A coarsification of the Wagner Hierarchy

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Abstract

In 1979, Klaus W. Wagner introduced a hierarchy on ω-regular sets that has length ω^ω. It is the one induced by the ordering on Deterministic Automata: $\mathcal{A} \leq_{1c} \mathcal{B}$ iff the set accepted by $\mathcal{A}$ is the inverse image of the set accepted by $\mathcal{B}$ by a continuous function. Lately, Jean-Eric Pin conjectured that there might be a coarser hierarchy with length ω and that it could be given by the same ordering on DA except that continuous is replaced by a slightly weaker notion 2-continuous. We provide a positive answer to this conjecture and give a description of this new hierarchy.

Keywords: ω-rationnal languages, automata, Wagner hierarchy, 2-continuous reduction.

1 Introduction

We recall that ω stands for the set of integers ($\mathbb{N}$). It is also, from the ordinal point of view, the first ordinal that is not finite. And ω_1 denotes the first uncountable ordinal.
Given any set Σ, called the alphabet, we write: Σ^* for the set of finite words from Σ (the set of sequences of finite length of elements from Σ), Σ^+ for the set of non empty finite words from Σ and Σ^ω for the set of ω-words (the set of functions from $\mathbb{N}$ to Σ). The empty word is denoted $\boldsymbol{\varepsilon}$ and the restriction of a word x to its n first letters is denoted $x \restriction n$. Given two words u and v, $u \subseteq v$ stands for "u is an initial segment of v". We write $lh(u)$ for the length of the word u.
The concatenation of words x and y is written $x \cdot y$. Given A some set of finite or infinite words and u any finite word, $u \cdot A$ stands for the set of all words $u \cdot y$ for $y \in A$. More generally, given A and B two sets of words, $A \cdot B$ denotes the sets of all words $x \cdot y$ for $x \in A$ and $y \in B$.
Finally, given any set $B \subseteq (\Sigma^\omega \cup \Sigma^*)$ and any finite word $u \in \Sigma^*$, we write $B_{(u)}$ to denote the set of all finite or infinite words x such that $u \cdot x \in B$.
Since we must deal with different kinds of alphabets, we use the the notation $A \subseteq \Sigma_A^\omega$ and $B \subseteq \Sigma_B^\omega$ to say that A and B a sets of infinite words on respectively the alphabet Σ_A and the alphabet Σ_B.
We also recall that T is a tree on the alphabet Σ if T is a subset of Σ^* that satisfies $(u \subseteq v \ \wedge \ v \in T) \ \longrightarrow \ u \in T$. We write $[T]$ the set of infinite branches of T ($[T] = \{x \in \Sigma^\omega : \ \forall n \in \mathbb{N} \ \ (x \restriction n) \in T\}$. A tree T is a non empty pruned tree if $T \neq \emptyset$ and every node in T has a proper extension in T. The axiom of Dependant Choice (DC) is the weaker version of the Axiom of Choice (AC) that states:

$$T \text{ is a non empty pruned tree } \longrightarrow \ [T] \neq \emptyset.$$

In 1979 Klaus W. Wagner introduced a hierarchy of lenght ω^ω on ω-regular sets, that is sets of ω words recognized by Deterministic Automata (DA) [18]. These ω-languages were crucial in the famous works on the decidability of the monadic second order logic with 1 successor (S1S) by Büchi and Landweber [1] [2].
Instead of providing a refinement of this hierarchy, this paper amazingly rather gives a coarsification of this hierarchy, which from length ω^ω falls down to length ω, the length of the integers.
We recall the definition of a DA reading ω-words:

Definition 1 *$\mathcal{A} = (Q, \Sigma,\ \Gamma_\perp,\ q_{init},\ \delta,\ \mathfrak{F})$ is some Deterministic Automaton if*

- *Q is the set of its states,*
- *Σ is its alphabet (necessarily finite). The case $|\Sigma| = 1$ being straightforward and meaningless, we assume that Σ contains at least two letters, among which are $\{0,1\}$.*
- *q_{init} is the initial state,*
- *δ is the transition function from $Q \times \Sigma$ to Q*
- *$\mathfrak{F} \subseteq \mathcal{P}(Q)$ is the set of accepting conditions (Müller conditions).*

Notation 2

(a) *An ω-word x is accepted by $\mathcal{A}$ iff the set of all the states which the* DA *runs through infinitely many times while reading x belongs to $\mathfrak{F}$. Formally, any infinite word $x = \prod_{n\in\mathbb{N}} a_n \in \Sigma^\omega$ induces an infinite word on the state alphabet Q: $y = \prod_{n\in\mathbb{N}} q_n \in Q^\omega$ such that q_0 is the initial state of $\mathcal{A}$ and $\delta(q_n, a_n) = q_{n+1}$ holds for any integer n.*

The word $x = \prod_{n\in\mathbb{N}} a_n$ is accepted by $\mathcal{A}$ iff $\{q \in Q :\ \forall n \exists m > n \quad q_m = q\}$ belongs to $\mathfrak{F}$.

(b) *We write $\ulcorner\mathcal{A}\urcorner$ for the set of all ω-words from Σ^ω that are accepted by $\mathcal{A}$.*

2 Introduction to the Wadge hierarchy

In the sixties W. W. Wadge [16][17] started to study a hierarchy of subsets of the Baire space (ω^ω =the space of mappings from $\mathbb{N}$ to $\mathbb{N}$) and more generally of subsets of Σ^ω (the space of mappings from $\mathbb{N}$ to Σ) where Σ is some at most countable alphabet. That particular topological space (ω^ω) is essential in Analysis since it is homeomorphic to $\mathbb{R}\setminus\mathbb{Q}$ (isomorphic to the Irrational space with respect to the topological structure). More generally the elements of Σ^ω are the functions from $\mathbb{N}$ to Σ, that is the ω-words on the alphabet Σ. We consider such spaces equipped with the usual topology, that is the product of the discrete topology on Σ. So, basic non empty open sets have the following form:

$$s \cdot \Sigma^\omega \quad \text{for some finite sequence } s \in \Sigma^*.$$

When Σ contains less than 2 letters, the problem of describing the subsets of Σ^ω becomes irrelevant, therefore in the sequel we assume that Σ contains at least two letters, and to simplify the notations, we call them 0 and 1.
Wadge was mainly interested in the class of Borel subsets of Σ^ω. Namely the smallest class that contains all the open sets and is closed under countable union and complementation. Borel sets

stand as the core of modern Analysis. They bear essential properties like measurability, the perfect set property and the Baire Property. As soon as they were introduced, Baire set down the Borel sets into a hierarchy that relies on enumerating the number of operations of countable union and complementation required to obtain them on the basis of open sets. Thus the so called Borel hierarchy looks like:

$$\utilde{\mathbf{\Sigma}}^{\circ}_{1} = \{Open\ sets\}$$

$$\text{(for any ordinal } \nu \geq 1) \quad \utilde{\mathbf{\Pi}}^{\circ}_{\nu} = \left\{ \Sigma^{\omega} \smallsetminus A : \quad A \in \utilde{\mathbf{\Sigma}}^{\circ}_{\nu} \right\}$$

$$\text{(for any ordinal } \nu > 1) \quad \utilde{\mathbf{\Sigma}}^{\circ}_{\nu} = \left\{ \bigcup_{n \in \omega} A_n : \ \forall n \in \omega \ \ A_n \in \bigcup_{\xi < \nu} \utilde{\mathbf{\Pi}}^{\circ}_{\xi} \right\}$$

The class $\utilde{\mathbf{\Pi}}^{\circ}_{1}$ is the collection of closed sets, $\utilde{\mathbf{\Sigma}}^{\circ}_{2}$ the collection of countable unions of closed sets, etc...
This Borel hierarchy looks like a ladder with two classes at each step ($\utilde{\mathbf{\Sigma}}^{\circ}_{\nu}$ and $\utilde{\mathbf{\Pi}}^{\circ}_{\nu}$). Each class contains all classes below it ($\utilde{\mathbf{\Sigma}}^{\circ}_{\nu}$ contains all sets in all $\utilde{\mathbf{\Pi}}^{\circ}_{\xi}$ and $\utilde{\mathbf{\Sigma}}^{\circ}_{\xi}$ for any $\xi < \nu$, same thing for $\utilde{\mathbf{\Pi}}^{\circ}_{\nu}$). And the length of the hierarchy is ω_1, the first uncountable ordinal:

$$\utilde{\mathbf{\Sigma}}^{\circ}_{1} \subsetneq \utilde{\mathbf{\Sigma}}^{\circ}_{2} \subsetneq \ldots \utilde{\mathbf{\Sigma}}^{\circ}_{n} \subsetneq \utilde{\mathbf{\Sigma}}^{\circ}_{n+1} \subsetneq \cdots \subsetneq \utilde{\mathbf{\Sigma}}^{\circ}_{\omega} \subsetneq \utilde{\mathbf{\Sigma}}^{\circ}_{\omega+1} \subsetneq \ldots \utilde{\mathbf{\Sigma}}^{\circ}_{\nu} \subsetneq \utilde{\mathbf{\Sigma}}^{\circ}_{\nu+1} \subsetneq \ldots\ldots$$

$$\utilde{\mathbf{\Pi}}^{\circ}_{1} \subsetneq \utilde{\mathbf{\Pi}}^{\circ}_{2} \subsetneq \ldots \utilde{\mathbf{\Pi}}^{\circ}_{n} \subsetneq \utilde{\mathbf{\Pi}}^{\circ}_{n+1} \subsetneq \cdots \subsetneq \utilde{\mathbf{\Pi}}^{\circ}_{\omega} \subsetneq \utilde{\mathbf{\Pi}}^{\circ}_{\omega+1} \subsetneq \ldots \utilde{\mathbf{\Pi}}^{\circ}_{\nu} \subsetneq \utilde{\mathbf{\Pi}}^{\circ}_{\nu+1} \subsetneq \ldots\ldots$$

Different improvements of the Borel hierarchy were introduced during the century. But by far, the most refined is the one due to Wadge that is based on the very simple topological notion of inverse image by a continuous function.
We recall that $f : \Sigma^{\omega}_{A} \longmapsto \Sigma^{\omega}_{B}$ is continuous if the inverse image of any open set is open. Or in other words, if it commutes on direct limits:

$$f(\lim_{i \in \mathbb{N}} x_i) = \lim_{i \in \mathbb{N}} f(x_i).$$

(where $x = \lim_{i \in \mathbb{N}} x_i$ holds iff $\forall n \in \mathbb{N} \ \ \exists i \in \mathbb{N} \quad \forall j \geq i \qquad x \upharpoonright n = x_j \upharpoonright n$).

Wadge introduced a very natural relation $\leq_{1c}$ between sets $A \subseteq \Sigma^{\omega}_{A}$ and $B \subseteq \Sigma^{\omega}_{B}$:

$$A \leq_{1c} B \iff_{def} \exists \ f : \Sigma^{\omega}_{A} \longmapsto \Sigma^{\omega}_{B} \ \ continuous \ \ s.t. \ \ f^{-1}B = A$$

In other words, x belongs to A iff $f(x)$ belongs to B. Intuitively $A \leq_{1c} B$ means that A is less complicated than B with regard to the topological structure. Whenever $A \leq_{1c} B$ we say that A continuously reduces to B because the problem of determining whether any word x belongs to A reduces to the problem of the continuous image of x belonging to B.
Directly related to the Wadge ordering ($\leq_{1c}$) are the strict ordering ($<_{1c}$) and the Wadge equivalence ($\equiv_{1c}$):

$$A <_{1c} B \iff_{def} A \leq_{1c} B \ \ but \ \ B \not\leq_{1c} A$$

$$A \equiv_{1c} B \iff_{def} A \leq_{1c} B \leq_{1c} A$$

when $A \equiv_{1c} B$ we say that A and B are Wadge equivalent.

For the rest of the section, we give a slight idea of what the Wadge hierarchy looks like, we don't give any proof and we warn the reader not to try to get them directly. They will be much easier to obtain with the tools of game theory that will be presented in next sections.
To give an idea of the Wadge ordering, consider any clopen (closed and open) set $A \subseteq \Sigma^\omega$ that is neither the empty set nor the whole space verifies

- $A \equiv_{1c} u \cdot \Sigma^\omega$ for any non empty finite word $u \in \Sigma^*$.
- $\emptyset <_{1c} A$ and $\Sigma^\omega <_{1c} A$
- $A \equiv_{1c} \Sigma^\omega \smallsetminus A$
- $A <_{1c} \{0^\omega\}$ and $A <_{1c} \Sigma^\omega \smallsetminus \{0^\omega\}$

Let also ${}^-A$ denote the complement of A: (${}^-A = \Sigma^\omega \smallsetminus A$) and given $\{\Sigma_+, \Sigma_-\}$ any partition of Σ into two **non empty** sets, we define $\boldsymbol{{}^\pm A}$ **as the set** $\boldsymbol{(\Sigma_+ \cdot A) \cup (\Sigma_- \cdot {}^-A)}$. There are two radically different kinds of sets: the ones that reduce to their complement (${}^-A \leq_{1c} A$) - for example any clopen set that is neither the empty set nor the whole space satisfies this property) such sets are called **1c-self dual** (they verify ${}^-A \equiv_{1c} A$). And the other ones that do not reduce to their complement and are called **1c-non self dual**. For example, the empty set is 1c-non self dual since no function satisfies $f^{-1}\Sigma^\omega = \emptyset$. Same remark holds with the whole space Σ^ω. Any closed set that is not open (e.g. $\{0^\omega\}$) and also any open but not closed set (e.g. $\Sigma^\omega \smallsetminus \{0^\omega\}$) is 1c-non self dual.
Moreover, if A is 1c-self dual, then $A \equiv_{1c} {}^\pm A$ holds. On the contrary, if A is 1c-non self dual, then $A \equiv_{1c} {}^\pm A$ fails. In that case, one has $A <_{1c} {}^\pm A$ and there is no B such that $A <_{1c} B <_{1c} {}^\pm A$. This means that with regards to the Wadge ordering, ${}^\pm A$ is (up to $\equiv_{1c}$) the least set strictly above A in case A is 1c-non self dual.
In a similar way, in order to catch the immediate "successor" of A in the Wadge hierarchy, in case A is 1c-self dual, we just have to consider the set $(\Sigma \smallsetminus \{0\})^* \cdot 0 \cdot A$. It satisfies $A <_{1c} (\Sigma \smallsetminus \{0\})^* \cdot 0 \cdot A$ and there is no B such that $A <_{1c} B <_{1c} (\Sigma \smallsetminus \{0\})^* \cdot 0 \cdot A$.

Now $\leq_{1c}$ could have been just any weird ordering between sets, but amazingly when restricted to Borel sets it turns out to be a pre-wellordering: it is not a total ordering ($\emptyset$ and ${}^-\emptyset$ are incomparable) but antichains have length at most 2 (there are no three elements incomparable with each other). Much more important is the fact it is wellfounded (there is no infinite descending chain $A_0 >_W A_1 >_W \ldots A_n >_W A_{n+1} >_W \ldots$). To be precise, the class $\bigcup_{\nu<\omega_1} \underset{\sim}{\boldsymbol{\Sigma}}^\circ_\nu$ of all Borel subsets of Σ^ω quotiented by the following equivalence relation

$$A \sim B \iff (A \equiv_{1c} B \ \vee \ A \equiv_{1c} \Sigma^\omega \smallsetminus B)$$

is well ordered by $<_{1c}$. This means is a total ordering on $\bigcup_{\nu<\omega_1} \underset{\sim}{\boldsymbol{\Sigma}}^\circ_\nu / \sim$ with no infinite descending chain. In other words the structure $\left(\bigcup_{\nu<\omega_1} \underset{\sim}{\boldsymbol{\Sigma}}^\circ_\nu / \sim, <_{1c}\right)$ is isomorphic to $(|WH|, <)$ where $|WH|$ is some ordinal that represents the height of the Wadge hierarchy and $<$ is the usual ordering on ordinals (for more details see [4]).

Since $<_{1c}$ is wellfounded on Borel sets, it induces a notion of height on these sets, called the **Wadge degree** and denoted $\boldsymbol{d^\circ_{1c}}$. At the bottom we find $\emptyset$ and ${}^-\emptyset$, since there is no set A such that $A <_{1c} \emptyset$ holds, and there is also no other set than the whole space which is incomparable with the empty set. So we set $\boldsymbol{d^\circ_{1c}(\emptyset) = d^\circ_{1c}({}^-\emptyset) = 0}$ and inductively, for any set A that is neither the empty set nor the whole space, we set $\boldsymbol{d^\circ_{1c}(A) = sup\{d^\circ_{1c}(B) + 1 : B <_{1c} A\}}$. As a quick tour, when Σ is finite, the Wadge ordering on Borel subsets of Σ^ω verifies:

- $d^\circ_{1c}(\emptyset) = d^\circ_{1c}({}^-\emptyset) = 0$,
- $d^\circ_{1c}(A) = 1$ iff A is clopen but different from the empty set and the whole space,
- $d^\circ_{1c}(A) = 2$ iff A is closed but not open (e.g. $\{0^\omega\}$) or if it is open but not closed (e.g. ${}^-\{0^\omega\}$),
- $d^\circ_{1c}(A)$ is odd iff A is 1c-self dual (${}^-A \equiv_{1c} A$),
- $d^\circ_{1c}(A)$ is even iff A is 1c-non self dual (limit ordinals are even),
- A and B are incomparable iff $d^\circ_{1c}(A) = d^\circ_{1c}(B)$ is even and ${}^-A \equiv_{1c} B$,
- $d^\circ_{1c}((\Sigma^* \cdot 0)^\omega) = \omega_1$.

3 2-continuous reduction

A mapping $\phi : \Sigma_A^\omega \longmapsto \Sigma_B^\omega$ is continuous if every pre-image of an open set is open, which we now write $\phi^{-1}A \in \underset{\sim}{\boldsymbol{\Sigma}}^\circ_1$ for any $A \in \underset{\sim}{\boldsymbol{\Sigma}}^\circ_1$. Generalizing this notion, we call **2-continuous** the functions that satisfy $\phi^{-1}A \in \underset{\sim}{\boldsymbol{\Sigma}}^\circ_2$ for any $A \in \underset{\sim}{\boldsymbol{\Sigma}}^\circ_2$. So, being 2-continuous is a weaker notion than being continuous. Since every continous function is 2-continuous, but there are 2-continuous functions that are not continuous. For example, consider the following mapping $\{0,1\}^\omega \longmapsto \{0,1\}^\omega$ defined by $\phi(0^\omega) = 0^\omega$ and $\phi(x) = 1^\omega$ for all $x \neq 0^\omega$. Given any open set $A \subseteq \{0,1\}^\omega$, only 4 possibilities may occur depending on if 0^ω or 1^ω belong to A:

(a) if neither 0^ω nor 1^ω belong to A, then $\phi^{-1}A = \emptyset$ which is open.

(b) if 0^ω and 1^ω both belong to A, then $\phi^{-1}A = \{0,1\}^\omega$ which is also open.

(c) if $0^\omega \notin A$ but $1^\omega \in A$, then $\phi^{-1}A = \{0,1\}^\omega \setminus \{0^\omega\}$ which is also open.

(d) if $0^\omega \in A$ but $1^\omega \notin A$, then $\phi^{-1}A = \{0^\omega\}$ which is closed, hence belongs to $\underset{\sim}{\boldsymbol{\Sigma}}^\circ_2$.

This shows that ϕ is 2-continuous, but since $\phi^{-1}0 \cdot \{0,1\}^\omega = \{0^\omega\}$, the inverse image of an open set is not open, hence it is not continuous. Further on, we will see precisely what 2-continuous mappings look like. They will be much easier to characterize once we relate them to a winning strategy for a player in a suitable game.

We introduce a new partial ordering on Borel sets by mean of 2 continuous functions.

Definition 3 *Let $A \subseteq \Sigma_A^\omega$ and $B \subseteq \Sigma_B^\omega$,*

$A \boldsymbol{\leq_{2c}} B \quad \Longleftrightarrow_{def}$ *there exists a 2-continuous $\phi : \Sigma_A^\omega \longmapsto \Sigma_B^\omega$ such that $(x \in A \longleftrightarrow \phi(x) \in B)$*

We write $A \boldsymbol{<_{2c}} B$ if $A \leq_{2c} B$ but $B \not\leq_{2c} A$. And also $A \boldsymbol{\equiv_{2c}} B$ when both $A \leq_{2c} B$ and $B \leq_{2c} A$.

Remarks 4 The relation $<_{2c}$ is a partial ordering on Borel sets, and $\equiv_{2c}$ is an equivalence relation. Moreover, $\leq_{2c}$ is a weaker ordering than $\leq_{1c}$, since $A \leq_{1c} B$ implies $A \leq_{2c} B$ but the contrary is generally false. For example, as subsets of $\{0,1\}^\omega$, $\{0^\omega\} \leq_{2c} 0 \cdot \{0,1\}^\omega$ but $\{0^\omega\} \not\leq_{1c} 0 \cdot \{0,1\}^\omega$.

In order to study this ordering $\leq_{2c}$ on Borel sets or more restrictively on the ones accepted by Deterministic Automata, we present a brief review of the major results concerning $\leq_{1c}$. We will not only mimic them but strongly rely on them.

4 A beginner's overview of the Wadge game

The main device in working with the Wadge hierarchy consists in replacing the notion of the inverse image by a continuous function by the one of a strategy for a player in a suitable game: the Wadge Game; so that the whole artillery developed in game theory becomes available.

Definition 5 (The Wadge game) *Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$ the Wadge game $\mathbf{W}(A\ ,\ B)$ is an infinite two player game where players take turn: I playing letters in Σ_A and II playing letters in Σ_B. Player I begins the run. Player II is allowed to skip provided he plays infinitely many letters; but player I is not. So that at the end of a run (in ω moves), I has produced an ω-word $x \in \Sigma_A^\omega$ and II has produced $y \in \Sigma_B^\omega$. We say*

$$\mathit{II} \text{ wins } \mathbf{W}(A\ ,\ B) \quad \Longleftrightarrow_{def} \ (x \in A \longleftrightarrow y \in B)$$

A strategy for player *II* is a mapping $\sigma :\ \Sigma_A^+ \longmapsto \Sigma_B \cup \{s\}$ where "s" is a letter that is not in the alphabet Σ_B (s stands for "skip"). Player *II* applies this strategy in the above game if he really plays $y' \in (\Sigma_B \cup \{s\})^\omega$ where both $y'(n) = \sigma(x \upharpoonright n+1)$ **and** the resukting infinite word y that one wants to know if it belongs to B or not is $y = y'$ once every occurrence of s has been erased. A strategy for *I* is defined similarly; and we say that σ is a winning strategy (abbreviated w.s.) for a player if that player always wins when using σ. We remark that a strategy is winning for a player in $\mathbf{W}(A\ ,\ B)$ iff it is winning for the same player in $\mathbf{W}(^-A\ ,\ ^-B)$. As usual in game theory, we write $\sigma* :\ \Sigma_A^\omega \longmapsto \Sigma_B^\omega$ the mapping induced by the strategy σ.
Wadge designed the game $\mathbf{W}(A\ ,\ B)$ so that

Proposition 6

(a) If σ is a strategy for II in a Wadge game, then $\sigma*$ is a continuous mapping.

(b) Given any continuous mapping $\phi :\ \Sigma_A^\omega \longmapsto \Sigma_B^\omega$, there exists a strategy σ for II in a Wadge game such that $\sigma* = \phi$.

In other words, given A and B Borel,

$$\mathit{II} \text{ has a w.s. in } \mathbf{W}(A\ ,\ B) \quad \Longleftrightarrow \quad A \leq_{1c} B.$$

Proof of proposition 6:

(a) It is enough to show that for any basic open set $u \cdot \Sigma_B^\omega$ (with $u \in \Sigma_B^*$), the set $\sigma *^{-1} u \cdot \Sigma_B^\omega$ is open. Then, get S the set of all finite words $v \in \Sigma_A^*$ such that $u = (\sigma * v)$ once every occurence of the letter s has been erased (in $\sigma * v$). By the very definition of σ, $\bigcup_{v \in S} v \cdot \Sigma_A^\omega = \sigma *^{-1} u \cdot \Sigma_B^\omega$ holds, showing it is an open set.

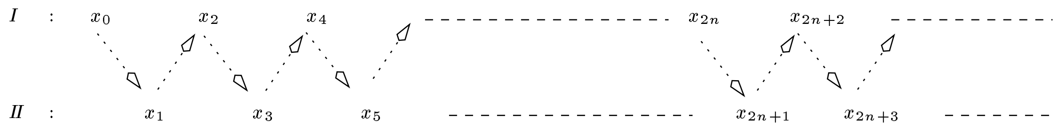
(b) We assume *II* is in position $u \in \Sigma_B^*$ and we describe *II*'s next move. Given any letter $a \in \Sigma_B$, the set $\phi^{-1} u \cdot a \cdot \Sigma_B^\omega$ is open (even clopen). So $\{\phi^{-1} u \cdot a \cdot \Sigma_B^\omega :\ a \in \Sigma_B\}$ forms a partition of $\phi^{-1} u \cdot \Sigma_B^\omega$ into clopen sets. So, sooner or later, *I* will reach a position v such that the basic open set $v \cdot \Sigma_A^\omega$ is included in some element of the partition. Thus, the strategy σ is to wait untill I reaches such a position, and then to play the one and only letter $a \in \Sigma_B$ that satisfies $v \cdot \Sigma_A^\omega \subseteq \phi^{-1}u \cdot a \cdot \Sigma_B^\omega$. Fortunately, player *II* is allowed to skip while I can't.

$\dashv$ 6

The Wadge game should be regarded as a particular instance of the more general two player infinite game with perfect information:

Definition 7 (The Gale & Stewart Game) *Let Σ be any set, and $A \subseteq \Sigma^\omega$, the game $\mathbb{G}(A)$ is a two player infinite game with perfect information where players take turn playing letters from Σ. Player I playing first. So in ω moves they produce an ω-word from Σ^ω, $x = \prod_{n\in\mathbb{N}} x_n$. Player I wins iff $x \in A$.*

Figure 1: the Gale & Stewart Game



The power of this game is entirely in the celebrated Martin's Borel Determinacy:

Theorem 8 (Martin) *If A is a Borel subset of Σ^ω, then $\mathbb{G}(A)$ is determined; i.e. one of the two players has a w.s.*

Proof of theorem 8: We refer the reader to Martin's articles [11] and [12] or to Kechris's book *Classical descriptive set theory* [7]. $\dashv$ 8

This means that one of the two players has a winning strategy. It is a very strong result, usually refered to as Borel Determinacy, which yields many consequences in various mathematical fields. One of them is Wadge Borel Determinacy. Given $A \subseteq \Sigma_A^\omega$ and $B \subseteq \Sigma_B^\omega$ both Borel, a proof of Wadge determinacy consists in defining a Borel set $C \subseteq (\Sigma_A \cup \Sigma_B \cup \{s\})^\omega$ such that a player has a winning strategy in $\mathbb{G}(C)$ iff the same player has a winning strategy in $\mathbf{W}(A\ ,\ B)$.
Let A' and A'' be the sets of all ω-words $x = \prod_{n\in\mathbb{N}} x_n$ on the alphabet $\Sigma_A \cup \Sigma_B \cup \{s\}$ that satisfy respectively $\prod_{n\in\mathbb{N}} x_{2n} \in A$ and $\prod_{n\in\mathbb{N}} x_{2n} \in \Sigma_A^\omega \smallsetminus A$. Both A' and A'' are Borel since they are the inverse image of A and ${}^-A$ by obvious continuous functions. Similarly, we define B' and B'' be the sets of all ω-words $x = \prod_{n\in\mathbb{N}} x_n$ on the alphabet $\Sigma_A \cup \Sigma_B \cup \{s\}$ that respectively satisfy $\prod_{n\in\mathbb{N}} x'_{2n+1} \in B$ and $\prod_{n\in\mathbb{N}} x'_{2n+1} \in \Sigma_B^\omega \smallsetminus B$, where x'_{2n+1} stands for the emptyword if $x_{2n+1} = s$, and for x_{2n+1} otherwise. For same reasons, B' and B'' are both Borel. Finally, we set $C = (A' \cap B') \cup (A'' \cap B'')$. It is a Borel set and whoever has a winning strategy in $\mathbb{G}(C)$ has a winning strategy in $\mathbf{W}(A\ ,\ B)$ - there is an obvious isomorphism between strategies for a player J in $\mathbb{G}(C)$ and strategies for the same player J in $\mathbf{W}(A\ ,\ B)$. Therefore Borel Determinacy implies Borel Wadge Determinacy, meaning that whenever A and B are both Borel, one of the two players necessarily have a winning strategy in $\mathbf{W}(A\ ,\ B)$.

An immediate consequence of Borel Wadge Determinacy is:

Proposition 9 *Let $A \subseteq \Sigma_A^\omega$ and $B \subseteq \Sigma_B^\omega$ be both Borel,*

$$A \not\leq_{1c} B \implies B \leq_{1c} {}^{-}A.$$

Proof of proposition 9: Since $A \not\leq_{1c} B$, *II* does not have a winning strategy in $\mathbf{W}(A\ ,\ B)$. Hence, by determinacy, player I has a winning strategy in that game. From that strategy, it is immediate to derive a winning strategy for *II* in $\mathbf{W}(B\ ,\ {}^{-}A)$.

$\dashv$ 9

This result directly leads to

Proposition 10 *[Wadge's lemma] If $A \subseteq \Sigma_A^\omega$ and $B \subseteq \Sigma_B^\omega$ are both Borel, then only one of the four possibilities occur:*

- *($A \leq_{1c} B$ and $B \not\leq_{1c} A$) which leads to $A <_{1c} B$*
- *($A \leq_{1c} B$ and $B \leq_{1c} A$) which leads to $A \equiv_{1c} B$*
- *($A \not\leq_{1c} B$ and $B \not\leq_{1c} A$) which leads to $A \equiv_{1c} {}^{-}B$*
- *($B \leq_{1c} A$ and $A \not\leq_{1c} B$) which leads to $B <_{1c} A$*

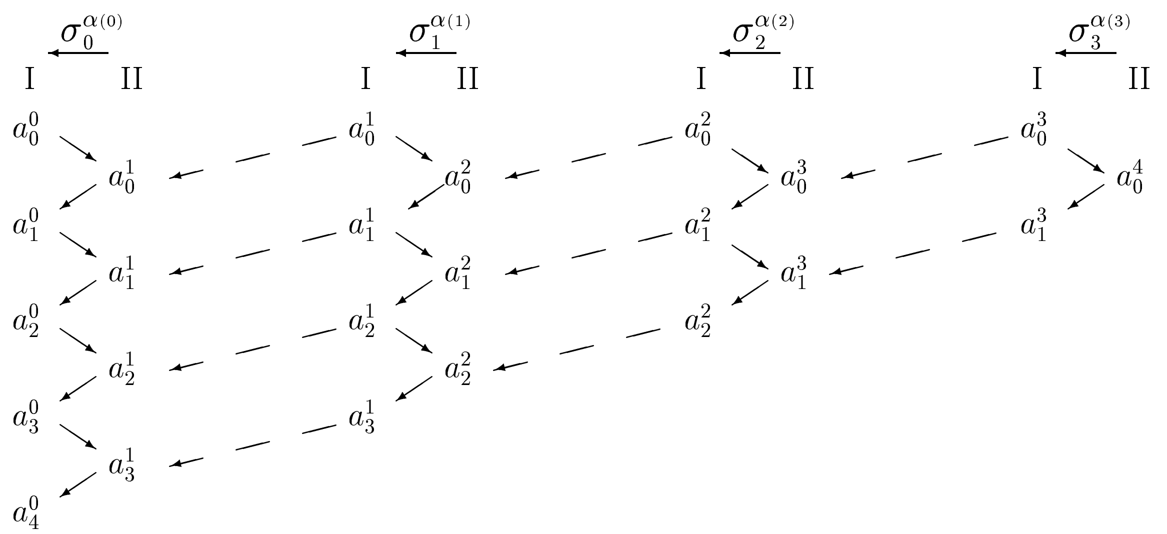
Proof of proposition 10: It is enough to check that $A \equiv_{1c} {}^{-}B$ holds when A and B are uncomparable by $\leq_{1c}$. Which is immediate by previous proposition. $\dashv$ 10

This last proposition shows that antichains have length at most 2. Next one states that $<_{1c}$ is a pre-wellordering.

Theorem 11 *[Martin, Wadge] the partial ordering $<_{1c}$ is wellfounded on Borel sets, which means there is no infinite sequence of Borel sets $(A_i)_{i\in\mathbb{N}}$ such that*

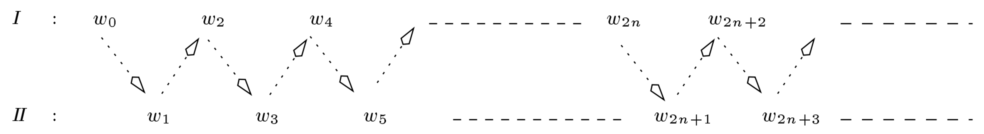
$$A_0 >_{1c} A_1 >_{1c} A_2 \ldots\ldots >_{1c} A_n >_{1c} A_{n+1} >_{1c} \ldots\ldots$$

Proof of theorem 11: The relation $A_i >_{1c} A_{i+1}$ means that $A_i \not\leq_{1c} A_{i+1}$ holds - therefore I has a winning strategy in the game $\mathbf{W}(A_i\ ,\ A_{i+1})$ - but it also means that $A_i \not\leq_{1c} {}^{-}A_{i+1}$ holds (otherwise A_i and A_{i+1} would be incomparable by $\leq_{1c}$). Now, for each integer n, let σ_n^0 and σ_n^1 be winning strategies for player I in games respectively $\mathbf{W}(A_n\ ,\ A_{n+1})$ and $\mathbf{W}(A_n\ ,\ {}^{-}A_{n+1})$. Given α any function from the integers to $\{0,1\}$, we consider ω Wadge games linked this way:



In the first game, player I applies the strategy $\sigma_0^{\alpha(0)}$ to II's play. Since it is a strategy for I, it gives the first letter a_0^0 before II has ever played anything, but then, applying $\sigma_0^{\alpha(0)}$ means to know II's first move a_0^1. Precisely, II copies I's moves in the second game. And so on. This means, in game number n, player I applies strategy $\sigma_n^{\alpha(n)}$ and II scrupulously copies I's moves in the game number $n+1$.
Intertwining stragies this way defines, for every ω-word $\alpha \in 2^\omega$, a unique ω-word on the alphabet Σ_{A_0}: $x_\alpha = \prod_{i\in\mathbb{N}} a_i^0$. We remark that, by construction and definition of a strategy for player one in a Wadge game, the first k letters of x_α only depend on the first k letters of α. More precisely, if $\alpha \restriction k = \alpha' \restriction k$ then $x_\alpha \restriction k = x_{\alpha'} \restriction k$. Hence, the mapping $\phi :\ \{0,1\}^\omega \longmapsto \Sigma_{A_0}^\omega$ defined by $\phi(\alpha) = x_\alpha$ is continuous. Therefore the inverse image of A_0 by this mapping is Borel. We set $B = \phi^{-1}A_0$ and we remark that by construction, if α and α' only differ in one position (i.e. $\exists! i\ \alpha(i) \neq \alpha'(i)$), then $\alpha \in B \iff \alpha' \notin B$.
In order to show that this leads to a contradiction, we introduce another game, very close to the Gale and Stewart one. Let $C \subseteq \Sigma_C^\omega$, the game $\mathbb{BP}(C)$ is a two player infinite game identical to the game $\mathbb{G}(C)$ except that both players'moves are finite non empty words instead of single letters. So I starts playing w_0, then II plays w_1 and so on. So that at the end of a run they have produced the ω-word $y = w_0 \cdot w_1 \cdot \ldots$, Player I wins iff $y \in C$.

Figure 2: The BP Game



It is almost immediate to see that $\mathbb{BP}(C)$ is determined as soon as C is Borel: the mapping $f :\ (\Sigma_C^*)^\omega \longmapsto \Sigma_C^\omega$ defined by $f(w_0, w_1, w_2, \ldots) = w_0 \cdot w_1 \cdot w_2 \cdot w_3 \cdot \ldots$ is obviously continuous, so the set $C' = f^{-1}C$ is Borel; hence $\mathbb{G}(C')$ is determined. Finally, it is straightforward to see that a player has a winning strategy in $\mathbb{G}(C')$ iff the same player has a strategy in $\mathbb{BP}(C)$.

Now, we come back to our original proof. Since B is Borel, we know that $\mathbb{BP}(B)$ is determined, hence I or II has a winning strategy.

- If I has a winning strategy τ, it compells him to play a finite word u_0 as first move. We recall $B_{(u_0)}$ denotes the set of all infinite words y such that $u_0 \cdot y \in B$. We show that I also has a winning strategy in $\mathbb{BP}(B_{(u_0)})$. Indeed, it consists in playing anything as first move (say v_0), then II plays some other word v_1. Then, until the game is over, I applies strategy τ, acting as if I's first move were u_0 and II's first move were $v_0 \cdot v_1$. This is obviously a winning strategy for I.

  But player II has also a winning strategy in $\mathbb{BP}(B_{(u_0)})$ which consists in applying τ, assuming he had played a word u_0' - which is exactly the same as u_0 except for one letter - right before the play started.

  This shows that both players have winning strategies, a contradiction.

- If II has a winning strategy τ, then I also has a winning strategy: play any first move, say u_0, then II plays u_1. Get any word u_1' that only differ from u_1 by one and only letter.

For the rest of the play, I applies $\mathcal{T}$ as if he were player $I\!I$ and I had just played $u_0 \cdot u'_1$ as first move.

This also shows that both players have winning strategies, a contradiction.

$\dashv$ 11

Definition 12 *We define the Wadge degree of Borel sets by induction:*

- $d^{\circ}_{1c}(\emptyset) = d^{\circ}_{1c}(^{-}\emptyset) = 0.$
- $d^{\circ}_{1c}(A) = \sup\{d^{\circ}_{1c}(B) : \ B <_{1c} A\}$ *(for any $A >_{1c} \emptyset$).*

5 Another hierarchy, other games

We need to translate the description of $\leq_{2c}$ in terms of the existence of a winning strategy for a player in a suitable game. This is what motivates next game.

Definition 13 (The $I\!I$ Finite Priority Wadge Game) *Let $A \subseteq \Sigma_A^{\omega}$, $B \subseteq \Sigma_B^{\omega}$, the game $\mathbf{FP}_{I\!I}(A\ ,\ B)$ is the same as the Wadge game $\mathbf{W}(A\ ,\ B)$ except that $I\!I$ may start from scratch finitely many times, erasing everything he has played and going back to the initial position.*
That is $\mathbf{FP}_{I\!I}(A\ ,\ B)$ is an infinite two player game where players take turn: I playing letters in Σ_A, $I\!I$ playing letters in Σ_B, Player I starts, player $I\!I$ is allowed to skip provided he plays infinitely many letters (player I can't skip). And at any time, $I\!I$ is allowed to erase everything he played since the beginning of the run, starting all over again. But if he does that infinitely many times, he loses the game. At the end of a run (in ω moves), I and $I\!I$ have respectively produced $x \in \Sigma_A^{\omega}$ and $y \in \Sigma_B^{\omega}$. We say

$$I\!I \text{ wins } \mathbf{FP}_{I\!I}(A\ ,\ B) \quad \Longleftrightarrow_{def} \ (x \in A \longleftrightarrow y \in B).$$

We write $A \leq_{\boldsymbol{FP_{I\!I}}} B$ when $I\!I$ has a winning strategy in $\mathbf{FP}_{I\!I}(A\ ,\ B)$.

We will see that whenever A and B are both Borel, determinacy leads to $A \leq_{FP_{I\!I}} B$ iff $A \leq_{2c} B$. For the moment, we introduce games where players must, along with playing letters as usual, answer questions about the future of the run. More precisely, on even moves, playing letters from an alphabet Σ, they must, on odd moves, provide a correct answer to a question of the form: will every position $u \in \Sigma^*$ you will reach be in a certain subtree Σ^*. In other words, the player will answer whether or not the infinite word he is actually building (on even moves) will be in some closed subset of Σ^{ω}. One may easily imagine that when a player's opponent start asking him questions like that, it becomes much more difficult to win. Of course it depends on the nature of the questions involved. For example, imagine a player in a Wadge game who is in charge of the closed set $\{0^{\omega}\}$, if his opponent asks him whether or not he will produce an infinite word that belongs to the empty set, the answer is simple and does not do any harm. But if he asks him whether or not he will stay in the closed set $\{0^{\omega}\}$, then the player must answer yes or no, so that the opponent doesn't need to wait until the end of the play to know if that word will be in the set the player is in charge of.

Starting with a player reaching positions in Σ^*, we define the space in which this player evolves when having to answer questions about the future of the run. We call such a space a **question tree**.

Notation 14 *For any finite or infinite word $x = x(0) \cdot x(1) \cdot x(2) \cdot x(3) \cdot x(4) \cdot \ldots$ we write $\frac{x}{2}$ for the word $x(0) \cdot x(2) \cdot x(4) \cdot \ldots = \langle x(2i) : i < lh(x)/2 \rangle$. And for a set A of finite or infinite word, $\frac{A}{2}$ stands for $\{\frac{x}{2} : x \in A\}$*

Definition 15 *Given an alphabet Σ, a question tree $\mathcal{T}$ on Σ is some* **non empty pruned tree** *that satisfies for any integer $u \in \mathcal{T}$ and any integer $n < lh(u)$:*

if n is even: *then $u(n) \in \Sigma$ (these are the nodes that correspond to the main run), and*

if n is odd: *then $u(n)$ is an auxiliary move with three different kinds of options :*

Option $\langle ! \rangle$: *in this case $u(n) = \langle ! \cdot v \rangle$ for some $v \in \Sigma$ and $\frac{u}{2} \subseteq v$. So v is some position in Σ^* that extends the position $\frac{u}{2}$. But then, we demand that any position w in $\mathcal{T}$ that extends u must verify $\frac{w}{2} \subseteq v$ or $v \subseteq \frac{w}{2}$. Moreover, we also require that $\mathcal{T}$ verifies the following condition: if $(u \upharpoonright n) \cdot \langle ! \cdot v' \rangle$ belongs to $\mathcal{T}$ with $v' \neq v$, then both $v' \subseteq v$ and $v \subseteq v'$ must fail. Or, to say it differently, T must satisfy the condition:*

$$\Big(u \in \mathcal{T} \ \wedge \ u \cdot \langle ! \cdot v \rangle \in \mathcal{T} \wedge \ u \cdot \big\langle ! \cdot v' \big\rangle \in \mathcal{T}\Big) \Longrightarrow v = v' \ \vee \ v \perp v'.$$

Option $\langle ? \rangle$: *in this case $u(n) = \langle ? \rangle$. This must be regarded as the option to avoid all other positions of the form $(u \upharpoonright n) \cdot \langle ! \cdot v \rangle$. Formally, this means that any w in $\mathcal{T}$ that extends $(u \upharpoonright n) \cdot \langle ? \rangle$ must satisfy*

$$v \subseteq \frac{w}{2} \ \textbf{fails for any } \boldsymbol{v} \textit{ such that } (u \upharpoonright n) \cdot \langle ! \cdot v \rangle \in \mathcal{T}.$$

Option $\langle - \rangle$: *this case should be regarded as no question asked at all. This option is relevant only to precisely describe what a question tree that asks finitely many questions along each of its infinite branches looks like. So, we only require $\mathcal{T}$ to satisfy:*

$$\textit{if } (u \upharpoonright n) \cdot \langle - \rangle \in \mathcal{T} \textit{ then } (u \upharpoonright n) \cdot \langle ? \rangle \notin \mathcal{T} \textit{ and for any } v \in \Sigma^* \ \ (u \upharpoonright n) \cdot \langle ! \cdot v \rangle \notin \mathcal{T},$$

in other words,

$$\big(lh(u') > n \ \wedge \ u' \in \mathcal{T} \ \wedge \ (u \upharpoonright n) = (u' \upharpoonright n)\big) \ \longrightarrow \ \big((u \upharpoonright n+1) = (u' \upharpoonright n+1)\big).$$

We remark that given any infinite word $x \in \Sigma^\omega$, there exists a unique infinite branch $y \in [\mathcal{T}]$ such that $\frac{y}{2} = x$.

To say it differently, from the point of view of a player who runs through branches of a question tree: on even moves, he simply plays letters from Σ, just like in the ordinary run. But on odd moves, everything looks like his opponent were asking him questions of the form: "do you intend to stay all run long in that closed subset of Σ^ω ? Or are you willing to exit it, which you should indicate by letting me know which one of the position that I submitted you, you will sooner or later reach". So, the player has the choice to say: "ok, I'm going to reach this position v" (among all the ones mentioned $\langle ! \cdot v' \rangle$). Or, on the contrary, taking option $\langle ? \rangle$ he may answer: "no, I won't reach any of these, I will rather stay out of the open set formed by the union of the basic open sets $v \cdot \Sigma^*$ for all positions v you are asking me about". Taking this option means remaining in the complement of this opens set, hence in a closed subset of Σ^ω. At last, when the option $\langle - \rangle$ is available, it is like no questions at all are asked. We simply need it for precisely define a question tree that asks finitely many questions along each of its branches. We say that a **question tree $\mathcal{T}$** is $\langle ? \rangle$-wellfounded if, for all $x \in [\mathcal{T}]$, there exists some integer n_x such that $x(2m+1) = \langle - \rangle$ for all $m \geq n_x$.

Definition 16 *Let $A \subseteq \Sigma_A^\omega$ and $\mathcal{T}$ be a question tree on Σ_A,*

$$x \in A^{\mathcal{T}} \iff_{def} \begin{cases} x \in [\mathcal{T}] \ \wedge \ \frac{x}{2} \in A \\ \vee \\ x \notin [\mathcal{T}] \wedge \exists x' \ \ (w \cdot x' \in [\mathcal{T}] \ \wedge \ \frac{w \cdot x'}{2} \in A) \\ \textit{where } w \textit{ is the longest initial part of } x \textit{ that belongs to } \mathcal{T}. \end{cases}$$

Remarks 17

(a) The definition of $A^{\mathcal{T}}$ may seem a little bit awkward. But we defined it this way so that player *II* still has a winning strategy in the restricted version of the Wadge game $\mathbf{W}\left(A^{\mathcal{T}}, A^{\mathcal{T}}\right)$, where *II* is not allowed to exit the question tree $\mathcal{T}$, while *I* can play anything he wants. Hence, a player J in charge of $A^{\mathcal{T}}$ in a Wadge game has a w.s. if and only if he has a w.s. that always remains inside $\mathcal{T}$. Therefore in the sequel we always consider that a strategy involving a set of the form $A^{\mathcal{T}}$ remains in the underlying question tree; in other words it restricts its moves to the legal ones.

(b) For any $A \subseteq \Sigma_A^\omega$ and any question tree $\mathcal{T}$ on Σ_A, the following obviously holds:

(i) ${}^-(A^{\mathcal{T}}) \equiv_{1c} ({}^-A)^{\mathcal{T}}$

(ii) $A^{\mathcal{T}} \leq_{1c} A$. Therefore, whenever A is Borel, $A^{\mathcal{T}}$ is Borel too (since it is the inverse image of a Borel set by a continuous function).

(c) For any Borel $A \subseteq \Sigma_A^\omega$, the set of all $A^{\mathcal{T}}$, where $\mathcal{T}$ ranges over all possible question trees on Σ_A is $<_{1c}$-wellfounded. Moreover, there cannot be two minimal **1c-non self dual** elements with one Wadge equivalent to the complement of the other. Indeed, towards a contradiction, assume that $\mathcal{T}$ and $\mathcal{T}'$ are two question trees on Σ_A that satisfy

$$\begin{cases} A^{\mathcal{T}} \equiv_{1c} {}^-(A^{\mathcal{T}'}) \\ \wedge \\ A^{\mathcal{T}} \leq_{1c} A^{\mathcal{T}''} \ \ \vee \ \ A^{\mathcal{T}'} \leq_{1c} A^{\mathcal{T}''} \text{ holds for any } \mathcal{T}'', \text{ question tree on } \Sigma_A. \end{cases}$$

From both $\mathcal{T}$ and $\mathcal{T}'$ one can easily build a question tree $\mathcal{S}$ on Σ_A which asks all questions demanded by both $\mathcal{T}$ and $\mathcal{T}'$. In other words, $\mathcal{S}$ satisfies: given any $x \in [\mathcal{S}]$, $y \in [\mathcal{T}]$ and $y' \in [\mathcal{T}']$, if $\frac{x}{2} = \frac{y}{2} = \frac{y'}{2}$ holds, then for any integer i, there exists integers j and j' such that $x(2j+1) = y(2i+1)$ (or $y(2i+1) = \langle ! \cdot v \rangle$ with $v \subsetneq \frac{x(2j+1)}{2}$ in which case $x(2j+1) = \left\langle ! \cdot \frac{x \restriction (2j+1)}{2} \right\rangle$) and $x(2j'+1) = y'(2i+1)$ (or $y'(2i'+1) = \langle ! \cdot v \rangle$ with $v \subsetneq \frac{x(2j+1)}{2}$ in which case $x(2j+1) = \left\langle ! \cdot \frac{x \restriction (2j+1)}{2} \right\rangle$). Then we see that $A^{\mathcal{S}} \leq_{1c} A^{\mathcal{T}}$ and $A^{\mathcal{S}} \leq_{1c} A^{\mathcal{T}'}$ both hold, therefore $A^{\mathcal{S}} <_{1c} A^{\mathcal{T}}$ and $A^{\mathcal{S}} <_{1c} A^{\mathcal{T}'}$ are both satisfied too; a contradiction. This shows that all minimal elements are Wadge equivalent. Hence there are only two possibilities for a $<_{1c}$-minimal element $A^{\mathcal{T}}$: either it is 1c-self dual **or** it is 1c-non self dual and its complement is not of the form $A^{\mathcal{T}'}$.

Notation 18 *Let $A \subseteq \Sigma_A^\omega$ be Borel,*
$\boldsymbol{A}^\perp$ *stands for a $<_{1c}$-minimal element of the form $A^{\mathcal{T}}$ where $\mathcal{T}$ ranges over all question trees over Σ_A.*

In fact, $A \longmapsto A^\perp$ is an operation only if we make it functional which requires the full Axiom of Choice. But that functional character is not needed in the proofs. We insist on the fact that $A^\perp$ is just a notational convenience and none of the foregoing proofs involving it require the Axiom of Choice.
The operation $^\perp$ preserves the Wadge ordering:

Proposition 19 *Let $A \subseteq \Sigma_A^\omega$ and $B \subseteq \Sigma_B^\omega$ be both Borel,*

$$A \leq_{1c} B \Longrightarrow A^\perp \leq_{1c} B^\perp.$$

Proof of proposition 19: We assume $I\!I$ has a w.s. σ in $\mathbf{W}(A\ ,\ B)$ and the question tree $\mathcal{T}_B$ on Σ_B verifies $B^{\mathcal{T}_B} = B^\perp$. We build a question tree $\mathcal{T}_A$ on Σ_A such that $I\!I$ has a w.s. in $\mathbf{W}\left(A^{\mathcal{T}_A}\ ,\ B^{\mathcal{T}_B}\right)$. By minimality of $A^\perp$, this leads to $A^\perp \leq_{1c} A^{\mathcal{T}_A} \leq_{1c} B^{\mathcal{T}_B} = B^\perp$, which gives the result.
We define $\mathcal{T}_A$ by describing which questions player $I\!I$ needs player I to answer in the Wadge game where I is in charge of A and $I\!I$ is in charge of $B^{\mathcal{T}_B}$, to win the game. Player $I\!I$'s only remaining concern if he applies the strategy σ on the main run is to give the correct answers to the questions he encounters. These questions take the form "will you end up in that closed set $C \subseteq \Sigma_B^\omega$ or not?". But the mapping $\sigma * \ \Sigma_A^\omega \longmapsto \Sigma_B^\omega$ induced by the strategy σ is a continuous function. Therefore the inverse image of C by this mapping is also a closed subset of Σ_A^ω. So $I\!I$ may ask I whether or not I's main run will be in $\sigma *^{-1} C$ (which means $\mathcal{T}_A$ is designed so that I necessarily must provide an answer to that question). If I gives a positive answer, then $I\!I$ gives the same one (which means he takes option $\langle ? \rangle$), and if I gives a negative answer, then $I\!I$ must give the same one, hence he must choose an option $\langle ! \cdot v \rangle$ for some v that exits C. But $I\!I$ can skip while I is not allowed to, thus waiting long enough, I will reach a position u in the main run that satisfies $\sigma * (u)$ is a finite word that exits C. This gives $I\!I$ the right answer.
This strategy is clearly winning for $I\!I$ in $\mathbf{W}\left(A^{\mathcal{T}_A}\ ,\ B^{\mathcal{T}_B}\right)$ since it is σ on the main run.

$\dashv$ 19

We are finished with the preliminaries. At last, we come to the main theorem that presents different equivalent descriptions of the ordering $\leq_{2c}$. Taking the right presentation of $\leq_{1c}$ for the right proof will be the only trick in the proofs of the rest of the paper.

Theorem 20 *Let $A \subseteq \Sigma_A^\omega$ and $B \subseteq \Sigma_B^\omega$ be Borel, the following are equivalent*

(a) $A \leq_{2c} B$

(b) $A^\perp \leq_{1c} B^\perp$

(c) $A \leq_{FP_{I\!I}} B$

(d) there exists a $\langle ? \rangle$-wellfounded question tree $\mathcal{T}$ on Σ_A such that $A^{\mathcal{T}} \leq_{1c} B$.

In order to prove this theorem, we need to introduce a few basic operations on sets that are very similar to basic operations on automata.

At this point, we want the reader to remark that given any set $A \subseteq \Sigma_A^\omega$ and any alphabet $\Sigma \supsetneq \Sigma_A$ one may find a set $B \subseteq \Sigma^\omega$ such that $B \equiv_{1c} A$ (e.g. define B as the inverse image of A by the homomorphism $h(a) = 0$ for $a \in \Sigma \smallsetminus \Sigma_A$ and $h(a) = a$ otherwise). So in the sequel we always assume that the alphabets involved are rich enough to allow the partitioning operations we need, or are simply enriched to fit the goal.

Definition 21 *Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$, and $\Sigma_A \subsetneq \Sigma_B$,*

$$A^{\longrightarrow}B =_{def} A \cup (\Sigma_A^* \cdot (\Sigma_B \setminus \Sigma_A) \cdot B).$$

A player in charge of $A^{\longrightarrow}B$ is like a player in charge of A endowed with an extra move that erases everything played before and turns A into B. But this extra move may be used only once and is materialized by playing a letter in $\Sigma_A \setminus \Sigma_B$ for the first time in the game. We say that a player, in charge of $A^{\longrightarrow}B$ in a Wadge game, "crosses the arrow" when he plays a letter in $\Sigma_B \setminus \Sigma_A$ for the first time.
The next operation allows a player to choose at first move the set he wants to be in charge of.

Definition 22 *Let $A, B \subseteq \Sigma^\omega$, assume $\{\Sigma_A, \Sigma_B\}$ is a partition of Σ in two non empty sets,*

$$\genfrac{}{}{0pt}{}{\nearrow A}{\searrow B} =_{def} (\Sigma_A \cdot A) \ \cup \ \Sigma_B \cdot B.$$

We write ${}^{\pm}A$ for the set $\genfrac{}{}{0pt}{}{\nearrow A}{\searrow {}^{-}A}$. We remark that when A is 1c-non self dual, the set ${}^{\pm}A$ is 1c-self dual and is the $<_{1c}$-least above A (and above ${}^{-}A$). We generalize definition 22 by allowing a player, at first move, to choose among countably many sets, the one he wants to be in charge of.

Definition 23 *Let $I \subseteq \Sigma^+$ be a set of finite non empty words that satisfies $\{i \cdot \Sigma^\omega : \ i \in I\}$ forms a partition of Σ^ω (i.e. $\forall u \in \Sigma^* \quad \exists i \in I \quad (i \subseteq u \text{ or } u \subseteq i \quad \wedge \quad \forall i, j \in I \ (i = j \ \vee \ i \perp j))$*
Let $A_i \subseteq \Sigma^\omega$ $(\forall i \in I)$,

$$\sum_{i \in I} A_i = \bigcup_{i \in I} i \cdot A_i.$$

We use definition 23 to get a description of 1c-self dual sets on the basis of 1c-non self dual ones.

Proposition 24 *Let $A \subseteq \Sigma_A^\omega$ and A be 1c-self dual, then*
there exists $(A_i)_{i \in I}$, $I \subseteq \Sigma_A^+$ as in definition 23, each $A_i \subseteq \Sigma_A^\omega$ 1c-non self dual and $A_i <_{1c} A$ such that

$$A \equiv_{1c} \sum_{i \in I} A_i.$$

Moreover, if Σ_A is finite then $A \equiv_{1c} {}^{\pm}A_i$ for some $i \in I$.

The proof is an easy consequence of:

Theorem 25 *[Martin, Wadge] Let $A \subseteq \Sigma_A^\omega$, A be Borel,*

$$A \text{ is 1c-non self dual} \iff \exists x \in \Sigma_A^\omega, \ \forall n \in \mathbb{N}, \ A_{(x \restriction n)} \equiv_{1c} A.$$

To say it in words, if A is 1c-non self dual, there exists $x \in \Sigma_A^\omega$ such that, for each integer n, player *II* has a w.s. in the game $\mathbf{W}\left(A\ ,\ A_{(u\upharpoonright n)}\right)$ which is the same game as the *II* imposed gane $\mathbf{W}(A\ ,\ A)$ where *II* starts in position already $x \upharpoonright n$.

Proof of proposition 24: The proof goes by induction on $d_{1c}^{\circ}(A)$.
We set

$$I_0 = \{x \upharpoonright n_x :\ x \in \Sigma_A^\omega\ \wedge\ n_x \text{ is the least integer such that } A_{(x\upharpoonright n_x)} <_{1c} A\}.$$

Clearly, if Σ_A is finite, then I_0 is finite too, and it is countable if Σ_A is countable.
Furthermore, for $u \in I_0$, $A_{(u)} <_{1c} A$ hence $A = \sum_{u \in I_0} A_{(u)}$; by induction hypothesis each $A_{(u)} \equiv_{1c} \sum_{i \in I_u} A_{(u)_i}$ with $A_{(u)_i} \subseteq \Sigma_A^\omega$, 1c-non self dual and $A_{(u)_i} <_{1c} A_{(u)}$. Thus

$$A \equiv_{1c} \sum_{u \in I_0} \left(\sum_{i \in I_u} A_{(u)_i} \right) = \sum_{u \cdot i\ \in I} A_{(u)_i}.$$

where $I = \{u \cdot i : u \in I_0, i \in I_u\}$ verifies: if Σ_A is finite, then each I_u is finite, hence I is finite too. And in that case, there exists u, i such that $A_{(u)_i}$ is of maximal Wadge degree; thus clearly $A \equiv_{1c} {}^{\pm}A_{(u)_i}$ holds.

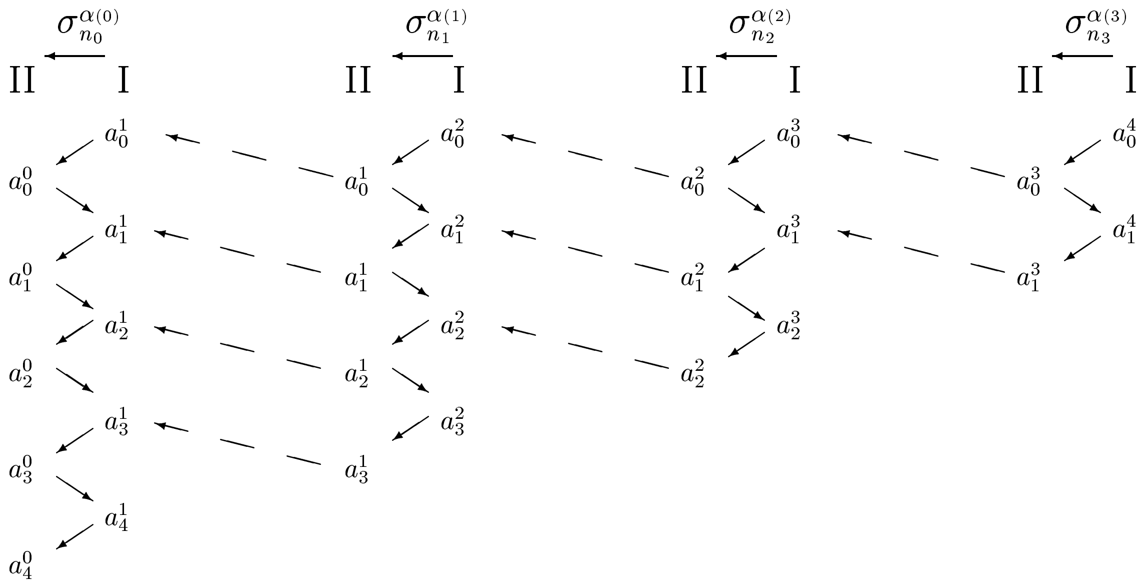
$\dashv$ 24

Proof of theorem 25: (See also [19]).
$(\Longrightarrow)$
The fact $A_{(x\upharpoonright n)} \leq_{1c} A$ is immediate. To show that $A_{(x\upharpoonright n)} \geq_{1c} A$ holds it is enough to consider the game $\mathbf{W}(A\ ,\ {}^{-}A)$ in which player *I* applies a w.s. σ and *II* skips indefinitely. The resulting sequence $x = \sigma * s^\omega$ satisfies $\forall n \in \omega,\ A_{(x\upharpoonright n)} \geq_{1c} A$.

$(\Longleftarrow)$
The proof is very close to Martin's proof that $<_{1c}$ is wellfounded on Borel sets. We prove

$$A \text{ is 1c-self dual } \Longrightarrow \forall x \in \Sigma_A^\omega,\ \exists n \in \omega,\ A_{(x\upharpoonright n)} \not\geq_{1c} A.$$

Towards a contradiction, assume that there exists x that satisfies $\forall n \in \omega,\ A_{(x\upharpoonright n)} \geq_{1c} A$. For each integer n let σ_n^0 be a w.s. for *II* in the *II* imposed game $\mathbf{W}(A\ ,\ A)$ where player *II*'s n first moves must be $x \upharpoonright n$ and σ_n^1 a w.s. for *II* in the *II* imposed game $\mathbf{W}(A\ ,\ {}^{-}A)$ where also player *II*'s n first moves must be $x \upharpoonright n$. Given $\alpha \in \{0,1\}^\omega$, consider the following combination of ω different Wadge games:



where n_k is the smallest integer greater than n_{k-1} such that player *II* having already played the n_k first values of x in game number k, player *II* in the first game has answered with at least k letters i.e.

$$\left(\sigma_{n_0}^{\alpha(0)} \circ \sigma_{n_1}^{\alpha(1)} \circ \ldots \circ \sigma_{n_k}^{\alpha(\mathrm{k})}\right) * (x \restriction n_k) \text{ has length at least } k.$$

(We need to proceed this way because in a Wadge game player *II* is allowed to skip, also the above picture is not totally accurate).

By construction, the mapping $f : \{0,1\}^\omega \longmapsto \Sigma_A^\omega$ defined by $f(\alpha) = \prod_{i\in\mathbb{N}} a_i^0$ is continuous. Thus $B = f^{-1}A$ is Borel, therefore the game $\mathbb{BP}(B)$ is determined, which means that one of the two players has a winning strategy.

- We first assume that I has a winning strategy $\mathcal{T}$ whose first move is the finite word u_0. Then, player I has also a winning strategy in $\mathbb{BP}(B_{(u_0)})$ which consists in playing anything as first move (say v_0), then, after *II* plays some other word v_1, I applies the strategy $\mathcal{T}$, acting as if I's first move were u_0 and *II*'s first move were $v_0 \cdot v_1$. This strategy is winning for I.

  But player *II* has also a winning strategy in $\mathbb{BP}(B_{(u_0)})$ which consists in applying $\mathcal{T}$, but assuming he played a word u_0' - which is exactly the same as u_0 except for one letter - right before the play started.

  This shows that both players have winning strategies, a contradiction.

- If *II* has a winning strategy $\mathcal{T}$, then I also has a winning strategy. Namely, play any non empty word u_0 as first move and then, once *II* plays his first move u_1, get any word u_1' that only differ from u_1 by a single letter; and for the rest of the play, apply $\mathcal{T}$ as if I were player *II* and I had just played $u_0 \cdot u_1'$ as first move. This is also a winning strategy for I, a contradiction.

$\dashv$ 25

Next notion is very "automaton-like", since it deals with the ability to start all over again as if nothing had been done so far: the initializability.

Definition 26 *Let $A \subseteq \Sigma_A^\omega$,*

$$A \textit{ is } \textbf{initializable} \quad \Longleftrightarrow_{def} \; A^{\longrightarrow}A \leq_{1c} A.$$

We emphasize that by the very definition of $^{\longrightarrow}$, the set $A^{\longrightarrow}A$ is the same as the set $A^{\longrightarrow}A'$ where $A' \subseteq \Sigma_{A'}^\omega$ with $\Sigma_A \subsetneq \Sigma_{A'}$ and $A \equiv_{1c} A'$.

If A is initializable then $A \equiv_{1c} \overbrace{A^{\longrightarrow}A^{\longrightarrow}\ldots^{\longrightarrow}A}^{n+1}$ holds for any integer n. We remark that an initializable set is necessarily 1c-non self dual because, as a consequence of proposition 24, every 1c-self dual set is obviously not initializable.

Next proposition should be regarded as a way to break the non initializable 1c-non self dual sets into smaller pieces. A trick that will be largely used in the forthcoming proofs.

Proposition 27 *Let $A \subseteq \Sigma_A^\omega$,*

$$\textit{if } {}^{-}A \not\equiv_{1c} A \textit{ and } A^{\longrightarrow}A \not\leq_{1c} A \textit{ then there exists } B \subseteq \Sigma_A^\omega, C \subseteq \Sigma_C^\omega \textit{ such that } \begin{cases} B, C <_{1c} A \\ B^{\longrightarrow}B \leq_{1c} B \\ C \geq_{1c} {}^{-}B \\ A \equiv_{1c} B^{\longrightarrow}C. \end{cases}$$

Proof of proposition 27: We set $T = \{u \in \Sigma_A^* : \; A_{(u)} \geq_{1c} A\}$. Since A is 1c-non self dual we remark that T is a non empty pruned tree. Hence, by Dependant Choice, $[T] \neq \emptyset$ holds. We define B by:

(a) If $[T] \smallsetminus A \neq \emptyset$ then $B = A \cap [T]$

(b) otherwise $B = \Sigma_A^\omega$

The reader should notice that B is designed so that player *II* has a w.s. in $\mathbf{W}(B \;,\; B)$ **that always remains in T**. This is obvious in case (b). And in case (a), the w.s. for *II* consists in copying *I*'s run as long as it remains inside T. If it exits T, then, at this time, *II* is still in a position $u \in T$ (and *I* is in position $u \cdot a \notin T$). Therefore, *II* plays some $y \in \Sigma_A^\omega$ such that $u \cdot y \in [T] \smallsetminus A$. It is true that there exists such a y because since $A \leq_{1c} A_{(u)}$ is satisfied, *II* has a w.s. in the game $\mathbf{W}\left(A \;,\; A_{(U)}\right)$; and in particular, this strategy is winning when *I* plays some infinite word $x_0 \in [T] \smallsetminus A$. This strategy gives such a y that necessarily belongs to $[T]$.
Obviously B is initializable in case (b). To show that B is initializable in case (a) it is enough to remark that for any $u \in T$ player *II* has a w.s. in $\mathbf{W}\left(A \;,\; A_{(u)}\right)$. In particular, when player *I* remains in T, this w.s. requires *II* to stay in T. Therefore this w.s. is also a w.s. for *II* in $\mathbf{W}\left(B \;,\; B_{(u)}\right)$ **as long as** *I* remains in T, But if *I* exits T, since at that time *II* is still in a position that is in T and $[T] \smallsetminus A \neq \emptyset$ holds, this position can be extended to an infinite branch (word) of $[T]$ which is not in A.
Obviously in both cases $B \leq_{1c} A$ holds and $B \leq_{1c} A$ fails. Otherwise A would be initializable. Hence we have $B <_{1c} A$.
We define U as the set of positions that exit T for the first time:

$$U = \{u \cdot a : \; u \in T, \; a \in \Sigma_A, \; u \cdot a \notin T\}.$$

We set $\Sigma_C = U \cup \Sigma_A$. We choose $u_0 \in U$ ($U \neq \emptyset$ since A is not initializable) and $a_0 \in \Sigma_A$. For any $x \in \Sigma_C^\omega$, we let $x(U/a_0)$ denote x after every occurrence of a letter in U has been replaced by a_0. We set

$$C = \sum_{l \in \Sigma_C} l \cdot A_l.$$

where A_l is defined by

- if $l \in u$ then $A_l = \{x \in \Sigma_C^\omega : \quad l \cdot (x(U/a_0)) \in A\}$
- if $l \notin u$ then $A_l = \{x \in \Sigma_C^\omega : \quad u_0 \cdot (x(U/a_0)) \in A\} = A_{u_0}$

By definition of U, for any $u \in U$, $A \not\leq_{1c} A_u$ holds. Therefore, since A is 1c-non self dual, one gets $A \not\leq_{1c} C$, thus $C <_{1c} A$.
We show that $A \equiv_{1c} B^{\longrightarrow}C$ holds by proving both $A \leq_{1c} B^{\longrightarrow}C$ and $A \geq_{1c} B^{\longrightarrow}C$ are verified:

(a) In the Wadge game $\mathbf{W}(A\ ,\ B^{\longrightarrow}C)$ the following strategy is winning for *II*. Player *II* stays in charge of B as long as I stays in T and, in that case, *II* copies I's run. If and when I exits T (i.e. comes to a position $u \cdot a$ with $u \in T$ and $u \cdot a \in U$) then *II* decides to be in charge of C and even more precisely to be in charge of A_u. This shows $A \leq_{1c} B^{\longrightarrow}C$.

(b) In the Wadge game $\mathbf{W}(B^{\longrightarrow}C\ ,\ A)$ the following strategy is winning for *II*.

(i) In both cases
(1) $[T] \smallsetminus A \neq \emptyset$ and B is defined by $B = A \cap [T]$
(2) $[T] \smallsetminus A = \emptyset$ then B is defined by $B = \Sigma_A^\omega$
as long as player I stays in charge of B, player *II* has a w.s. against I that remains inside T. At any move if I decides to become in charge of C, then *II*, still being in T, is in a position that is equivalent to the initial one and therefore can easily beat I.

It just remains to prove that $C \geq_{1c} {}^{-}B$ holds. Towards a contradiction, we assume $C \leq_{1c} B$, which leads to

$$A \equiv_{1c} B^{\longrightarrow}C \leq_{1c} B^{\longrightarrow}B \equiv_{1c} B.$$

which contradicts $B <_{1c} A$.

$\dashv$ 27

Before we come to the proof of theorem 20, we still need to state a last proposition that describes how $A^\perp$ looks like, depending on whether A is initializable or not. More precisely, it precisely tells us what are the conditions required by A to ensure that $A^\perp$ is 1c-non self dual.

Proposition 28 *Let $A \subseteq \Sigma_A^\omega$, A Borel,*

(a) if A is 1c-self dual, then $A^\perp$ is 1c-self dual:

$$A \equiv_{1c} {}^{-}A \Longrightarrow A^\perp \equiv_{1c} {}^{-}A^\perp$$

(b) if A is 1c-non self dual but not initializable, then $A^\perp$ is self dual:

$$(A \not\equiv_{1c} {}^{-}A \ \wedge \ A \not\equiv_{1c} A^{\longrightarrow}A) \Longrightarrow A^\perp \equiv_{1c} {}^{-}A^\perp$$

(c) if A is initializable, then $A^\perp$ is 1c-non self dual:

$$A \equiv_{1c} A^{\longrightarrow}A \Longrightarrow A^\perp \not\equiv_{1c} {}^{-}A^\perp.$$

Proof of proposition 28:

(a) If $A \equiv_{1c} {}^{-}A$, then by proposition 19, $A^{\perp} \equiv_{1c} ({}^{-}A)^{\perp} \equiv_{1c} {}^{-}(A^{\perp})$

(b) The proof is by induction on $d^{\circ}_{1c}(A)$. By proposition 27 there exists $B \subseteq \Sigma_A^{\omega}$ initializable and $C \subseteq \Sigma_A^{\omega}$ such that $B,\ C <_{1c} A,\ {}^{-}B \leq_{1c} C$ and $A \equiv_{1c} B^{\longrightarrow}C$. Easily one can see that the following holds

$$(B^{\longrightarrow}C)^{\perp} \equiv_{1c} \left\langle \begin{matrix} \nearrow B^{\perp} \\ \searrow C^{\perp} \end{matrix} \right.$$

- if $C \not\geq_{1c} B$ then, by determinacy, $C \equiv_{1c} {}^{-}B$ thus $\left\langle \begin{matrix} \nearrow B^{\perp} \\ \searrow C^{\perp} \end{matrix} \right. \equiv_{1c} {}^{\pm}C^{\perp}$ which is 1c-self dual.
- if $C \geq_{1c} B$ then $\left\langle \begin{matrix} \nearrow B^{\perp} \\ \searrow C^{\perp} \end{matrix} \right. \equiv_{1c} \left\langle \begin{matrix} \nearrow C^{\perp} \\ \searrow C^{\perp} \end{matrix} \right. \equiv_{1c} C^{\perp}$. But in that case, since both $C \geq_{1c} B$ and $C \geq_{1c} {}^{-}B$ hold, C cannot be initializable otherwise it would lead to $B^{\longrightarrow}C \leq_{1c} C$, hence $B^{\longrightarrow}C \leq_{1c} C$, which gives $A \equiv_{1c} C$ contradicting the fact A is not initializable. Granted with C not being initializable and satisfying $C <_{1c} A$, we apply the induction hypothesis in order to get $C^{\perp}$ is 1c-self dual.

(c) We assume $A^{\perp} = A^{\mathcal{T}}$. It is sufficient to show there exists $x \in [\mathcal{T}]$ that verifies $A^{\mathcal{T}} \leq_{1c} (A^{\mathcal{T}})_{(x\restriction n)}$ for all integer n. We set

$$S = \{w \in \Sigma_A^{*} :\ A \leq_{1c} A_{(w)}\}.$$

We remark that S is a non empty pruned tree because A is 1c-non self dual. We inductively define x by:

- $x(0)$ is anything so that $x(0) \in S$.
- assume $x(0) \cdot x(1) \cdot \ldots \cdot x(2n)$ has been defined and all the promises made states that the run will be in the closed set F_n **and** the next necessary moves are going to be v_n so that the run will also take place inside the basic clopen set

$$O_n = x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot v_n \cdot \Sigma_A^{\omega}$$

where:

$$O_n = \bigcap_{i<n,\ \ x(2i+1)=\langle ! \cdot w_i \rangle} w_i \cdot \Sigma_A^{\omega}$$

and

$$F_n = \bigcap_{i<n,\ \ x(2i+1)=\langle ? \rangle} (\Sigma_A^{\omega} \smallsetminus P_i)$$

where $P_i = \bigcap_{x\restriction(2i+1)\ \cdot \langle ! \cdot w \rangle\ \in S} w \cdot \Sigma_A^{\omega}$ (for each integer i such that $x(2i+1) = \langle ? \rangle$).

(By convention v_0 is the empty word and $O_0 = F_0 = \Sigma_A^{\omega}$). From the position $x \restriction 2n$, legal moves are now restricted to the closed set $F_n \cap O_n$.

We assume that by construction the following holds:

$$[S] \cap O_n \subseteq F_n \ \text{ and } \ O_n \cap [S] \neq \emptyset \text{ (i.e. } x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot v_n \in S)$$

We set

(i) $x(2n+1) = \langle ! \cdot x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) w_n \rangle$ iff there exists $w_n \in \Sigma_A^*$ such that $v_n \subseteq w_n$ **and** $x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot w_n \in S$. One has

$$O_{n+1} = x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot w_n \cdot \Sigma_A^\omega \text{ and } F_{n+1} = F_n.$$

(ii) $x(2n+1) = \langle ? \rangle$ otherwise. Notice that necessarily $[S] \cap O_n \subseteq \Sigma_A^\omega \smallsetminus P_n$. Then

$$O_{n+1} = O_n \text{ and } F_{n+1} = F_n \cap (\Sigma_A^\omega \smallsetminus P_n).$$

In both cases the following requirement holds:

$$O_{n+1} \cap [S] \subseteq F_{n+1} \text{ and } O_{n+1} \cap [S] \neq \emptyset.$$

- On even moves: $x(2n+2) =$ anything such that

$$x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot x(2n+2) \cdot \Sigma_A^\omega \subseteq [S] \cap O_{n+1}.$$

We now show that x works, i.e. $A^\perp \leq_{1c} A^\perp[x \restriction k]$ holds for each integer k. It is enough to show that the property holds for k even. We set $k = 2n$.

First we remark that A being initializable there exists a strategy $\mathcal{T}$ that **always remains inside S** and is winning for *II* in the Wadge game

$$\mathbf{W}\left(A\ ,\ A_{(x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot v_n)}\right).$$

This is true because $A \leq_{1c} A_{(x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot v_n)}$ holds since the word $x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot v_n$ belongs to S. Elsewhere, $A^{\longrightarrow}A \leq_{1c} A$ holds because A is initializable. Therefore there exists a w.s. for *II* in the game

$$\mathbf{W}\left(A^{\longrightarrow}A\ ,\ A_{(x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot v_n)}\right).$$

But in that game, as long as I does not "cross the arrow" (that is plays a letter that is not in Σ_A) the w.s. requires *II*'s run to remain inside S (otherwise I would have a winning strategy). From that w.s. we easily derive a strategy that remains in S and is winning for *II* in the game

$$\mathbf{W}\left(A\ ,\ A_{(x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n) \cdot v_n)}\right).$$

Hence there exists a strategy $\mathcal{T}$ that is winning for *II* in

$$\mathbf{W}\left(A\ ,\ A_{(x(0) \cdot x(2) \cdot x(4) \cdot \ldots \cdot x(2n))}\right).$$

such that $\forall y \in \Sigma_A^\omega \quad \mathcal{T} * y \in O_n \cap [S]$.

By minimality of $A^\perp$ it is enough to show that there exists a question tree $\mathcal{S}$ on Σ_A such that *II* has a w.s. in the game

$$\mathbf{W}\left(A^{\mathcal{S}}\ ,\ A^{\mathcal{T}}{}_{(x(0) \cdot x(1) \cdot x(2) \cdot \ldots \cdot x(2n))}\right).$$

In that game, we let *II* apply the strategy $\mathcal{T}$ on even moves. These moves are legal and *II* is sure to win, provided he gives the right answer on odd moves. But to ensure this, it is enough, whenever he needs to know if his run will be in some closed set, that *II* "asks" his opponent to answer this very question. This is simply done by imposing I to answer the question whether or not he will remain in the inverse image of this closed set by the continuous function induced by the strategy. Therefore, we have shown:

$$A^\perp \leq_{1c} A^{\mathcal{S}} \leq_{1c} A^{\mathcal{T}}{}_{(x \restriction 2n+1)} = A^\perp[x \restriction k+1]$$

$\dashv$ 28

At last we give a full proof of theorem (a) which gave the different equivalent versions of the partial ordering $\leq_{2c}$.

Proof of theorem 20:

(a)$\Longrightarrow$(b): we let $\mathcal{T}_B$ be a question tree on Σ_B such that $B^{\mathcal{T}_B} \equiv_{1c} B^{\perp}$. We define a question tree on Σ_A such that $A^{\mathcal{T}_A} \leq_{1c} B^{\mathcal{T}_B}$. By minimality, one shows

$$A^{\perp} \leq_{1c} A^{\mathcal{T}_A} \leq_{1c} B^{\mathcal{T}_B} \leq_{1c} B^{\perp}.$$

Claim 29 *If $C \subseteq \Sigma_C^{\omega}$, $C \in \underset{\sim}{\mathbf{\Sigma}}^{\circ}_2 \cap \underset{\sim}{\mathbf{\Pi}}^{\circ}_2$, then there exists a $\langle ? \rangle$-wellfounded question tree $\mathcal{T}_C$ on Σ_C such that $C^{\mathcal{T}_C} \leq_{1c} {}^{\pm}\emptyset$.*

Proof of claim 29: By hypothesis, $C = \bigcup_{i \in \mathbb{N}} F_i$ and ${}^{-}C = \bigcup_{i \in \mathbb{N}} G_i$, where each F_i and G_i is closed. We define $\mathcal{T}_C$ by describing a w.s. for *II* in $\mathbf{W}(C\ ,\ {}^{\pm}\emptyset)$ where player *II* asks player *I* questions about whether or not *I*'s run will end in some closed subsets. *II*'s problem to be sure to win is to know after finitely many *I*'s moves, if *I*'s run will be in C or in ${}^{-}C$. But, *II* clearly only needs to ask finitely many questions to know that. Namely, "will you end up in F_0?", "will you end up in G_0?", "will you end up in F_1?", "will you end up in G_1?", and so on until *I* answers "yes".

$\dashv$ 29

Let $\phi:\ \Sigma_A^{\omega} \longmapsto \Sigma_B^{\omega}$ be a 2-continuous function such that $x \in A \longleftrightarrow \phi(x) \in B$.

We assume $\Sigma_B = \{b_0, b_1, b_2, \dots\}$ (countable alphabet). We describe $\mathcal{T}_A$ by letting *II* ask *I* directly the questions he needs *I* to answer in order to be able to apply ϕ on the main run.

First of all, *II* wants to know which first letter he should play. He wants to kwnow if it is b_0 or not. So he just needs to ask *I*. This works because, since ϕ is 2-continuous, the inverse image of any open set is both $\underset{\sim}{\mathbf{\Sigma}}^{\circ}_2$ and $\underset{\sim}{\mathbf{\Pi}}^{\circ}_2$. Hence $\phi^{-1}(b_o \cdot \Sigma_B^{\omega}) \in \underset{\sim}{\mathbf{\Sigma}}^{\circ}_2 \cap \underset{\sim}{\mathbf{\Pi}}^{\circ}_2$. Therefore, by previous claim, asking finitely many questions about whether or not *I* will end up in closed subsets, *II* will know if *I*'s run will be in $\phi^{-1} b_o \cdot \Sigma_B^{\omega}$ or not. If it comes out that *I* answers "yes", then *II* plays b_0 as first move; otherwise, he just needs to ask finitely many questions to know if he should play b_1; and so on... So, after finitely many questions asked, *II* will know that he must play some b_{i_0} (for some integer i_0) as first move. Asking finitely many questions is not a problem for player *II* since he is allowed to skip. All we want to say is that, since *I* can't skip, *I* must keep on going from a position to another in $\mathcal{T}_A$, and we make sure that $\mathcal{T}_A$ is designed so that player *I* encounters the questions that player *II* wants him to answer during this process.

So, if *II* knows how to play the first move, he knows how to play any move from the main run: the same idea works since, assuming *II* has already a position in the main run: $b_{i_0} \cdot b_{i_1} \cdot b_{i_2} \cdot \ldots \cdot b_{i_k}$, all the sets $\phi^{-1}(b_{i_0} \cdot b_{i_1} \cdot b_{i_2} \cdot \ldots \cdot b_{i_k} \cdot b_j \cdot \Sigma_B^{\omega})$, for any integer j, are in $\underset{\sim}{\mathbf{\Sigma}}^{\circ}_2 \cap \underset{\sim}{\mathbf{\Pi}}^{\circ}_2$. This settles the problem of *II*'s main run. For the auxiliary moves, *II* must choose, at each step, whether the main run will end in a closed set or not. But that, he may directly ask *I*. Imagine that *II* wants to know if his main run will be in some closed set C, he just needs to ask *I* if *I*'s main run will be in $\phi^{-1}C$, which he can do by asking finitely many question ($\phi^{-1}C \in \underset{\sim}{\mathbf{\Sigma}}^\circ_2 \cap \underset{\sim}{\mathbf{\Pi}}^\circ_2$). If I answers "yes", he answers "yes" too, which means he takes option $\langle ? \rangle$. On the contrary, if I finally answers "no", then $I\!I$ must indicate which position in the main run he will reach that exits the closed set C. But, since he knows for sure that he will exit C, he just needs to wait long enough to see this position come, on his main run. This means, $I\!I$ keeps on asking questions about his main run, until he gets to know the long enough position he will reach that exits C. Then, $I\!I$ just has to give the correct answer in the auxiliary move. So, we have described some question tree $\mathcal{T}_A$ on Σ_A such that in the game $\mathbf{W}\left(A^{\mathcal{T}_A}\ ,\ B^{\mathcal{T}_B}\right)$, player $I\!I$ applies a strategy that verifies: $I\!I$'s main run is exactly the image of I's main run by ϕ. Since $x \in A \longleftrightarrow \phi(x) \in B$ holds, this strategy is winning for $I\!I$.

(b)$\Longrightarrow$(c): we assume $A^\perp \leq_{1c} B^\perp$. We need to show that $A \leq_{FP_{I\!I}} B$ is satisfied. We do this by induction on $d^\circ_{1c}(A)$.

if $d^\circ_{1c}(A) = 0$: then $A = \emptyset$ or ${}^-A = \emptyset$, and the result is straightforward.

if A is initializable: then $A \leq_{1c} B$ holds. To see this, towards a contradiction, assume that $A \not\leq_{1c} B$ holds. From $A \not\leq_{1c} B$, one gets $B \leq_{1c} {}^-A$. By proposition 19, this leads to $B^\perp \leq_{1c} ({}^-A)^\perp \leq_{1c} {}^-(A^\perp)$, which gives $A^\perp \leq_{1c} {}^-(A^\perp)$, showing that $A^\perp$ is 1c-self dual, a contradiction since A is initializable.

if A is 1c-non self dual but not initializable: then by proposition 27 there exists

$$A' \subseteq \Sigma_A^\omega, C \subseteq \Sigma_C^\omega \text{ such that } \begin{cases} A', C <_{1c} A \\ A'^{\longrightarrow} A' \leq_{1c} A' \\ C \geq_{1c} {}^-A' \\ A \equiv_{1c} A'^{\longrightarrow} C. \end{cases}$$

Since operation ${}^\perp$ preserves $\leq_{1c}$, we have both $A'^\perp \leq_{1c} B^\perp$ and $C^\perp \leq_{1c} B^\perp$. Since $A', C <_{1c} A$ holds, we can apply the induction hypothesis, which leads to $A' \leq_{FP_{I\!I}} B$ and $C \leq_{FP_{I\!I}} B$. From a w.s. for $I\!I$ in $\mathbf{FP}_{I\!I}(A'\ ,\ B)$ and a w.s. for $I\!I$ in $\mathbf{FP}_{I\!I}(C\ ,\ B)$, it is obvious to design a w.s. for $I\!I$ in $\mathbf{FP}_{I\!I}(A'^{\longrightarrow} C\ ,\ B)$, and finally a w.s. for $I\!I$ in $\mathbf{FP}_{I\!I}(A\ ,\ B)$.

if A is 1c-self dual: then, by proposition 24, there exists $(A_i)_{i\in I}$, $I \subseteq \Sigma_A^+$ as in definition 23, each $A_i \subseteq \Sigma_A^\omega$ 1c-non self dual and $A_i <_{1c} A$ such that

$$A \equiv_{1c} \sum_{i\in I} A_i.$$

Since $\leq_{1c}$ is preserved by operation ${}^\perp$, ${A_i}^\perp \leq_{1c} B^\perp$ holds for every $i \in I$. And since $A_i <_{1c} A$ is satisfied, by induction hypothesis, we get $A_i \leq_{FP_{I\!I}} B$ for each $i \in I$. Given winning strategies for $I\!I$ in games $\mathbf{FP}_{I\!I}(A_i\ ,\ B)$, it is easy to get a w.s. for $I\!I$ in $\mathbf{FP}_{I\!I}\left(\sum_{i\in I} A_i\ ,\ B\right)$, from which we derive a w.s. for $I\!I$ in $\mathbf{FP}_{I\!I}(A\ ,\ B)$.

(c)$\Longrightarrow$(d): given "$\curvearrowleft$" a letter not in Σ_B that symbolizes the erasing process, the w.s. for $I\!I$ in $\mathbf{FP}_{I\!I}(A\ ,\ B)$ can be regarded as a w.s. for $I\!I$ in the Wadge game $\mathbf{W}(A\ ,\ B')$ where B' is a subset of $(\Sigma_B \cup \{\curvearrowleft\})^\omega$ defined by $B' = B \cup (\Sigma_B \cup \{\curvearrowleft\})^* \cdot \curvearrowleft \cdot B$. For each integer n we define C_n as the subset of $(\Sigma_B \cup \{\curvearrowleft\})^\omega$ of all infinite words that contain at most n occurences of the letter "$\curvearrowleft$". We remark that C_n is a closed subset of $(\Sigma_B \cup \{\curvearrowleft\})^\omega$. We assume that σ is a w.s. for $I\!I$ in $\mathbf{W}(A\ ,\ B')$, and $\sigma*$ is the continuous function induced by this strategy. For each integer n, $\sigma *^{-1} C_n$ is a closed subset of Σ_A^ω. We describe a $\langle ? \rangle$-wellfounded question tree $\mathcal{T}$ on Σ_A by letting *II*, in the game $\mathbf{W}(A\ ,\ B)$, directly ask player *I* questions about whether or not *I*'s run will end in some closed subset of Σ_A^ω.

Player *II* first asks *I* if he will remain in $\sigma *^{-1} C_0$, if *I* answers "yes", then *II* knows for sure that applying σ, he will never need to play the eraser, therefore he might as well use σ in $\mathbf{W}\left(A^{\mathcal{T}_A}\ ,\ B\right)$, and he certainly doesn't need to ask any more questions to win the game. If *I* answers "no", then *II* asks *I* if he will remain in $\sigma *^{-1} C_1$, if he will remain in $\sigma *^{-1} C_2$, and so on, until *I* answers "yes" to a question of the form "will you remain in $\sigma *^{-1} C_n$. At that point, *II* doesn't need any more answers, all he needs is to wait long enough so that *I* reaches a position u in the main run, that verifies $\sigma * (u) \in (\Sigma_B^* \cdot \curvearrowleft)^n$, then he just has to apply σ to *I*'s main run, forgetting about $\sigma * (u)$.

(d)$\Longrightarrow$(a): let $\mathcal{T}$ be a $\langle ? \rangle$-wellfounded question tree on Σ_A and σ be a w.s. for *II* in the game $\mathbf{W}\left(A^{\mathcal{T}}\ ,\ B\right)$. We recall that for any infinite word $x \in \Sigma_A^\omega$, there exists a unique infinite word $x' \in [\mathcal{T}]$ such that $\frac{x'}{2} = x$. We define the mapping $\phi :\ \Sigma_A^\omega \longmapsto \Sigma_B^\omega$ by $\phi(x) = \sigma * (x')$. By construction,

$$\phi(x) \in B \iff \sigma * (x') \in B \iff x' \in A^{\mathcal{T}} \iff x \in A.$$

Since $\phi^{-1}B = A$ holds, we only need to show that ϕ is 2-continuous. So, given O any open subset of Σ_B^ω, we need to show that $\phi^{-1}O \in \underset{\sim}{\mathbf{\Sigma}}^\circ_2 \cap \underset{\sim}{\mathbf{\Pi}}^\circ_2$. We know that $\phi^{-1}O$ is an open subset of $[\mathcal{T}]$, which means that there exists $(u_i)_{i\in J}$, countably many finite words from $\mathcal{T}$ such that $\sigma *^{-1} O = \bigcup_{i\in J} u_i \cdot [\mathcal{T}_{(u_i)}]$. We prove that $\phi^{-1}O$ (which is exactly $\frac{\sigma *^{-1} O}{2}$) is a $\mathbf{\Delta}^\circ_2$-subset of Σ_A^ω.

Since $\mathcal{T}$ is $\langle ? \rangle$-wellfounded, $\sigma *^{-1} O$ can be written $\sigma *^{-1} O = \bigcup_{i\in I} v_i \cdot \mathcal{T}_{(v_i)}$ where i is countable, each subtree $\mathcal{T}_{(v_i)}$ contains no question at all and finally $i \neq j \longrightarrow v_i \bot v_j$. We remark that

(a) $\frac{v_i \cdot \mathcal{T}_{(v_i)}}{2}$ is a closed subset of Σ_A^ω because it contains exactly all words in $\frac{v_i}{2} \cdot \Sigma_A^\omega$ that satisfy the conditions stipulated by all the auxiliary moves of v_i.

(b) If $i \neq j$, then there exists some $w \in \Sigma_A^*$ such that

$$\frac{v_i \cdot \mathcal{T}_{(v_i)}}{2} \cap w \cdot \Sigma_A^\omega = \emptyset \quad \text{and} \quad \frac{v_j \cdot \mathcal{T}_{(v_j)}}{2} \subseteq w \cdot \Sigma_A^\omega$$

or the same relation holds swapping i and j.

Therefore $\phi^{-1}O = \frac{\sigma *^{-1} O}{2} = \frac{\bigcup_{i\in I} v_i \cdot \mathcal{T}_{(v_i)}}{2} = \bigcup_{i\in I} \frac{v_i \cdot \mathcal{T}_{(v_i)}}{2}$ is a countable union of closed subsets, hence it is a $\underset{\sim}{\mathbf{\Sigma}}^\circ_2$-subset of Σ_A^ω. We only need to show that it is also a $\underset{\sim}{\mathbf{\Pi}}^\circ_2$-subset.

First we recall that closed sets are also $\underset{\sim}{\mathbf{\Pi}}^\circ_2$-sets and that a finite union of $\underset{\sim}{\mathbf{\Pi}}^\circ_2$-sets is $\underset{\sim}{\mathbf{\Pi}}^\circ_2$. We remark that any v_i can be sliced into three parts:

- the main run part: $\frac{v_i}{2}$,

- all commitments about remaining in closed subsets (this is when $v_i(k) = \langle ? \rangle$), that can be summarized in the commitment about staying in the intersection of these closed set (a closed set itself) that we denote C_i,
- all commitments about reaching further positions (this is when $v_i(k)$ is of the form $\langle ! \cdot u' \rangle$), that can also be summarized in saying that the future main run is restricted to a basic clopen set of the form $w_i \cdot \Sigma_A^\omega$, for some $w_i \in \Sigma_A^*$.

Finally we set

$$\phi^{-1}O \;=\; \frac{\sigma *^{-1} O}{2} \;=\; \bigcup_{i\in I} \frac{v_i \cdot \mathcal{T}_{(v_i)}}{2} = \underbrace{\bigcap_{i\in I} \underbrace{\left(\underbrace{\left(\bigcup_{\substack{j\in I\\ j\neq i}} \underbrace{w_j \cdot \Sigma_A^\omega}_{\mathbf{\Delta}_1^\circ} \right)}_{\mathbf{\Sigma}_1^\circ} \cup \underbrace{\left(\underbrace{w_j \cdot \Sigma_A}_{\mathbf{\Delta}_1^\circ} \cap \underbrace{C_i}_{\mathbf{\Pi}_1^\circ} \right)}_{\mathbf{\Pi}_1^\circ} \right)}_{\mathbf{\Delta}_2^\circ}}_{\mathbf{\Pi}_2^\circ}.$$

This shows that $\phi^{-1}O$ is $\mathbf{\Pi}_2^\circ$, therefore it is both $\mathbf{\Sigma}_2^\circ$ and $\mathbf{\Pi}_2^\circ$, which finishes the proof.

$\dashv$ 20

As a consequence, we see that

Corollary 30

(a) the partial ordering $<_{2c}$ is a pre-wellordering on Borel sets with antichains of length at most 2.

(b) The game $\mathbf{FP}_{II}(A\ ,\ B)$ is determined as soon as A and B are both Borel.

Proof of corollary 30: It is an immediate consequence of theorem 20, left to the reader.

$\dashv$ 30

Since $<_{2c}$ is wellfounded on Borel sets, we can define the $2c$-degree of a Borel set in a similar way we defined the $1c$-degree:

Definition 31 *Let $A \subseteq \Sigma_A^\omega$ be Borel,*

- $d_{2c}^\circ(A) = 0 \iff_{def} (A = \emptyset \ \vee\ A = \Sigma_A^\omega)$,
- *for $A >_{2c} \emptyset$, $d_{2c}^\circ(A) = \sup\{d_{2c}^\circ(B) :\ B <_{2c} A\}$.*

6 A brief description of the 2c-hierarchy

In a similar way 1c-self dual sets are defined, we say a set A is **2c-self dual** if $A \equiv_{2c} {}^{-}A$ holds. And we say a set is **2c-non self dual** if it is not 2c-self dual. We get this surprising following characterization of 2c-non self dual sets.

Proposition 32 *Let $A \subseteq \Sigma_A^\omega$ be Borel,*

$$A \text{ is 2c-non self dual} \iff A \text{ is initializable.}$$

Proof of proposition 32: Applying theorem 20 and proposition 28 we obtain:

$$A \not\leq_{2c} {}^{-}A \longleftrightarrow A^{\perp} \not\leq_{1c} ({}^{-}A)^{\perp} \longleftrightarrow A^{\perp} \not\leq_{1c} {}^{-}(A^{\perp}) \longleftrightarrow A \text{ is initializable.}$$

$\dashv$ 32

Another amazing result is that the weak ordering $\leq_{2c}$ can be replaced by the the much stronger one $\leq_{1c}$, as soon as one of the two sets involved in the relation is initializable.

Lemma 33 *Let $A \subseteq \Sigma_A^\omega$ and $B \subseteq \Sigma_B^\omega$ be both Borel,*

$$A \text{ or } B \text{ initializable} \implies (A \leq_{2c} B \longleftrightarrow A \leq_{1c} B)\,.$$

Proof of lemma 33: assuming A or B is initializable, it is enough to show that $A \leq_{2c} B \longrightarrow A \leq_{1c} B$ holds. Which comes to showing $A \leq_{FP_{II}} B \longrightarrow A \leq_{1c} B$, which is straightforward when at least one of the two sets involved is initializable.

$\dashv$ 33

In a very similar way we described 1c-self dual sets on the basis of 1c-non self dual one, we can give a description of 2c-self dual sets on the basis of initializable ones, that it on the basis of 2c-non self dual sets.

Proposition 34 *Let $A \subseteq \Sigma_A^\omega$ be Borel,*

$$A \text{ is 2c-self dual} \iff \text{there exists} \begin{cases} B <_{1c} A, \quad B \text{ initializable s.t. } A \equiv_{2c} {}^{\pm}B \\ \qquad\qquad\text{or} \\ \\ (B_i)_{i\in\mathbb{N}},\ \text{each } B_i <_{1c} A \text{ and initializable,} \\ \text{s.t. } \forall i \exists j \quad B_i <_{1c} B_j \text{ and } A \equiv_{2c} \sum_{i\in I} B_i. \end{cases}$$

This proposition can be summarized by saying that A is 2c-self dual iff there exists a countable set i and a sequence $(B_i)_{i\in\mathbb{N}}$ such that $A \equiv_{2c} \sum_{i\in I} B_i$ with each $B_i <_{1c} A$ and initializable, and either there exists i and j such that $B_i \leq_{1c} {}^{-}B_j$ and for all integer k $B_k \leq_{1c} B_i$ or $B_k \leq_{1c} B_j$, or the sequence $(B_i)_{i\in\mathbb{N}}$ is $<_{1c}$-unbounded.

Proof of proposition 34: The ($\Longleftarrow$) direction is obvious since A is clearly not initializable. We prove the ($\Longrightarrow$) direction. The proof goes by induction on $d^{\circ}_{1c}(A)$. By proposition 32, we know that A is 2c-non self dual iff A is not initializable.

(a) If A is 1c-self dual, then there exists a countable set i and 2c-non self dual sets $C_i <_{1c} A$ such that $A \equiv_{1c} \sum_{i\in I} C_i$. By induction hypothesis, there exists countable sets J_i and initializable sets $(B_{i_j})_{j\in J_i}$ such that $C_i \equiv_{2c} \sum_{j\in J_i} B_{i_j}$. Therefore $A \equiv_{2c} \sum_{i\in I}(\sum_{j\in J_i} B_{i_j})$ holds, and two cases appear:

- either $(B_{i_j})_{i\in I,\ j\in J_i}$ is $<_{1c}$-unbounded which gives the result.
- Or $(B_{i_j})_{i\in I,\ j\in J_i}$ is not $<_{1c}$-unbounded, in which case (since there cannot be a $<_{1c}$-maximal element otherwise A would be equivalent to that element hence initializable) there are two $<_{1c}$-maximal elements, one being equivalent to the complement of the other, which also gives the result.

(b) If A is 1c-non self dual but not initializable,then by proposition 27:

$$\text{there exists } B \subseteq \Sigma_A^\omega, C \subseteq \Sigma_C^\omega \text{ such that } \begin{cases} B, C <_{1c} A \\ B^{\longrightarrow} B \leq_{1c} B \\ C \geq_{1c} {}^{-}B \\ A \equiv_{1c} B^{\longrightarrow} C. \end{cases}$$

Clearly, since $A \equiv_{1c} B^{\longrightarrow}C$, one gets $A \equiv_{FP_{II}} \left\langle\begin{array}{l} \nearrow B \\ \searrow C \end{array}\right.$, hence $A \equiv_{2c} \left\langle\begin{array}{l} \nearrow B \\ \searrow C \end{array}\right.$. Applying the induction hypothesis, just like in case (a) gives the result .

$\dashv$ 34

Remark 35

- If B is initializable and $A \equiv_{2c} {}^{\pm}B$, then $A \geq_{1c} {}^{\pm}B$.
- If each B_i is initializable and $A \equiv_{2c} \sum_{i\in I} B_i$, then $A \geq_{1c} \sum_{i\in I} B_i$.

Proof of remark 35: Clearly, if B is initializable and $A \equiv_{2c} {}^{\pm}B$, then $A \geq_{1c} B$ holds (otherwise $A \geq_{2c} B$ would not hold) and $A \geq_{1c} {}^{-}B$ too for same reasons. Therefore $A \geq_{1c} {}^{\pm}B$ is verified. Same argument works for the second case.

$\dashv$ 35

So, describing the hierarchy comes to describing the hierarchy of initializable sets. The ones at the lowest level are $\emptyset$ and its complement (the whole space), and for the ω next ones, we present a simple operation that generates them.

Definition 36 *Let $A \subseteq \Sigma_A^\omega$, O_+, O_- two different letters not in Σ_A; set $\Sigma = \Sigma_A \cup \{O_+, O_-\}$,*

$$A^\diamond =_{def} A \ \cup \ \Sigma \cdot O_+ \cdot A \ \cup \ \Sigma \cdot O_- \cdot (\Sigma_A^\omega \setminus A)$$

This operation $A \longmapsto A^\diamond$ preserves the Wadge ordering. This means if $A \leq_{1c} B$ then $A^\diamond \leq_{1c} B^\diamond$. We also remark that $A^\diamond$ is necessarily initializable. We claim that it is (up to $\equiv_{1c}$ and complementation) the $<_{1c}$-least initializable set above A.

Proposition 37 *Let $A \subseteq \Sigma_A^\omega$ be Borel and $B \subseteq \Sigma_B^\omega$ be Borel,*

(a)

$$B <_{2c} A^\diamond \iff B \leq_{2c} {}^{\pm}A$$

(b) If B is initializable,

$$B <_{1c} A^\diamond \iff (B \leq_{1c} A \ \vee \ B \leq_{1c} {}^{-}A)$$

Proof of proposition 37:

(a) The $(\Longleftarrow)$ direction is straightforward. We prove the $(\Longrightarrow)$ direction. By hypothesis, $B <_{2c} A^\diamond$ is satisfied, but since $A^\diamond$ is initializable, $<_{2c}$ can be replaced by $<_{1c}$. Hence $B \leq_{1c} A^\diamond$ and $B \leq_{1c} {}^{-}(A^\diamond)$ holds. We set $A' \subseteq \Sigma^\omega$ by $A' = A^\diamond \cup (\Sigma_A^* \cdot \{O_+, O_-\})^\omega$. So that A' is just $A^\diamond$ with all infinite words with infinitely many letters not in Σ_A. Clearly, $A' \equiv_{1c} {}^{-}(A^\diamond)$. Now, from a w.s. for *II* in $\mathbf{W}(B\ ,\ A^\diamond)$ and a w.s. for *II* in $\mathbf{W}(B\ ,\ A')$, it is easy to derive a w.s. for *II* in $\mathbf{W}(B\ ,\ A^\diamond)$ that **never** plays infinitely many letters not in Σ_A (hence it is also a w.s. for *II* in $\mathbf{W}(B\ ,\ A')$. Then, from such a strategy one gets a w.s. for *II* in $\mathbf{FP}_{II}(B\ ,\ {}^{\pm}A)$, which shows that $B \leq_{FP_{II}} {}^{\pm}A$. Therefore $B \leq_{2c} {}^{\pm}A$ holds.

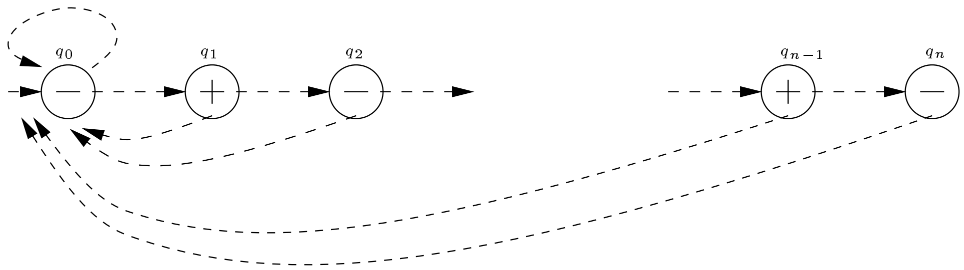
(b) The $(\Longleftarrow)$ direction is straightforward. We prove the $(\Longrightarrow)$ direction. We proceed as in (a). By hypothesis, $B \leq_{1c} A^\diamond$ and $B \leq_{1c} {}^{-}(A^\diamond)$ is satisfied, we set $A' \subseteq \Sigma^\omega$ by $A' = A^\diamond \cup (\Sigma_A^* \cdot \{O_+, O_-\})^\omega$. So that A' is just $A^\diamond$ with all infinite words with infinitely many letters not in Σ_A. And from winning strategies for *II* in respectively $\mathbf{W}(B\ ,\ A^\diamond)$ and $\mathbf{W}(B\ ,\ A')$, we get a w.s. for *II* in $\mathbf{W}(B\ ,\ A^\diamond)$ that always plays finitely many letters not in Σ_A. From it, one gets a w.s. for *II* in $\mathbf{FP}_{II}(B\ ,\ {}^{\pm}A)$, which shows that $B \leq_{FP_{II}} {}^{\pm}A$. Knowing that B is initializable leads to $B \leq_{1c} {}^{\pm}A$, hence $B \leq_{1c} A$ or $B \leq_{1c} {}^{-}A$.

$\dashv$ 37

Corollary 38 $A^\diamond$ is, up to $\equiv_{1c}$ and complementation, the next initializable set above A.

We remark that given any deterministic automaton $\mathcal{A}$, any automaton $\mathcal{B}$ that bears the same graph as $\mathcal{A}$ satisfies $\ulcorner\mathcal{A}\urcorner \equiv_{1c} \ulcorner\mathcal{B}\urcorner$, hence $\ulcorner\mathcal{A}\urcorner \equiv_{2c} \ulcorner\mathcal{B}\urcorner$. So, when dealing with Deterministic Automata, only the graph of the automaton matters. Any two labelling that both make sure they corespond to Deterministic Automata are equivalent. So from now on, we identify the graph of the automaton with the automaton itself.

Definition 39 *For each $n \in \mathbb{N}$ we define $\ominus^n$ as the following (graph of an) automaton:*



It has $n+1$ states: $q_0, q_1, \ldots, q_{n-1}, q_n$, with q_0 being the initial state. For each state q_i there is exactly an arrow going from q_i to q_{i+1} (except for q_n) and an arrow from q_i to q_0. And the set of accepting sets of states is $\{\{q_i :\ i < k\} :\ k \leq n \ \wedge\ k = n \mod 2\}$. So that $\ominus^0$ is the automaton with a single state that rejects every infinite word, hence recognizes the empty set.

We remark that, whatever the labelling is, the set accepted by the automaton $\ominus^n$ is initializable (every position is $\equiv_{1c}$ to the initial one).

Proposition 40 *Let* $A \subseteq \Sigma_A^\omega$,

$$(A \equiv_{1c} \ulcorner\ominus^{n}\urcorner \quad \vee \quad A \equiv_{1c} {}^{-}\ulcorner\ominus^{n}\urcorner \quad \Longrightarrow \quad A^{\diamond} \equiv_{1c} \ulcorner\ominus^{n+1}\urcorner$$

Proof of proposition 40: By playing both underlying Wadge games, the proof is straightforward and left to the reader. $\dashv$ 40

Proposition 41 *Let* $\mathcal{A}$ *be a DA,*

$$\mathcal{A} \textit{ has exactly } n+1 \textit{ states} \quad \Longrightarrow \quad (\ulcorner\mathcal{A}\urcorner \leq_{1c} \ulcorner\ominus^{n}\urcorner \quad \vee \quad \ulcorner\mathcal{A}\urcorner \leq_{1c} {}^{-}\ulcorner\ominus^{n}\urcorner$$

Proof of proposition 41: The proof goes by induction on n.

- If $n = 0$ then $\ulcorner\mathcal{A}\urcorner = \emptyset$ or ${}^{-}\ulcorner\mathcal{A}\urcorner = \emptyset$ which gives the result.
- We assume the hypothesis holds for $n - 1 \geq 0$. The set of states of $\mathcal{A}$ is $Q = \{q'_o, \ldots, q'_n\}$ and we assume w.l.o.g. that it is rejecting ($Q \notin \mathfrak{F}$, otherwise we work with ${}^{-}\mathcal{A}$ instead of $\mathcal{A}$). We pick any state q'_i and we define $\mathcal{A}_i$ as the DA which graph contains all states from Q except q'_i and with an arrow going from any states to any state, with the accepting conditions $\mathfrak{F}_i = \{R \setminus \{q'_i\} : \ R \in \mathfrak{F}\}$. The initial state of $\mathcal{A}_i$ being any state (it really doesn't matter since from any two states, there is a path going from one to the other and back again).

  Clearly, $\mathcal{A}_i$ is (the graph of a) DA with n states. Therefore, by induction hypothesis, one has $\ulcorner\mathcal{A}_i\urcorner \leq_{1c} \ulcorner\ominus^{n-1}\urcorner$ or $\ulcorner\mathcal{A}_i\urcorner \leq_{1c} \ulcorner\ominus^{n-1}\urcorner$. Hence, by proposition 40, $\ulcorner\mathcal{A}_i\urcorner \leq_{1c} \ulcorner\ominus^{n}\urcorner$ holds. Elsewhere, as long as player I doesn't reach a position that coresponds to being in state q'_i, there is a winning strategy for $I\!I$ in $\mathbf{W}(\ulcorner\mathcal{A}\urcorner, \ulcorner\mathcal{A}_i\urcorner)$. Hence, with same restrictions for I there is a winning strategy σ_i for $I\!I$ in $\mathbf{W}(\ulcorner\mathcal{A}_i\urcorner, \ulcorner\ominus^{n}\urcorner)$. Now, granted with the strategies σ_i's, we build the following strategy for $I\!I$ in $\mathbf{W}(\ulcorner\mathcal{A}\urcorner, \ulcorner\ominus^{n}\urcorner)$:

  (a) $I\!I$ starts with $j = 1$ and

  (b) let $i = j \mod n+1$ (with $i \leq n$). As long as I doesn't reach a position that coresponds to being in state q'_i, $I\!I$ applies the strategy σ_i. If I reaches a position that coresponds to being in state q'_i, then $I\!I$ reaches a position that coresponds to going first in the right mosts state of $\ominus^n$ (the state q_n) and then going back to the initial state (q_0); then $I\!I$ sets $j = i + 1$ and starts (b) all over again.

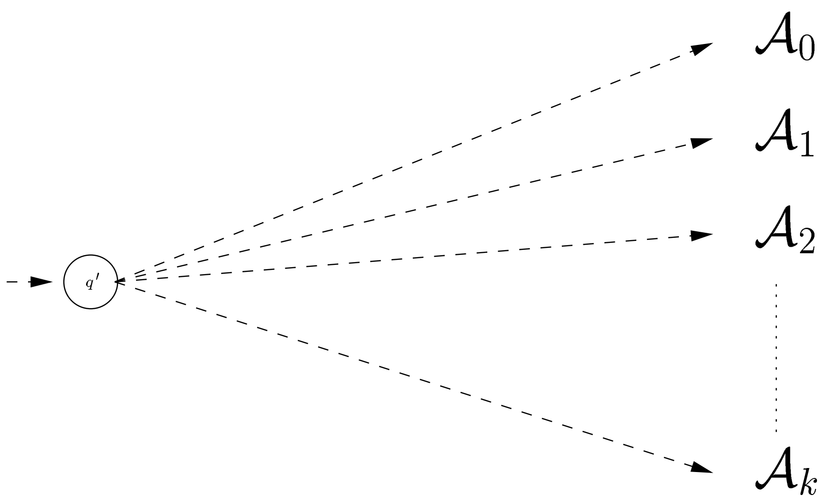
  This strategy is clearly winning for $I\!I$.

$\dashv$ 41

7 Computing the 2c-degree of an automaton

Remark 42 Given $\mathcal{A} = (Q, \Sigma, \Gamma_\perp, q_{init}, \delta, \mathfrak{F})$ a DA, we consider the equivalence relation on Q defined by $q \equiv q'$ iff there is a path in the graph of $\mathcal{A}$ going from q to q' and back to q. Let $Q_0, \ldots Q_k$ be the equivalence classes of Q. For each such class Q_i, we define the subautomaton $\mathcal{A}_i = (Q_i, \Sigma, \Gamma_\perp, q_{init_i},, \delta_i, \mathfrak{F})$ where q_{init_i} is any n state from Q_i and δ_i is any transition function from $Q_i \times \Sigma$ to Q_i that makes it a deterministic automaton, once given into account that the graph of $\mathcal{A}_i$ is exactly the graph of $\mathcal{A}$ restricted to the set of states Q_i (i.e. only arrows going from q to q', with both q and q' are preserved). The set of accepting conditions is $\{Q' \in \mathfrak{F} : \ Q' \subseteq Q_i\}$.

Figure 3: $\mathcal{B}$



Let $\mathcal{B}$ be any deterministic automaton such that its graph is like in next figure:
There is one new state q', that is not in any Q_i, and that stands for the initial state of $\mathcal{B}$. For any i there is one arrow going from q' to q_{init_i}, the initial state of $\mathcal{A}_i$ (and also the whole graph of $\mathcal{A}_i$).
It is clear that $I\!I$ has winning strategies in both games $\mathbf{FP}_{I\!I}(\ulcorner\mathcal{A}\urcorner, \ulcorner\mathcal{B}\urcorner)$ and $\mathbf{FP}_{I\!I}(\ulcorner\mathcal{B}\urcorner, \ulcorner\mathcal{A}\urcorner)$ which proves that $\ulcorner\mathcal{A}\urcorner \equiv_{2c} \ulcorner\mathcal{B}\urcorner$ holds. We remark also that every $\ulcorner\mathcal{A}_i\urcorner$ is initializable, and that there are only two possibilities: either there is an integer i such that $\ulcorner\mathcal{A}_i\urcorner \leq_{1c} \ulcorner\mathcal{A}_j\urcorner$ holds for all j (in which case $\ulcorner\mathcal{A}\urcorner \equiv_{2c} \ulcorner\mathcal{A}_i\urcorner$), or there are two integers i, j with $\ulcorner\mathcal{A}_i\urcorner \equiv_{1c} {}^{-}\ulcorner\mathcal{A}_j\urcorner$ and $\ulcorner\mathcal{A}_k\urcorner \leq_{1c} \ulcorner\mathcal{A}_i\urcorner$ or $\ulcorner\mathcal{A}_k\urcorner \leq_{1c} \ulcorner\mathcal{A}_j\urcorner$ holds for all h (in which case $\ulcorner\mathcal{A}\urcorner \equiv_{2c} {}^{\pm}\ulcorner\mathcal{A}_i\urcorner$).

So, we see the whole problem of determining the class of an automaton reduces to determining the class of the subautomata $\mathcal{A}_i$ (a work that was already done in [18]). We briefly restate it.

Definition 43 *Given an automaton $\mathcal{A} = (Q, \Sigma, \Gamma_\perp, q_{init}, \delta, \mathfrak{F})$, so that for any two states $q, q' \in Q$ there is a path going from q to q' and back to q,*

(a) a **loop** *is some $Q' = \{q_{i_0}, \dots, q_{i_l}\} \subseteq Q$ such that for $(a_0, \dots, a_l)$, some letters from Σ: $\delta(a_j, q_{i_j}) = q_{i_{j+1}}$ holds for all $j < l$ and $\delta(a_l, q_{i_l}) = q_{i_0}$.*

(b) Let $\mathfrak{L}$ be the lattice of all loops of $\mathcal{A}$ ordered by inclusion. For any loop Q', we define $ht_{\mathfrak{L}}(Q')$ by

- *if there is no loop $Q'' \subsetneq Q'$ then $ht_{\mathfrak{L}}(Q') = 0$*
- *otherwise,*
  - *if $Q' \in \mathfrak{F}$ then* $ht_{\mathfrak{L}}(Q') = max \left\{ \begin{array}{c} \{ht_{\mathfrak{L}}(Q'') \;:\; Q'' \subsetneq Q' \;\wedge\; Q'' \in \mathfrak{F}\} \\ \cup \\ \{ht_{\mathfrak{L}}(Q'') + 1 : \; Q'' \subsetneq Q' \;\wedge\; Q'' \notin \mathfrak{F}\} \end{array} \right.$

- *if* $Q' \notin \mathfrak{F}$ *then* $ht_{\mathfrak{L}}(Q') = max \begin{cases} \{ht_{\mathfrak{L}}(Q'')+1 : \; Q'' \subsetneq Q' \;\wedge\; Q'' \in \mathfrak{F}\} \\ \qquad\qquad\cup \\ \{ht_{\mathfrak{L}}(Q'') \quad : \; Q'' \subsetneq Q' \;\wedge\; Q'' \notin \mathfrak{F}\} \end{cases}$

Proposition 44 *Given an automaton* $\mathcal{A} = (Q, \Sigma, \; \Gamma_{\perp}, \; q_{init}, \; \delta, \; \mathfrak{F})$*, so that for any two states* $q, q' \in Q$ *there is a path going from* q *to* q' *and back to* q*,*

$$\ulcorner\mathcal{A}\urcorner \equiv_{1c} \ulcorner\ominus^{ht_{\mathfrak{L}}(Q)}\urcorner \textit{ if } Q \notin \mathfrak{F} \textit{ and } \ulcorner\mathcal{A}\urcorner \equiv_{1c} {}^{-}(\ulcorner\ominus^{ht_{\mathfrak{L}}(Q)}\urcorner \textit{ if } Q \in \mathfrak{F}.$$

Proof of proposition 44: the proof goes by induction on $ht_{\mathfrak{L}}(Q)$.

If $\boldsymbol{ht_{\mathfrak{L}}(Q) = 0}$:

- if $Q \in \mathfrak{F}$, then every loop is accepting, so that the automaton accepts every word. which means $\ulcorner\mathcal{A}\urcorner \equiv_{1c} {}^{-}\ominus$.
- if $Q \notin \mathfrak{F}$, then every loop is rejecting, so that the automaton recognizes the empty set. Hence $\ulcorner\mathcal{A}\urcorner \equiv_{1c} \ominus$.

If $\boldsymbol{ht_{\mathfrak{L}}(Q) = n+1}$:

- if $Q \in \mathfrak{F}$, then we get all maximal loops $Q_0, \ldots, Q_p$ that satisfy $ht_{\mathfrak{L}}(Q_i) = n$ **and** $Q_i \notin \mathfrak{F}$. By maximal loop, we mean there is no loop Q' that satisfy both $Q_i \subsetneq Q'$ and $ht_{\mathfrak{L}}(Q_i) = ht_{\mathfrak{L}}(Q_i) = n$ **and** both Q_i and Q' are not in $\mathfrak{F}$. (We also notice that there exists at least one such Q_i, otherwise $ht_{\mathfrak{L}}(Q) = n+1$ would fail). For each $i \leq p$, define $\mathcal{A}_i$ as any deterministic automaton which graph is the same as the graph of $\mathcal{A}$ restricted to the sole states in Q_i, and with initial state, any of the states in Q_i. The conditions of acceptance of $\mathcal{A}_i$ are the same ones of $\mathcal{A}$, which means it is $\mathfrak{F}_i = \{Q' \cap Q_i : \; Q' \in \mathfrak{F}\}$. By induction hypothesis, for each $i \leq p$, $\ulcorner\mathcal{A}_i\urcorner \equiv_{1c} \ulcorner\ominus^{ht_{\mathfrak{L}}(Q_i)}\urcorner = \ulcorner\ominus^{n}\urcorner$ holds.

  Showing that $\ulcorner\mathcal{A}\urcorner \geq_{1c} {}^{-}\ulcorner\ominus^{n+1}\urcorner$ is satisfied is almost immediate, once considered that both $\ulcorner\mathcal{A}\urcorner \geq_{1c} \ulcorner\ominus^{n}\urcorner$ and $\ulcorner\mathcal{A}\urcorner \geq_{1c} {}^{-}\ulcorner\ominus^{n}\urcorner$ hold and that $\ulcorner\mathcal{A}\urcorner$ is initializable:

  - $\ulcorner\mathcal{A}\urcorner \geq_{1c} \ulcorner\ominus^{n}\urcorner$ is satisfied since $\ulcorner\mathcal{A}\urcorner \geq_{1c} \ulcorner\mathcal{A}_i\urcorner \geq_{1c} \ulcorner\ominus^{n}\urcorner$ is verified, and
  - $\ulcorner\mathcal{A}\urcorner \geq_{1c} {}^{-}\ulcorner\ominus^{n}\urcorner$ holds since there is a w.s. for *II* in the game $\mathbf{W}\,(\ulcorner\mathcal{A}_i\urcorner\,,\,\ulcorner\mathcal{A}\urcorner)$ (for any $i \leq p$), which consists in, every time player *I* runs through a loop $Q' \subseteq Q_i$, then *II* runs through a loop $Q'' \subseteq Q$ that satisfies $ht_{\mathfrak{L}}(Q'') = ht_{\mathfrak{L}}(Q') + 1$ **and** $Q'' \in \mathfrak{F} \longleftrightarrow Q' \notin \mathfrak{F}$.

  On the other hand, in order to show that $\ulcorner\mathcal{A}\urcorner \leq_{1c} {}^{-}\ulcorner\ominus^{n+1}\urcorner$ holds, we design the following strategy for *II* in $\mathbf{W}\left(\ulcorner\mathcal{A}\urcorner\,,\,{}^{-}\ulcorner\ominus^{n+1}\urcorner\right)$:

  (a) start with $i = 0$,

  (b) as long as *I* only reaches states that are among Q_i, then apply a winning strategy for *II* in $\mathbf{W}\,(\ulcorner\mathcal{A}_i\urcorner\,,\,\ulcorner\ominus^{n}\urcorner)$ - this strategy never reaches the right most state of ${}^{-}(\ominus^{n+1})$. If *I* reaches a state that is not in Q_i, then *II* goes through the right most state of ${}^{-}(\ominus^{n+1})$ and *II* starts again (b) with $i = i+1$ if $i < p$ and $i = 0$ otherwise.

  This strategy is winning because, either there is an integer i such that *I* remains inside Q_i after a certain time, in which case *II* wins, or there is no such integer i, so there is a minimal loop Q' such that *I* runs through this loop infinitely many times.

This loop satisfies $Q' \not\subseteq Q_i$ for any $i \leq p$, hence since each Q_i is maximal (in the sense mentionned above), we know that $Q' \in \mathfrak{F}$ holds, which proves that the strategy works in this case.

- if $Q \notin \mathfrak{F}$, then the proof is the same *mutatis mutandis.*

$\dashv$ 44

Theorem 45 *Let $\mathcal{A}$ be a DA and $n \in \mathbb{N}$,*

(a) *(i)* $d^{\circ}_{2c}(\ulcorner\mathcal{A}\urcorner) = 2n$ *iff* $\ulcorner\mathcal{A}\urcorner \equiv_{2c} \ulcorner\ominus^{n}\urcorner$ *or* $^{-}\ulcorner\ominus^{n}\urcorner$

(ii) $d^{\circ}_{2c}(\ulcorner\mathcal{A}\urcorner) = 2n+1$ *iff* $\ulcorner\mathcal{A}\urcorner \equiv_{2c} {}^{\pm}\ulcorner\ominus^{n}\urcorner$.

(b) $d^{\circ}_{2c}(\ulcorner\mathcal{A}\urcorner) < \omega$,

Proof of proposition 45: The proof is an immediate consequence of propositions 37, 40 and 41.

$\dashv$ 45

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