum_{i=k}^{0}B_i\bullet\nu_i+E\bullet\omega^{n+1}+G\right)$ for some $\varrho\in\{-,+,\pm\}$ and $G\leq_W E$
- Player *III* is in charge of $\ddot{\ulcorner}\mathcal{A}_q^Z\left\langle \Upsilon_{C_j}/j<g\right\rangle^\frown\langle C_j/g\leq j\leq h\rangle\ddot{\urcorner}$ and reduces player *II*'s run (he applies a w.s. against *II*).
- Player *IV* is in charge of $\ddot{\ulcorner}\mathcal{A}_q^Z\left\langle \Upsilon_{C_j}/j<g+1\right\rangle^\frown\langle C_j/g+1\leq j\leq h\rangle\ddot{\urcorner}$ and copies *III*'s run except when mentioned differently.

As long as the positions u reached by *II* satisfy:

$$\mathfrak{CS}_{(u)} \geq_W {}^{-}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1} + B_i) \text{ or } (B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1} + B_i)$$

then everything is just fine because *III* cannot have "chosen" to be in charge of C_g. Otherwise it would be losing since B_i is initializable and

$$C_g <_W (B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1} + B_i).$$

Now if *II* reaches a position u such that

$$\mathfrak{CS}_{(u)} \not\geq_W {}^{-}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1} + B_i)$$

or

$$\mathfrak{CS}_{(u)} \not\geq_W (B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1} + B_i)$$

is verified, then the following holds:

$$\mathfrak{CS}_{(u)} \leq_W {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1}) = \Upsilon_{C_g}.$$

We notice that *III* always being one move behind *II* is necessarily still in some position that reduces

$${}^{-}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1} + B_i) \text{ or } (B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1} + B_i).$$

In particular he cannot have "chosen" to be in charge of C_g because the following holds

$$(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \mu_{i+1} + B_i) >_W C_g.$$

Three alternatives remain:

(a) Either *III* is already in a position where he has "chosen" to be in charge of some C_i or Υ_{C_i} (for some $i \neq g$); in which case the strategy for that consists in keeping on copying *III*'s run until the end, is winning for *IV*.

(b) Or *III* has not "chosen" to be in charge of some C_i or Υ_{C_i} (for some $i \neq g$) and *IV*'s actual position can be extended into some position v such that $\mathcal{A}_q^Z[v] = (q_g, \perp)$ with no $t < lh(v)$ and no $q \in K$ verifying $\mathcal{A}_q^Z[v \restriction t] = (q, \perp)$. Then *IV* reaches that position so that everything goes as if he was starting a new game being in charge of Υ_{C_g} which allows him to reduce *II* and is clearly winning.

(c) Or *III* has not "chosen" to be in charge of some C_i or Υ_{C_i} (for some $i \neq g$) and *IV*'s position cannot be extended to some position v such that $\mathcal{A}_q^Z[v] = (q_g, \perp)$ with no no $q \in K$ verifying $\mathcal{A}_q^Z[v \restriction t] = (q, \perp)$. Then, here also, *IV*'s strategy consists in copying *III*'s run until the end.

$$\textbf{(c)(ii)}\ \boldsymbol{\mu_i \neq 0}\ : \begin{cases} C_g >_W {}^{\pm}(B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1} + B_i \bullet \theta) \\ \qquad\qquad \text{and} \\ \Upsilon_{C_g} = {}^{\pm}(\underbrace{B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1}}_{B'} + B_i \bullet \theta) \end{cases} \text{ where:}$$

$$\theta = \omega^l \cdot n_l + \ldots + \omega^{j+1} \cdot n_{j+1} + \omega^j \cdot m_j$$

With j being the greatest integer such that $n_j \neq m_j$ in the comparison of the Cantor Normal Forms of base ω of ν_i and μ_i:

$$\nu_i = \omega^l \cdot n_l + \ldots + \omega^0 \cdot n_0 \quad \text{and} \quad \mu_i = \omega^l \cdot m_l + \ldots + \omega^0 \cdot m_0$$

We write B' for $B_k \bullet \mu_k + \ldots + B_{i+1} \bullet \nu_{i+1}$. W remark that $\theta \leq \mu_i < \theta + \omega^j$ holds and here also we consider a combination of Wadge games that involves four different players.

- Player I is in charge of $\ulcorner \mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g \right\rangle ^\frown \langle C_j/g \leq j \leq h \rangle \urcorner$ and plays anything he wants.
- Player $I\!I$ is in charge of a **canonical set** ($\mathfrak{CS}$) Wadge equivalent to

  $$\ulcorner \mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g \right\rangle ^\frown \langle C_j/g \leq j \leq h \rangle \urcorner$$

  and applies a w.s. against I:
  Here also the canonical set ($\mathfrak{CS}$) takes one of the following forms:
  (a) $\mathfrak{CS} = {}^{\varrho}\left(\sum_{i=k}^{0} B_i \bullet \nu_i + E \bullet \omega^{n+1} \right)$ for some $\varrho \in \{-, +, \pm\}$.
  (b) $\mathfrak{CS} = {}^{\varrho}\left(\sum_{i=k}^{0} B_i \bullet \nu_i + E \bullet \omega^{n+1} + G \right)$ for some $\varrho \in \{-, +, \pm\}$ and $G \leq_W E$
- Player $I\!V$ is in charge of $\ulcorner \mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g+1 \right\rangle ^\frown \langle C_j/g+1 \leq j \leq h \rangle \urcorner$ and always copies $I\!I\!I$'s run except when mentioned .
- Player $I\!I\!I$ is in charge of $\ulcorner \mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g \right\rangle ^\frown \langle C_j/g \leq j \leq h \rangle \urcorner$ and reduces player $I\!I$'s run (he applies a w.s. against $I\!I$) with an additional condition.
- Player $I\!V$ is in charge of $\ulcorner \mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g+1 \right\rangle ^\frown \langle C_j/g+1 \leq j \leq h \rangle \urcorner$ and, all along, copies $I\!I\!I$'s run except when mentioned.

(a) **If $\boldsymbol{j = 0}$** then $\theta + \omega^j = \theta + 1$.

$I\!V$ copies $I\!I\!I$'s run as long as $I\!I$'s position u verifies

$$\mathfrak{CS}_{(u)} \geq_W B' + B_i \bullet (\theta + 1) \text{ or } {}^{-}(B' + B_i \bullet (\theta + 1)).$$

We notice that $B' + B_i \bullet (\theta + 1) >_W C_g$ holds, therefore $I\!I\!I$ cannot "choose" C_g. If $I\!I$ reaches some position u such that

$$\mathfrak{CS}_{(u)} \not\geq_W B' + B_i \bullet (\theta + 1) \text{ or } {}^{-}(B' + B_i \bullet (\theta + 1)),$$

then necessarily $\mathfrak{CS}_{(u)} \leq_W {}^{\pm}(B' + B_i \bullet \theta)$ holds. Now same old story:

(i) Either $I\!I\!I$ is already in a position where he has "chosen" to be in charge of some C_i or Υ_{C_i} (for some $i \neq g$); in which case the strategy that consists in keeping on copying $I\!I\!I$'s run until the end, is winning for $I\!V$.

(ii) Or $I\!I\!I$ did not "choose" to be in charge of some C_i or Υ_{C_i} (for some $i \neq g$) and $I\!V$'s actual position can be extended to some position v such that $\mathcal{A}_q^Z[v] = (q_g, \perp)$ with no $t < lh(v)$ and no $q \in K$ verifying $\mathcal{A}_q^Z[v \upharpoonright t] = (q, \perp)$. Then $I\!V$ reaches that position. And from now on $I\!V$ directly reduces $I\!I$'s run.

(iii) Or *III* did not "choose" to be in charge of some C_i or Υ_{C_i} (for some $i \neq g$) and *IV*'s position cannot be extended to some position v such that $\mathcal{A}_q^Z[v] = (q_g, \perp)$ with no $t < lh(v)$ and no $q \in K$ verifying $\mathcal{A}_q^Z[v \upharpoonright t] = (q, \perp)$. Then here also *IV*'s strategy consists in copying *III*'s run until the end.

(b) **If $j > 0$**

We consider a combination of Wadge games with five different players:

- Player I is in charge of $\ulcorner \mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g \right\rangle^\frown \langle C_j/g \leq j \leq h \rangle \urcorner$ and plays anything he wants.
- Player *II* is in charge of a **canonical set ($\mathfrak{CS}$)** Wadge equivalent to

$$\ulcorner \mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g \right\rangle^\frown \langle C_j/g \leq j \leq h \rangle \urcorner$$

and applies a w.s. against I:

The canonical set ($\mathfrak{CS}$) mentioned above takes one of the following form:

(i) $\mathfrak{CS} = {}^{\varrho}\left(\sum_{i=k}^{0} B_i \bullet \nu_i + E \bullet \omega^{n+1}\right)$ for some $\varrho \in \{-, +, \pm\}$.

(ii) $\mathfrak{CS} = {}^{\varrho}\left(\sum_{i=k}^{0} B_i \bullet \nu_i + E \bullet \omega^{n+1} + G\right)$ for some $\varrho \in \{-, +, \pm\}$ and $G \leq_W E$

- Player *III* is in charge of the same canonical set $\mathfrak{CS}$ and copies *II*'s run except when mentioned.
- Player *IV* is in charge of $\ulcorner \mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g \right\rangle^\frown \langle C_j/g \leq j \leq h \rangle \urcorner$ and reduces player *III*'s run (he applies a w.s. against *III*).
- Player V is in charge of $\ulcorner \mathcal{A}_q^Z \left\langle \Upsilon_{C_j}/j < g+1 \right\rangle^\frown \langle C_j/g+1 \leq j \leq h \rangle \urcorner$ and reduces *IV*.

Player *III* copies *II*'s run as long as *II* only reaches positions of the form u that satisfy:

$$\mathfrak{CS}_{(u)} \geq_W B' + B_i \bullet (\theta + \omega^j) \text{ or } {}^-(B' + B_i \bullet (\theta + \omega^j)).$$

If *II* reaches a position u that satisfies

$$\mathfrak{CS}_{(u)} <_W B' + B_i \bullet (\theta + \omega^j) \text{ and } {}^-(B' + B_i \bullet (\theta + \omega^j)),$$

then for some integer t the position u also verifies

$$\mathfrak{CS}_{(u)} \leq_W B' + B_i \bullet (\theta + \omega^{j-1} \cdot t).$$

Then *III* reaches some position u' that satisfies

$$\mathfrak{CS}_{(u')} \leq_W B' + B_i \bullet (\theta + \omega^{j-1} \cdot (t + m_{j-1} + 1)),$$

and as long as *II* does not reach a position v satisfying

$$\mathfrak{CS}_{(v)} \leq_W {}^{\pm}(B' + B_i \bullet \theta)$$

Player *III*, **always keeping $B_i \bullet (\theta + \omega^{j-1} \cdot (m_{j-1} + 1))$ in advance**, reduces player *II*. This means that when *II* is in some position u such that

$$B' + B_i \bullet (\theta + \xi) \leq_W \mathfrak{CS}_{(u)} <_W B' + B_i \bullet (\theta + \xi + 1)$$

for some $\xi < \omega^{j-1} \cdot t$, then *III* is in some position v that satisfies both

$$B' + B_i \bullet (\theta + \omega^{j-1} \cdot (m_{j-1} + 1) + \xi) \leq_W \mathfrak{CS}_{(u)}$$

and

$$\mathfrak{CS}_{(u)} <_W B' + B_i \bullet (\theta + \omega^{j-1} \cdot (m_{j-1} + 1) + \xi + 1).$$

But since $B' + B_i \bullet (\theta + \omega^{j-1} \cdot (m_{j-1} + 1)) >_W C_g$ holds, player *IV* has not "chosen" C_g yet. So

(i) either *IV* already "chose" some other Υ_{C_i} or C_i (for some $i \neq g$); in which case V just needs to copy *IV*'s run until the end.

(ii) Or *IV* did not "choose" any C_i or Υ_{C_i} and there is **no extending** position w such that $\mathcal{A}_q^Z[w] = (q_g, \perp)$ with no $r < lh(w)$ and no $q \in K$ verifying $\mathcal{A}_q^Z[w \restriction r] = (q, \perp)$. In that case also, player V just needs to copy *IV*'s run until the end to be certain to win.

(iii) Or *IV* did not "choose" any C_i or Υ_{C_i} yet and there exists some extending position w such that $\mathcal{A}_q^Z[w] = (q_g, \perp)$ with no $r < lh(w)$ and no $q \in K$ verifying $\mathcal{A}_q^Z[w \restriction r] = (q, \perp)$. In which case V reaches that position so that everything goes now as if he is in charge of Υ_{C_g}, a set that allows him to directly reduce *II*'s run.

⊣45 ⊣44

Everything is now set up to generalize theorem 38.

Theorem 46 *Let* $\mathcal{A} = (K, \Sigma, \Gamma_\perp, q_{init}, \perp, \delta, \mathfrak{F})$ *be some* DPDA, *E be some initializable set; and k be an integer. Assume $0 \leq \nu_i < \omega_1$ $(i \leq k)$ with $0 \neq \nu_k$. Assume also $0 < \nu < \omega_1$ and B_i stands for $E^{\natural_{i+1}}$,*

$$\ulcorner \mathcal{A} \urcorner \equiv_W \Big(\underbrace{(\sum_{i=k}^{0} B_i \bullet \nu_i)}_{B} + (E \bullet \nu) \Big) \implies \nu < \omega^\omega$$

Proof of theorem 46: Let K be the set $\{q_j : j \leq h\}$. Towards a contradiction we assume that $\nu \geq \omega^\omega$ holds.

We define a function $\phi : K \times \Gamma \times \mathfrak{S}(B)^{h+1} \longmapsto \omega$ by :
for each $q \in K$, $Z \in \Gamma$ and $\vec{C} \in \mathfrak{S}(B)^{h+1}$,

- If $\ddot{\ulcorner} \mathcal{A}_q^Z(\vec{C}) \ddot{\urcorner} \geq_W B + E \bullet \omega^\omega$ or $\ddot{\ulcorner} \mathcal{A}_q^Z(\vec{C}) \ddot{\urcorner} \geq_W {}^-(B + E \bullet \omega^\omega)$, then $\phi(q, p, \vec{C}) = 0$.
- If $\ddot{\ulcorner} \mathcal{A}_q^Z(\vec{C}) \ddot{\urcorner} <_W B + E \bullet \omega^\omega$, then $\phi(q, p, \vec{C}) = inf\{n : \ddot{\ulcorner} \mathcal{A}_q^Z(\vec{C}) \ddot{\urcorner} \leq_W B + E \bullet \omega^n\}$.

We also define the function $\psi : K \times \Gamma_\perp \longmapsto \omega$ by

- for any $q \in K$ and $Z \in \Gamma$
  - If $\ulcorner \mathcal{A}_q^Z \urcorner \geq_W B + E \bullet \omega^\omega$ or $\ulcorner \mathcal{A}_q^Z \urcorner \geq_W {}^-(B + E \bullet \omega^\omega)$, then $\psi(q, Z) = 0$.
  - If $\ulcorner \mathcal{A}_q^Z \urcorner <_W B + E \bullet \omega^\omega$, then $\psi(q, Z) = \inf\{n : \ulcorner \mathcal{A}_q^Z \urcorner \leq_W B + E \bullet \omega^n\}$.

- for any $q \in K$
  - If $\ulcorner \mathcal{A}_q^{\perp} \urcorner \geq_W B+E \bullet \omega^\omega$ or ${}^{-}(B+E \bullet \omega^\omega)$ then $\psi(q, \perp) = 0$.
  - If $\ulcorner \mathcal{A}_q^{\perp} \urcorner <_W B+E \bullet \omega^\omega$ then $\psi(q, \perp) = \inf\{n : \ulcorner \mathcal{A}_q^{\perp} \urcorner \leq_W B+E \bullet \omega^n\}$.

We set

$$n = \sup\left(\{\phi(q, p, \vec{C}) : q \in K, Z \in \Gamma, \vec{C} \in \mathfrak{S}(B)^{h+1}\} \cup \{\psi(q, Z) : q \in K, Z \in \Gamma_\perp\}\right).$$

We remark that n is necessarily finite.
By lemma 39 there exists some non empty word $u \in \Sigma^*$ and some $\varrho \in \{^+, ^-\}$ such that

$$\ulcorner \mathcal{A} \urcorner_{(u)} \equiv_W {}^{\varrho}(B+E \bullet (\omega^{n+1} + 1)).$$

Let $\mathcal{A}[u] = (q'', w'')$. We notice that

- $w'' = \perp$ is impossible since by definition of n the following holds:

  $$\ulcorner \mathcal{A}_{q''}^{\perp} \urcorner \leq_W B+E \bullet \omega^n \quad \text{or} \quad \ulcorner \mathcal{A}_{q''}^{\perp} \urcorner \geq_W B+E \bullet \omega^\omega \quad \text{or} \quad \ulcorner \mathcal{A}_{q''}^{\perp} \urcorner \geq_W {}^{-}(B+E \bullet \omega^\omega).$$

  So w'' is of the form $w'' = \perp \cdot w' \cdot p'$ for some $p' \in \Gamma$ and $w' \in \Gamma^*$.

- Since $\ulcorner \mathcal{A}_{q''}^{p'} \urcorner \leq_W B+E \bullet \omega^n$ holds, there exists an initial segment $w \cdot p$ of $w' \cdot p'$ (i.e. $w \cdot p \subseteq w' \cdot p'$) and a state $q \in K$ such that $(q, \perp \cdot w \cdot p)$ is reachable from $(q', \perp \cdot w' \cdot p')$: there exists $v \in \Sigma^*$ s.t. $\mathcal{A}[u \cdot v] = (q, \perp \cdot w \cdot p))$ verifying **both conditions** (a) and (b) below:

  (a) $\ulcorner \mathcal{A}_q^{\perp \cdot w \cdot Z} \urcorner \geq_W B+E \bullet \omega^{n+1} \quad \vee \quad \ulcorner \mathcal{A}_q^{\perp \cdot w \cdot Z} \urcorner \geq_W {}^{-}(B+E \bullet \omega^{n+1})$
  (b) $\forall q' \in K \quad \ulcorner \mathcal{A}_q^{\perp \cdot w} \urcorner <_W B+E \bullet \omega^{n+1} \quad \vee \quad \ulcorner \mathcal{A}_q^{\perp \cdot w} \urcorner >_W \ulcorner \mathcal{A}_q^{\perp \cdot w \cdot Z} \urcorner$.

  In words, each reachable configuration of the machine $(q', \perp \cdot w \cdot p)$ corresponds to some position strictly below $B+E \bullet \omega^{n+1}$.

Since $K = \{q_{init} : i \leq h\}$, for each integer i define:

$$\begin{aligned} C_i &= \ulcorner \mathcal{A}_{q_{init}}^{\perp \cdot w} \urcorner \quad \text{if } \ulcorner \mathcal{A}_{q_{init}}^{\perp \cdot w} \urcorner \leq_W B+E \bullet \omega^{n+1} \\ &= \emptyset \quad \text{otherwise} \end{aligned}$$

Clearly $\ulcorner \mathcal{A}_q^{\perp \cdot w \cdot Z} \urcorner \leq_W \ddot{\ulcorner} \mathcal{A}_q^Z \langle C_i / i \leq h \rangle \ddot{\urcorner}$ holds. For each i, let Υ_{C_i} be the set related to C_i introduced in definition 43.

By construction, we know that the following holds, by choice of n:

$$\ddot{\ulcorner} \mathcal{A}_q^Z \left\langle \Upsilon_{C_i} / i \leq h \right\rangle \ddot{\urcorner} \quad \leq_W \quad B+E \bullet \omega^n.$$

And by lemma 44

$$\ddot{\ulcorner} \mathcal{A}_q^Z \left\langle \Upsilon_{C_i} / i \leq h \right\rangle \ddot{\urcorner} \quad \equiv_W \quad \ddot{\ulcorner} \mathcal{A}_q^Z \langle C_i / i \leq h \rangle \ddot{\urcorner}$$

which leads to the following contradiction:

$$B+E \bullet \omega^{n+1} \text{ or } {}^{-}(B+E \bullet \omega^{n+1}) \quad \leq_W \quad \ulcorner \mathcal{A}_q^{\perp \cdot w \cdot Z} \urcorner \quad \leq_W \quad B+E \bullet \omega^n.$$

$\dashv$ 46

Lemma 47 *Let* $\mathcal{A} = (K, \Sigma, \ \Gamma_\perp, \ q_{init}, \perp, \ \delta, \ \mathfrak{F})$ *be some* DPDA*;* t, k *be two integers with* $k \neq 0$, $0 \leq \nu_i < \omega_1 \quad (i \leq k)$, $\ \nu_k \neq 0$, $0 < \nu < \omega_1$ *and* $\varrho \in \{^+, \ ^-, \ ^\pm\}$,

$$\ulcorner \mathcal{A} \urcorner \equiv_W {}^\varrho\Big(\underbrace{(\sum_{i=k}^{0} \ominus^{t+i} \bullet \nu_i)}_{B} + (\ominus^t \bullet \nu)\Big) \implies \forall i < k \quad \nu_i < \omega^\omega \ \wedge \ \nu < \omega^\omega$$

Proof of lemma 47: Towards a contradiction:

(a) We assume that there exists some i such that ν_i is greater or equal than ω^ω. Let i be the greatest such integer. We get any *DPDA* $\mathcal{A}'$ that verifies $\ulcorner \mathcal{A}' \urcorner \equiv_W \ominus^{t+i}$. Now the *DPDA* $\mathcal{A}'' = \mathcal{A} +_{\mathbb{A}} \mathcal{A}'$ satisfies

$$\ulcorner \mathcal{A}'' \urcorner \ \equiv_W \ {}^\varrho\Big((\sum_{j=k}^{0} \ominus^{t+j} \bullet \nu_j) + (\ominus^t \bullet \nu)\Big) \ + \ \ominus^{t+i} \ \equiv_W \ (\sum_{j=k}^{i+1} \ominus^{t+j} \bullet \nu_j) + (\ominus^{t+i} \bullet (\nu_i + 1))$$

(i) If $i = k$, we are left with $\ulcorner \mathcal{A}'' \urcorner \equiv_W \ominus^{t+k} \bullet (\nu_k + 1)$ where $\nu_k + 1 > \omega^\omega$ holds; which contradicts theorem 38.

(ii) If $i < k$, we are left with $\ulcorner \mathcal{A}'' \urcorner \equiv_W (\sum_{j=k}^{i+1} \ominus^{t+j} \bullet \nu_j) + (\ominus^{t+i} \bullet (\nu_i + 1))$ which contradicts theorem 46.

(b) We assume that $\nu_i < \omega^\omega$ is verified for all i but $\nu \geq \omega^\omega$.

(i) Case $\varrho = +$ contradicts theorem 46.

(ii) In case $\varrho = -$, we get ${}^-\mathcal{A} = (K, \Sigma, \ \Gamma_\perp, \ q_{init}, \perp, \ \delta, \ \mathcal{P}(K) \setminus \mathfrak{F})$ the complement of $\mathcal{A}$ which accepts the very ω-sequences that $\mathcal{A}$ rejects. Same argument as above applied to ${}^-\mathcal{A}$ gives a contradiction.

(iii) In case $\varrho = \pm$, there exists a non empty finite word $u \in \Sigma^*$ such that

$$\ulcorner \mathcal{A} \urcorner_{(u)} \equiv_W B + (\ominus^t \bullet \nu).$$

From $\mathcal{A}[u] = (q, \perp \cdot \gamma)$, one builds the automaton $\mathcal{A}'$ which recognizes $\ulcorner \mathcal{A} \urcorner_{(u)}$ (as the same as $\mathcal{A}$ except that it starts in initial state q with $\perp \cdot \gamma$ written in the stack) but contradicts theorem 46.

$\dashv$ 47

It's now child's play to get the result.

Theorem 48 *Let* $\mathcal{A} = (K, \Sigma, \ \Gamma_\perp, \ q_{init}, \perp, \ \delta, \ \mathfrak{F})$ *be some* DPDA, *there exists an integer* k, *integers* $n_k > \ldots > n_0$, *ordinals* $0 < \nu_i < \omega^\omega$ *and* $\varrho \in \{+, \ -, \ \pm\}$ *such that*

$$\ulcorner \mathcal{A} \urcorner \ \equiv_W \ {}^\varrho\,(\ominus^{n_k} \bullet \nu_k + \ldots + \ominus^{n_0} \bullet \nu_0) = {}^\varrho\left(\sum_{i=k}^{0} \ominus^{n_i} \bullet \nu_i\right)$$

Proof of theorem 48: $\ulcorner \mathcal{A} \urcorner$ is a boolean combination of $\underset{\sim}{\mathbf{\Sigma}}^\circ_2$ sets, or in other words there exists an integer n such that $\ulcorner \mathcal{A} \urcorner \leq_W \emptyset^{\natural n} \equiv_W \ulcorner \ominus^n \urcorner$. By [6], [7], any subset of Σ^ω which satisfies this condition is Wadge equivalent to a set of the form ${}^\varrho\,(\ominus^{n_k} \bullet \xi_n + \ldots + \ominus^{n_0} \bullet \xi_0)$ where each factor ξ_i is countable. (Note that the fact Σ is finite is required to get such a precise result). By previous lemma 47 each factor is strictly less than ω^ω.

$\dashv$ 48

Theorem 48 offers an immediate proof of theorem 31 which stated that for any *DPDA* $\mathcal{A}$ there exists some unique $\alpha < \omega^{(\omega^2)}$ and unique $\varrho \in \{-,+,\pm\}$ such that $\ulcorner\mathcal{A}\urcorner \equiv_W \ {}^{\varrho}\ulcorner\mathbb{A}(\alpha)\urcorner$:

Proof of theorem 31: If $\mathcal{A}$ is some *DPDA* then, by theorem 48, the set $\ulcorner\mathcal{A}\urcorner$ is Wadge equivalent to a set ${}^{\varrho}(\ominus^{n_k}\cdot\nu_k + \ldots + \ominus^{n_0}\cdot\nu_0)$ with $n_k > \ldots > n_0$. But this set is clearly unique (see [6] or [7]) and so is also ϱ. Set $\alpha = (\omega^\omega)^{n_k}\cdot\nu_k + \ldots + (\omega^\omega)^{n_0}\cdot\nu_0$. By definition, $\mathbb{A}(\alpha) \equiv_W (\ominus^{n_k}\cdot\nu_k + \ldots + \ominus^{n_0}\cdot\nu_0)$, therefore $\ulcorner\mathcal{A}\urcorner \equiv_W {}^{\varrho}\ulcorner\mathbb{A}(\alpha)\urcorner = \ulcorner{}^{\varrho}\mathbb{A}(\alpha)\urcorner$.

$\dashv$ 31

12 conclusion

We showed that the Wadge hierarchy of *DPDA* enriches Wagner's hierarchy of *DA*. Its height is $\omega^{(\omega^2)}$ instead of ω^ω because the existence of a stack allows to "count" up to ω. And by repeating this process finitely many times (either by increasing the number of states or the number of the letters from the pushdown alphabet) it climbs up to ω^ω.
We apologize for our proof is rather technical and uneasy. It requires a certain knowledge of the Wadge hierarchy and some familiarity with game theory. Anyways, we believe that there might be a different proof, a more "analytical" one, that would rely on the direct study of the given *DPDA*. O. Finkel makes progress in that direction with the hope that such a proof should be imitating Wagner's one, proceeding by closer and closer encounter to the *DPDA* $\mathcal{A}$: at first by studying the possibilities of intricate possible "loops" this would lead to some integer n such that

$$\ulcorner\ominus^n\urcorner \text{ or } {}^{-}\ulcorner\ominus^n\urcorner \ \leq_W \ulcorner\mathcal{A}\urcorner <_W \ulcorner\ominus^{n+1}\urcorner.$$

The second step would be to determine - also by analyzing the definition of $\mathcal{A}$ - the ordinal ν such that

$$\ulcorner\ominus^n \bullet \nu\urcorner \ \text{ or } \ {}^{-}\ulcorner\ominus^n \bullet \nu\urcorner \ \leq_W \ulcorner\mathcal{A}\urcorner <_W \ulcorner\ominus^n \bullet (\nu+1)\urcorner.$$

And going on and on this way, after finitely many steps one should be able to reach the ordinal α and the variable ϱ such that $\ulcorner\mathcal{A}\urcorner \equiv_W {}^{\varrho}\mathbb{A}(\alpha)$. We even believe that this work could be done by defining another relation on *DPDA* ($\leq_M$), that would be equivalent to $\leq_W$ ($\mathcal{A} \leq_M \mathcal{B} \iff \ulcorner\mathcal{A}\urcorner \leq_W \ulcorner\mathcal{B}\urcorner$) but would rather refer to the definition of the *DPDA* (to its transition function or in simple words its mechanism instead of the language it recognizes).

Nevertheless, we also believe that our work on the Wadge hierarchy by the introduction of the conciliatory hierarchy and the definition of the operations that are the set theoretic counterpart of the ordinal ones, should make possible a study of transfinite machines. For example, we already have got partial results with *DA* reading words of length ω^n. Such machines can recognize sets that are boolean combinations of $\underset{\sim}{\mathbf{\Sigma}}^\circ_{2n}$-sets and generate a hierarchy whose length is the first fixpoint of exponentiation of base ω. Not to mention the works of O. Finkel on Non Deterministic PushDown Automata (see [10] and [12]).
Besides, we also conjecture that the Wadge hierarchy of languages recognized by Deterministic Turing Machines (with the condition of acceptance being to go through a certain set of states infinitely many times) is just the extension of the Wagner hierarchy and the *DPDA* hierarchy where coefficients run through all effective ordinals. That is, $\ulcorner\mathcal{M}\urcorner$ is the set accepted by the

Turing Machine $\mathcal{M}$ iff there exists an integer k, integers $n_k > \ldots > n_0$, ordinals $0 < \nu_i < \omega_1^{ck}$ and $\varrho \in \{+, -, \pm\}$ such that

$$\ulcorner\mathcal{M}\urcorner \equiv_W {}^{\varrho}(\ominus^{n_k} \bullet \nu_k + \ldots + \ominus^{n_0} \bullet \nu_0) = {}^{\varrho}\left(\sum_{i=k}^{0} \ominus^{n_i} \bullet \nu_i\right)$$

Where ω_1^{ck} is the first non recursive ordinal. The height of such a hierarchy would be $(\omega_1^{ck})^{\omega}$.
We want to say a word about a hierarchy finer than the Wadge one: the Lipschitz hierarchy. It consists in exactly the same definition as the one of the Wadge hierarchy except that the notion of continuous function is replaced by the one of Lipschitz function. In terms of games, we say that $A \leq_L B$ holds iff *II* has a winning strategy in the restricted Wadge game where I is in charge of A and *II* is in charge of B and **none of the players has the right to skip**. Both hierarchies have same non self dual (for $\leq_L$) sets. And when dealing with finite alphabets, the only difference between this hierarchy and the Wadge one relies on the fact that instead of having only one self dual set of the form ${}^{\pm}A$ between a non self dual set A and its successor, in the Lipschitz hierarchy there is a whole sequence of ω self dual ones: the $\langle {}^{n\pm}A / n \in \mathbb{N} \rangle$.
Given $\{\Sigma_+, \Sigma_-\}$ a partition of Σ in two non empty sets, $A \subseteq \Sigma^\omega$ a non self dual set; and an integer n, ${}^{n\pm}A$ is defined by:

$${}^{n\pm}A = (\Sigma^n \cdot \Sigma_+ \cdot A) \ \cup \ (\Sigma^n \cdot \Sigma_- \cdot {}^{-}A)$$

So if A and B are both non self dual with $d_W^\circ B = (d_W^\circ A) + 1$, in the Lipschitz hierarchy one has

$$A <_L {}^{0\pm}A <_L {}^{1\pm}A <_L {}^{2\pm}A <_L \ldots <_L {}^{n\pm}A <_L {}^{(n+1)\pm}A <_L \ldots\ldots <_L B$$

and every self dual set between A and B takes the form ${}^{n\pm}A$ (up to Lipschitz equivalence).
Clearly, if A is recognized by some *DPDA* $\mathcal{A}$, then, on the basis of $\mathcal{A}$, it is straightforward to build a *DPDA* ${}^{n\pm}\mathcal{A}$ that recognizes exactly ${}^{n\pm}A$. This settles the problem of the Lipschitz hierarchy for *DPDA*.

At last we want to mention that the Wadge hierarchy of non deterministic finite machines remains an open problem, although progress are made in that direction too (see [10] and [11]).

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