Game Logic is Strong Enough for Parity Games

Dietmar Berwanger*
Mathematische Grundlagen der Informatik
RWTH Aachen

Abstract

This paper is concerned with the expressive power of Parikh's Game Logic interpreted in Kripke structures. We show that the syntactical alternation hierarchy of this logic is strict, by expressing the winning condition for parity games with n priorities. Moreover, it follows that Game Logic is not captured by any finite level of the μ-calculus alternation hierarchy. We can further conclude that model checking for the μ-calculus is efficiently solvable iff this is possible for Game Logic.

1 Introduction

Game Logic was developed by Parikh [Par85] for reasoning about neighborhood models, a generalisation of Kripke structures, viewing accessibility as a relation between *sets* of worlds. His formalism extends propositional logic by addition of modal operators whose meanings are assigned by games. Specifically, path-forming games for two players are described and the modality reflects the issue of a player to have a winning strategy.

Broadening the game-theoretic perspective, the preservation properties for bisimulation shown by Pauly [Pau00] conveyed Game Logic to the scope of process theory. Since bisimulation captures the idea of behavioral process equivalence, the framework of program specification and verification relies on description formalisms which respect this equivalence. In that framework, program states are often associated to the worlds of a Kripke structure while the state transitions are modeled by the (binary) accessibility relation.

When interpreted in Kripke structures, as we do in this paper, Game Logic (GL) resembles the Propositional Dynamic Logic (PDL) of Fischer and Ladner. Actually, the only game construction of GL not shared by PDL is the *dualization* operator, expressing a role interchange between the players. However, as we shall prove, this ability of modal negation, invests GL with hardly foreseeable expressive power.

A useful yardstick for the measure of expressiveness over Kripke models is provided by the modal μ-calculus (L_μ), the extension of basic modal logic by least and greatest fixed point constructions. On the one hand, as demonstrated by Janin and Walukiewicz [JW96], this logic is as strong as monadic second-order logic in describing programs up to bisimulation equivalence. On the other hand, L_μ displays a hierarchical structure, induced by the number of syntactical

*E-mail: berwanger@informatik.rwth-aachen.de

alternations between fixed point operators, which Bradfield [Bra96] and Lenzi [Len96] have shown to be semantically strict.

Interestingly, most of the formalisms commonly used for process description allow translations into low levels of the L_μ alternation hierarchy. On its first level this hierarchy already captures, for instance, PDL as well as CTL, another popular process logic, while their expressive extensions ΔPDL and CTL* do not exceed the second level.

Still, the low levels of the hierarchy do not exhaust the significant properties expressible in L_μ. A comprehensive example of formulae strictly distributed over all levels of the alternation hierarchy regards parity games. Thus, strictly on level n, there is a formula stating that the first player has a winning strategy in parity games with n priorities.

Clearly, Game Logic over Kripke structures is subsumed by L_μ. Moreover, it turns out that every GL-formula can be translated into an equivalent L_μ-formula which uses at most two monadic variables. Though, by alternately nesting the scope of the two variables, formulae of any syntactical alternation level can be obtained. But note, that the syntactical appearance of a formula does not provide direct evidence of its hierarchical position. For instance, the fragment L_2 of the μ-calculus, introduced in [EJS93], covers all levels of the syntactic hierarchy, but, as it was shown in [NS99], all its formulae can be equivalently translated into formulae on the second level. In the light of this antecedence and the aforementioned results about process logics, one might conjecture that

(1) the two-variable fragment of L_μ is subsumed by a finite level of the alternation hierarchy and, therefore,

(2) GL is subsumed by a finite level of the L_μ alternation hierarchy.

We prove that this is not the case. GL, and hence also the two-variable fragment of L_μ are sufficiently powerful to express winning conditions of parity games with arbitrary high indices. Thus, GL contains formulae at arbitrary levels of the alternation hierarchy of the μ-calculus, and the alternation hierarchy of GL is strict.

1.1 Outline

In the background section, we give a brief introduction into Game Logic and parity games. We assume that the reader is familiar with the μ-calculus. For an extensive survey see, e. g., [AN01] or [Eme97].

In the third section we show that in L_μ the formula expressing that the first player has a winning strategy in parity games with n priorities can be equivalently written with only two variables. As a consequence, we obtain that the alternation hierarchy of two-variable fragment of L_μ is strict and not subsumed by any finite level of the L_μ alternation hierarchy.

Towards a common framework for comparing GL to L_μ, we introduce, in Section 4, an interpretation of Game Logic in terms of parity games. At hand with this, we construct in Section 5 a GL-formalization of the winning condition in parity games with n priorities. Besides strictness of the GL-alternation hierarchy, this construction also shows that model checking for L_μ can be done in polynomial time iff efficient model checking for Game Logic is possible.

2 Background

2.1 Game Logic over Kripke structures

Definition 1. Departing from a set P of atomic propositions p and a set G of atomic games g, the formulae of GL are composed according to the following rules

$$\varphi := \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi \ ,$$
$$\gamma := g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^{\mathrm{d}}.$$

Throughout this paper we consider interpretations of these formulae over unimodal Kripke structures $\mathcal{A}, v$ with $\mathcal{A} = (A, E, (A_p)_{p\in P})$. Consequently, G consists of a single atomic game g.

Intuitively, the meaning of GL-formulae in a rooted Kripke structure can be understood by way of games between two players, angel and demon. Typically, $\langle\gamma\rangle\varphi$ expresses that angel has a strategy to play the game γ starting at v in such a way that either φ is true when the game ends, or the game breaks and demon fails. The game-forming rules can be read as follows. $\gamma_1;\gamma_2$ means: play γ_1 first, then γ_2; in $\gamma_1 \cup \gamma_2$ angel first chooses one of γ_1 and γ_2 and then the chosen game is played; in γ^*, the game γ is reiterated, while angel can decide before each round whether a new round is to be played; in $(\varphi?)$ an independent observer checks whether φ holds. If so, the play just ends, otherwise it breaks and angel looses. In the atomic game g angel can move to some position reachable from the current position. Finally, the dual game γ^{d} means that the two players swap their roles and then γ is played.

We will not formally define the semantics here. Instead, in Section 4 we provide an interpretation of GL in terms of parity games, which is equivalent to the standard semantics over Kripke structures as introduced in [Pau99].

To enhance readability we assign precedence to the operators: unary operators bind tighter than binary ones and ; binds tighter than $\cup$. Additionally, we define dual operators as a shorthand

$$\varphi_1 \wedge \varphi_2 := \neg(\neg\varphi_1 \vee \neg\varphi_2) \qquad \top := \neg\bot$$
$$\gamma_1 \cap \gamma_2 := (\gamma_1^{\mathrm{d}} \cup \gamma_2^{\mathrm{d}})^{\mathrm{d}} \qquad \gamma^\circ := (\gamma^{\mathrm{d}})^{*\,\mathrm{d}}$$

Using this notation, each game can be equivalently written in such a way that tests apply only to $\bot$, $\top$, atomic, or negated atomic propositions as follows:

$$(\varphi_1 \vee \varphi_2)? \equiv \varphi_1? \cup \varphi_2? \qquad (\neg\varphi)? \equiv (\varphi?^{\,\mathrm{d}}; \bot?) \cap \top?$$
$$(\varphi_1 \wedge \varphi_2)? \equiv \varphi_1? \cap \varphi_2? \qquad (\langle\gamma\rangle\varphi)? \equiv \gamma; \varphi?$$

This transformation allows us to detangle formulae and games for inductive reasoning. Further, we can exploit the equivalences

$$\neg\langle\gamma\rangle\varphi \equiv \langle\gamma^{\mathrm{d}}\rangle\neg\varphi \qquad \varphi?^{\mathrm{d}} \equiv (\neg\varphi?\,; \bot?^{\,\mathrm{d}}) \cup \top?$$
$$(\gamma_1;\gamma_2)^{\mathrm{d}} \equiv \gamma_1^{\mathrm{d}};\gamma_2^{\mathrm{d}} \qquad (\gamma^*)^{\mathrm{d}} \equiv (\gamma^{\mathrm{d}})^\circ$$

to bring any GL-formula into a *normal form* where negation applies only to atomic propositions, and game dualization only to atomic and surrender $(\bot?)$ games.

2.2 Parity games and the modal μ-calculus

Definition 2. The formulae of the modal μ-calculus L_μ are constructed with atomic propositions p from a given set P and propositional variables X from a given set Q according to the grammer:

$$\varphi := \bot \mid p \mid X \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \mu X.\varphi,$$

where the fixed point rule $\mu X.\varphi$ applies to formulae $\varphi(X)$ in which the free variable X appears only positively, that is, under an number of negations.

To define the meaning of a formula φ under a variable assignment χ in a given Kripke structure $\mathcal{A} = (A, E, (A_p)_{p\in P})$, we describe its *extension* $[\![\varphi]\!]_\chi$, that is, the set of worlds in $\mathcal{A}$ where φ holds when its free variables are interpreted according to χ. Thus, atomic propositions p extend to the sets A_p in $\mathcal{A}$, $\bot$ to the empty set, and the extension of free variables X is given by $\chi(X)$. For the propositional and modal operators we have

$$\begin{aligned}
[\![\neg\varphi]\!]_\chi &:= A \setminus [\![\varphi]\!]_\chi\,, \\
[\![\varphi_1 \vee \varphi_2]\!]_\chi &:= [\![\varphi_1]\!]_\chi \cup [\![\varphi_2]\!]_\chi\,, \\
[\![\Diamond\varphi]\!]_\chi &:= \{\, a \;:\; (\exists b.\,(a,b) \in E)\, b \in [\![\varphi]\!]_\chi \,\}\,.
\end{aligned}$$

For a formula $\mu X.\varphi$, consider the operator $\varphi(\cdot)$ which maps every $B \subseteq A$ to the extension $[\![\varphi]\!]_{\chi[X/B]}$ obtained with the assignment χ when the value of X is set to B. By the requirement on φ to contain X only positively, this operator is monotone and hence it has a least fixed point, by the Knaster-Tarski theorem. This provides the semantics of the μ operator:

$$[\![\mu X.\varphi]\!]_\chi := \bigcup \{\, B \subseteq A \;:\; B = [\![\varphi]\!]_{\chi[X/B]} \,\}.$$

The operators $\wedge$, $\Box$, and μ are introduced as abbreviations to the dual of $\vee$, $\Diamond$, and μ, respectively. In this way, $\mu X.\varphi := \neg\nu X.\neg\varphi[\neg X/X]$ is interpreted as the greatest fixed point of $\varphi(\cdot)$.

It is not hard to see that every GL-formula can be translated into an equivalent L_μ-sentence. Furthermore, the following construction, quoted from a private communication of Marc Pauly, shows that the image of this translation can be kept within the two-variable fragment of L_μ.

First, games are translated into L_μ-formulae with one free variable X or Y, which are repeatedly reused. This is accomplished by two mappings, $\cdot^X$ and $\cdot^Y$, between GL-games and L_μ-formulae:

$$\begin{aligned}
g^X &:= \Diamond X & \qquad g^Y &:= \Diamond Y \\
(\gamma_1 \cup \gamma_2)^X &:= \gamma_1^X \vee \gamma_2^X & (\gamma_1 \cup \gamma_2)^Y &:= \gamma_1^Y \vee \gamma_2^Y \\
(\gamma_1 ; \gamma_2)^X &:= \gamma_1^X[X := \gamma_2^X] & (\gamma_1 ; \gamma_2)^Y &:= \gamma_1^Y[Y := \gamma_2^Y] \\
(\varphi?)^X &:= \varphi^\sharp \wedge X & (\varphi?)^Y &:= \varphi^\sharp \wedge Y \\
(\gamma^d)^X &:= \neg\gamma^X[X := \neg X] & (\gamma^d)^Y &:= \neg\gamma^Y[Y := \neg Y] \\
(\gamma^*)^X &:= \mu Y.X \vee \gamma^Y & (\gamma^*)^Y &:= \mu X.Y \vee \gamma^X
\end{aligned}$$

At hand with these, the translation $\cdot^\sharp$ associates to any GL-formula an L_μ-sentence as follows:

$$\begin{aligned}
p^\sharp &:= p\\
(\neg\varphi)^\sharp &:= \neg\varphi^\sharp\\
(\varphi_1 \vee \varphi_2)^\sharp &:= \varphi_1^\sharp \vee \varphi_2^\sharp\\
(\langle\gamma\rangle\varphi)^\sharp &:= \gamma^X[X := \varphi^\sharp]
\end{aligned}$$

Please note, that the translation rule for choice games introduces two occurences of the free variable. In the substitution process for the game modality these are both replaced by the same expression, thus leading to a possibly exponential blow-up of the obtained L_μ-sentence. However, this phenomen can be avoided by translating into the equational μ-calculus rather than its linear variant.

An important framework for the analysis of L_μ is provided by parity games.

Definition 3. A *parity game* is given as a rooted Kripke structure $\mathcal{G}, v_0$ with

$$\mathcal{G} = (V, V_\Diamond, E, \Omega),$$

where V is a set of positions with a designated subset $V_\Diamond$, $E \subseteq V \times V$ is a transition relation, and $\Omega = (\Omega_i)_{1\leq i\leq n}$ is a coloring of V with priorities $1, \dots, n$ determining the winning condition. We denote the set $V \setminus V_\Diamond$ by $V_\Box$; the number n of priorities is called the *index* of $\mathcal{G}$.

In a play of $\mathcal{G}, v_0$ two players, $\Diamond$ and $\Box$, move a token along the transitions of E starting from v_0. Once a position v is reached, player $\Diamond$ performs the move if $v \in V_\Diamond$, otherwise player $\Box$. If the current position allows no further transitions, then the player in turn to move looses. In case this never happens, the play is infinite. Since there are finitely many priorities, the token will meet some of them infinitely often. We look at the least among these priorities: if it is an even number, player $\Diamond$ wins, otherwise he looses.

For these games we only need a simple notion of a strategy.

Definition 4. A *memoryless strategy* for player $\Diamond$ in the parity game $\mathcal{G}, v_0$ is a function $\sigma : V_\Diamond \to V$ assigning to each position $v \in V_\Diamond$ some successor $w \in vE$.

The strategy is *winning* if player $\Diamond$ wins every play of the game $\mathcal{G}^\sigma, v_0$ obtained by removing from $\mathcal{G}$ the transitions (v, w) from $v \in V_\Diamond$ to $w \notin \sigma(v)$.

Theorem 5 (Memoryless determinacy [EJ91])**.** *In any parity game, either player $\Diamond$ or player $\Box$ has a memoryless winning strategy.*

Parity games are tightly connected with the μ-calculus, some authors even say parity games *are* μ-calculus. This assumption has several reasons. For instance, as proven by Emerson, Jutla, and Sistla, model checking for the μ-calculus can be reduced to the problem of deciding the winner of a parity game.

Theorem 6 ([EJS93])**.** *For every structure $\mathcal{A}, v$ and formula $\psi \in \mathrm{L}_\mu$ a parity game $\mathcal{G}(\mathcal{A}, \psi, v)$ can be constructed in linear time, such that player $\Diamond$ has a winning strategy in this game iff $\mathcal{A}, v \models \psi$.*

For ψ given in an adequate normal form, so that negation applies to atomic propositions only and every fixed point variable has a unique binding definition, the game $\mathcal{G}(\mathcal{A}, \psi, v)$ can be described as follows. The set of positions is

$$V = \{(\varphi, a) \ : \ \varphi \text{ is a subformula of } \psi \text{ and } a \in A\},$$

whereof $V_\Diamond$ holds the positions $(\varphi_1 \vee \varphi_2, a)$, $(\Diamond\varphi, a)$, (p, a) with $a \notin A_p$, and $(\neg p, a)$ with $a \in A_p$. All plays start at (ψ, v) and transitions in E are such that

- no moves are possible from (α, a) where α is atomic or negated atomic;
- from $(\varphi_1 \vee \varphi_2, a)$ or $(\varphi_1 \wedge \varphi_2, a)$ transitions lead to (φ_1, a) and (φ_2, a);
- from $(\Diamond\varphi, a)$ or $(\Box\varphi, a)$ there are transitions to all positions (φ, b) with b a successor of a.
- from $(\mu T.\varphi, a)$ or $(\nu T.\varphi, a)$ there is a transition to (φ, a);
- from (T, a) with T a fixed point variable defined as $(\mu T.\varphi, a)$ or $(\nu T.\varphi, a)$ a transition leads to (φ, a).

In order to define the coloring Ω of V, let us assume that the fixed point variables of ψ appear in the order $T_1, \dots, T_k$. Then, each position (T_i, a) is assigned to priority $2i + 1$ if T_i is a least fixed point variable or to priority $2i + 2$ if T_i is a greatest fixed point variable. All remaining positions receive priority $2k + 2$. A sparer assignment could manage with n priorities when the formula ψ is on the n-th L_μ alternation level.

As a converse to the above result, Emerson and Jutla showed that the problem of establishing whether player $\Diamond$ has a winning strategy in a given parity game can be equivalently viewed as a model checking problem for L_μ.

Theorem 7 ([EJ91])**.** *There is a formula* $\mathrm{W}^n \in \mathrm{L}_\mu$*, such that in any parity game* $\mathcal{G}, v_0$ *with* n *priorities player* $\Diamond$ *has a winning strategy iff* $\mathcal{G}, v_0 \models \mathrm{W}^n$*.*

We give a variant of the formula W^n here.

For convenience, let us abbreviate the formula expressing that player $\Diamond$ can ensure that a position where φ holds is reached in one move by

$$\triangleright\varphi := (V_\Diamond \wedge \Diamond\varphi) \vee (V_\Box \wedge \Box\varphi).$$

Further, we write $\Omega_{>i}$ for $\bigvee_{k=i+1}^{n} \Omega_k$. Empty disjunctions, like $\Omega_{>n}$, are by default false. In longer formulae we sometimes omit the $\wedge$ symbol and write, e.g., $\Omega_i \triangleright\varphi$ for $\Omega_i \wedge \triangleright\varphi$.

For simplicity, let us assume that n is odd.

Definition 8. The formula W^n expressing that player $\Diamond$ has a winning strategy in a parity game with n priorities is

$$\mathrm{W}^n := \mu Z_1 \nu Z_2 \dots \mu Z_n. \bigvee_{i=1}^{n} \Omega_i \wedge \triangleright Z_i.$$

To understand this expression, let us consider the formulae $W_i(\varphi)$ describing those positions from which player $\Diamond$ can ensure that

1. either he wins while no priority less than i is ever played, or
2. some position where φ holds is being reached.

We obtain for odd i

$$W_i(\varphi) := \mu Z.W_{i+1}(\varphi \vee \Omega_i \wedge \triangleright Z),$$

and for even i

$$W_i(\varphi) := \nu Z.W_{i+1}(\varphi \vee \Omega_i \wedge \triangleright Z).$$

Thus, the above expression for W^n is given by $W_1(\bot)$.

As shown by Bradfield, these formulae are hard instances of the μ-calculus alternation hierarchy.

Theorem 9 ([Bra96])**.** *For any number n, the formula W^n is contained on the n-th level of the μ-calculus hierarchy but there is no formula equivalent to W^n at level $n-1$.*

3 Deciding the winner in parity games with two variables

In this section we rule out the expectation that the weakness of GL can be proved via its translation into the two-variable fragment of L_μ.

Given n, consider the following formulae for $i = 1, \dots, n$

$$\varphi_i(X) := \mu Y.\big((\Omega_i \wedge \triangleright Y) \vee (\Omega_{<i} \wedge X) \vee (\Omega_{>i} \wedge \varphi_{i+1}(Y))\big)$$

when i is odd and, otherwise,

$$\varphi_i(Y) := \nu X.\big((\Omega_i \wedge \triangleright X) \vee (\Omega_{<i} \wedge Y) \vee (\Omega_{>i} \wedge \varphi_{i+1}(X))\big).$$

By removing the conjunctions with $\Omega_{<1}$ and $\Omega_{>n}$ from φ_1 we obtain an L_μ-sentence which, obviously, does not use more than two variables. Let us denote this sentence by $\mathrm{W}^n_{(2)}$.

Example. For $n = 3$ we get

$$\mu X.\big(\Omega_1 \triangleright X \vee \Omega_{>1} \nu Y.\big(\Omega_2 \triangleright Y \vee \Omega_{<2} X \vee \Omega_{>2}(\mu X.\Omega_3 \triangleright X \vee \Omega_{<3} Y)\big)\big).$$

Proposition 10. *On every parity game of index n we have $\mathrm{W}^n_{(2)} \equiv \mathrm{W}^n$.*

Proof. For $\mathcal{A}, v$ a parity game of index n, let us consider the model checking games $\mathcal{G} := \mathcal{G}(\mathcal{A}, \mathrm{W}^n, v)$ and $\mathcal{G}_{(2)} := \mathcal{G}(\mathcal{A}, \mathrm{W}^n_{(2)}, v)$. Our aim is to show that Player $\Diamond$ either has a winning strategy in both games or in none of them.

Inspecting the structure of $\mathcal{G}$, we can observe that a winning strategy for player $\Diamond$ is completely specified by the transitions of type $(a, \Diamond Z_i) \to (b, Z_i)$ where i is the priority of a. At every other position (c, η) where $\Diamond$ has to choose, her choice is already prefigured in the label of c. Of course, knowing the meaning of W^n, it is no surprise that the choices in the game $\mathcal{G}$ boil down to choices in $\mathcal{A}$.

On the other hand, it is easy to verify that the actual choices in the game $\mathcal{G}_{(2)}$ are structure choices as well. Here, they are of the shape $(a, \Diamond Z) \to (b, Z)$ with $Z = X$ when the priority of a is even and $Z = Y$ otherwise.

But this means that any winning strategy of Player $\Diamond$ in $\mathcal{G}$ can be projected without loss onto a winning strategy on $\mathcal{A}$, which on his part completely describes a winning strategy in $\mathcal{G}_{(2)}$.

In the same way, it follows that a winning strategy for Player $\Diamond$ in $\mathcal{G}_{(2)}$ can be translated via $\mathcal{A}$ into a winning strategy in $\mathcal{G}$. $\square$

Since $\mathrm{W}^n_{(2)}$ and W^n are both on the n-th level of the L_μ-hierarchy the following result is immediate.

Theorem 11. *The alternation hierarchy of the two-variable fragment of* L_μ *is strict and not contained in any finite level of the full logic.*

4 Parity semantics for Game Logic

In order to provide a common ground for comparing Game Logic to L_μ we will give a reading of the semantics of GL in terms of parity games. In contrast to the standard semantics, we expect from this approach to give us access not only to the truth value of a GL-sentence in a given structure, but also to its justification in the form of a proof or rejection.

To be more precise, for a given structure $\mathcal{A}$ and a formula $\psi \in \mathrm{L}_\mu$, a winning strategy for player $\Diamond$ in the model checking game $\mathcal{G}(\mathcal{A}, \psi, v)$ can be viewed as a proof of $\mathcal{A}, v \models \psi$ in an interactive proof system. Acting as a prover, player $\Diamond$ can convince the verifier, player $\Box$, that ψ holds at v, by choosing according to its strategy whenever a disjunction or an existential subformula of ψ is considered. In the same way, a memoryless winning strategy for player $\Box$ can be seen as a rejection of $\mathcal{A}, v \models \psi$.

Thus, a model checking game for GL similar to the corresponding game for L_μ, could open the way for the translation of proofs of GL-formulae into proofs of L_μ-formulae and vice versa, in order to compare their model classes. As the standard semantics of GL the meaning of games is given by predicate transformers, functions on the powerset of the universe, a straightforward approach to verify the validity of a sentence by constructing such functions would be very inefficient. In contrast to these, winning strategies for model checking games are considerably smaller objects.

At this point we could rely on the games obtained via the translation of GL into L_μ, but we choose to avoid this detour for a better understanding.

Definition 12. Given a formula $\psi \in \mathrm{GL}$ in negation normal form, we define its *closure* $\mathrm{cl}(\psi)$ as the smallest set which contains ψ and is closed under the following operations:

(i) taking of subformulae: for each $\varphi \in \mathrm{cl}(\psi)$ any subformula η of φ is also contained in $\mathrm{cl}(\psi)$;

(ii) game choice: for each $\langle \gamma_1 \cup \gamma_2 \rangle \varphi \in \mathrm{cl}(\psi)$ we have $\{\langle \gamma_1 \rangle \varphi, \langle \gamma_2 \rangle \varphi\} \subseteq \mathrm{cl}(\psi)$ and likewise for $\cap$;

(iii) unrolling: for each $\langle \gamma^* \rangle \varphi \in \mathrm{cl}(\psi)$ also $\langle \gamma; \gamma^* \rangle \varphi \in \mathrm{cl}(\psi)$ and likewise for $^\circ$;

(iv) splitting: for each $\langle \gamma_1; \gamma_2 \rangle \varphi \in \mathrm{cl}(\psi)$ also $\langle \gamma_1 \rangle \langle \gamma_2 \rangle \varphi \in \mathrm{cl}(\psi)$.

It is not hard to verify that $|\mathrm{cl}(\psi)|$ is bounded by $O(|\psi|)$, where $|\psi|$ is the number of symbols in ψ.

Definition 13. To any rooted structure $\mathcal{A}, v_0$ and formula $\psi \in \mathrm{GL}$ we associate a parity game $\mathcal{G}(\mathcal{A}, \psi)$ with positions

$$V := \{(\varphi, a) \ : \ \psi \in \mathrm{cl}(\psi) \text{ and } a \in A\}.$$

Thereof, player $\Diamond$ holds all positions where the formula is of the shape

$$\bot \mid \varphi_1 \vee \varphi_2 \mid \langle \alpha ? \rangle \varphi \mid \langle g \rangle \varphi \mid \langle \gamma_1 \cup \gamma_2 \rangle \varphi \mid \langle \gamma^* \rangle \varphi$$

with α standing for $\bot$, $\top$ and atomic propositions, possibly negated. Additionally, $V_\Diamond$ includes

$$\{(p,a) \ : \ a \notin A_p\} \cup \{(\neg p, a) \ : \ a \in A_p\}.$$

The remaining positions belong to player $\Box$. All plays start at position (ψ, v_0). The transitions are given as follows.

- From positions $(\bot, a)$, $(\top, a)$, (p, a), or $(\neg p, a)$ no moves can be done.
- From $(\varphi_1 \vee \varphi_2, a)$ or $(\varphi_1 \wedge \varphi_2, a)$ two transitions lead to (φ_1, a) and (φ_2, a).
- From $(\langle \alpha ? \rangle \varphi, a)$ there is a transition to (φ, a), if one of the following holds:
  - α is $\top$ or $\top^{\mathrm{d}}$;
  - α is p or p^d and $a \in A_p$;
  - α is $\neg p$ or $(\neg p)^d$ and $a \notin A_p$.

  Otherwise, no moves can be done.
- From $(\langle \gamma_1 \cup \gamma_2 \rangle \varphi, a)$ transitions lead to $(\langle \gamma_1 \rangle \varphi, a)$ and $(\langle \gamma_2 \rangle \varphi, a)$.
- From $(\langle g \rangle \varphi, a)$ there are transitions to each of $\{(\varphi, b) \mid (a,b) \in E_g\}$.
- From $(\langle \gamma^* \rangle \varphi, a)$ or $(\langle \gamma^\circ \rangle \varphi, a)$ two transitions lead to (φ, a) and $(\langle \gamma ; \gamma^* \rangle \varphi, a)$.
- From $(\langle \gamma_1 ; \gamma_2 \rangle, a)$ there is a transition to $(\langle \gamma_1 \rangle \langle \gamma_2 \rangle \varphi, a)$.

In order to assign the priorities we first introduce a measure for the nesting of alternating stars in GL-games.

Definition 14. For GL-games in negation normal form we define the following *star alternation hierarchy*:

(i) The first level of the hierarchy, $\Sigma_0^* = \Pi_0^*$, consists of the *- and $^\circ$-free games.

(ii) For every higher level, Σ_{i+1}^* is formed by closing $\Sigma_i^* \cup \Pi_i^*$ under $;$, $\cap$, $\cup$ and *.

(iii) The level Π_{i+1}^* is obtained dually, by closing under $^\circ$ instead of *.

Only the positions of the form $(\langle \gamma^* \rangle \varphi, a)$ or $(\langle \gamma^\circ \rangle \varphi, a)$ receive significant priority colorings. Towards this, we look at the least i such that $\gamma \in \Sigma_i \cup \Pi_i$ and assign $(\langle \gamma^* \rangle \varphi, a)$ to Ω_{2i+1} or, respectively, $(\langle \gamma^\circ \rangle \varphi, a)$ to Ω_{2i+2}. All remaining positions are set to some irrelevantly high priority.

Instead of $\mathcal{G}(\mathcal{A}, \psi), (\psi, v_0)$ we will usually write $\mathcal{G}(\mathcal{A}, \psi, v_0)$.

Theorem 15. *A formula $\psi \in$ GL holds in a rooted structure $\mathcal{A}, v_0$ iff player $\Diamond$ has a winning strategy in the game $\mathcal{G}(\mathcal{A}, \psi, v_0)$.*

We will not give the proof in this place. A straightforward approach uses the translation of GL into L_μ.

Note, that the number of positions in $G(\mathcal{A}, \psi, v_0)$ is bounded by $O(|A| \cdot |\psi|)$. Since the problem, whether player $\Diamond$ has a winning strategy in a parity game is known to be in NP $\cap$ Co-NP we can immediately conclude

Corollary 16. *The model checking problem for* GL *over finite structures is in* NP $\cap$ Co-NP.

5 The alternation hierarchy of Game Logic

The alternation levels of the L_μ hierarchy keep track of the number of nested alternations between least and greatest fixed point operators. In GL, when formulae are represented in negation normal form, this corresponds to the nesting of * and $^\circ$ operators within the game modalities. We can thus extend the star hierarchy over games from Definition 14 to a hierarchy over formulae.

Definition 17. The *alternation hierarchy* of GL is the sequence $(\Sigma_i)_{i<\omega}$ of sets consisting of formulae ψ in negation normal form, such that

$$\psi \in \Sigma_i \quad \text{iff} \quad \{\, \gamma \;:\; \langle\gamma\rangle\varphi \in \mathrm{cl}(\psi)\,\} \subseteq \Sigma_i^* \cup \Pi_i^* .$$

To populate this hierarchy we follow the lines of the construction of $\mathrm{W}_{(2)}$ to formulate the winning condition for player $\Diamond$ in a parity game $\mathcal{A} = (V, V_\Diamond, E, \Omega)$ with n priorities.

Like in the explanation to Definition 8, we will construct games γ_i over the Kripke stucture $\mathcal{A}$ for which the angelic player has a strategy iff he can ensure in each play of the parity game $\mathcal{A}$ that he either wins, or the play reaches a priority less than i. Then, γ_1 will be the formula we are looking for.

Let f be the composite game corresponding to the $\triangleright$-operator in the previous section:

$$f := V_\Diamond ?; g \cup V_\Box ?; g^{\mathrm{d}}$$

Assuming n is odd, consider the sequence $(\gamma_i)_{1\le i\le n}$ of GL-games starting with

$$\gamma_n := (\Omega_n ?; f)^*; \Omega_{<n}?$$

and, for any even index $i < n$,

$$\gamma_i := \big(\Omega_{\ge i}?^{\,\mathrm{d}}; (\Omega_i ?; f \cup \Omega_{>i}?; \gamma_{i+1})\big)^\circ; \Omega_{<i}?^{\,\mathrm{d}}$$

while for $i < n$ odd,

$$\gamma_i := (\Omega_i ?; f \cup \Omega_{>i}?; \gamma_{i+1})^*; \Omega_{<i}?$$

Before we proceed, let us understand which options the players actually have in a game γ_i. First, for a player to hold the star (or circle) means only little choice, since he *can* stop iterating γ_i only when some priority less then i is seen. This is required by the guards $\Omega_{<i}?$ and $\Omega_{<i}?^{\,\mathrm{d}}$ at the exit point. But note that, in that case he *is forced* to stop iterating. For the demonic player this is stated explicitely by the condition $\Omega_{\ge i}$ at the entrance of the iteration, when i is even. For the angelic player however, this guard needs not to be set as he has to make his choices in such a way that f is finally being played and it doesn't help to cheat at that point: if the current position in $\mathcal{A}$ has priority j he will always choose towards reaching the subgame $(\Omega_j ?; f)$ in γ_j. Since all $\cup$ choices are determined by the value of j, entering a game γ_i with $i < j$ would lead player $\Diamond$ to fail the test after his next $\cup$ choice. Thus, the actual choices take place in the structure, that is, when f is being played.

We are interested in γ_1. Please note, that the meaning of any formula $\langle\gamma_1\rangle\varphi$ does not depend on φ, since γ_1 is either finished by surrender or it never ends. Thus, we denote $\langle\gamma_1\rangle\top$ by W_*^n.

Example. For $n = 3$ we obtain, by replacing $\Omega_{<1}$ with $\perp$ and ommitting the ; operator,

$$\left\langle \left(\Omega_1 ? f \cup \Omega_{>1} ? \left(\Omega_{\geq 2} ?^{\mathrm{d}} \left(\Omega_2 ? f \cup \Omega_{>2} ? (\Omega_3 ? f)^* \Omega_{<3} ? \right) \right)^{\circ} \Omega_{<2} ?^{\mathrm{d}} \right)^{*} \perp \right\rangle \top$$

Proposition 18. *For every parity game $\mathcal{A}, v$ of index n we have*

$$\mathcal{A}, v \models \mathrm{W}^n_* \quad \textit{iff} \quad \mathcal{A}, v \models \mathrm{W}^n.$$

Proof. Our intention is to translate the proof of W^n on $\mathcal{A}, v$ into a proof of W^n_* on the same structure, and vice versa. Towards this, we look at the model checking games resulting from the two formulae $\mathcal{G} := \mathcal{G}(\mathcal{A}, \mathrm{W}^n, v)$, and respectively, $\mathcal{G}_* := \mathcal{G}(\mathcal{A}, \mathrm{W}^n_*, v)$. Hereon, the proofs appear as winning strategies. Thus, we can rephrase our aim in terms of parity games: If player $\Diamond$ has a winning strategy in $\mathcal{G}$ then he also has a strategy in $\mathcal{G}_*$ (and we are able to construct it), and vice versa.

Let us assume that $\mathcal{A}, v$ satisfies W^n, i.e., player $\Diamond$ has a winning strategy σ in $\mathcal{A}, v$. Hence, he also has a winning strategy τ in $\mathcal{G}$. As we have seen in the proof of Proposition 10, the relevant advices of τ are all transitions of the type $(a, \Diamond Z_i) \to (b, Z_i)$ where is i the priority of a. In other words, the strategy τ of player $\Diamond$ in $\mathcal{G}$ is uniquely determined by his strategy σ in $\mathcal{A}, v$.

Now, getting back to GL, in the light of the above remarks concerning the freedom of choice in γ_i, we can see that for any winning strategy of player $\Diamond$ in $\mathcal{G}_*$ the relevant choices are structure choices of the type $(a, g; \xi_i) \to (b, \xi_i)$ where, for i the priority of a (assumed odd),

$$\xi_i = \langle (\Omega_i ?; f \cup \Omega_{>i} ?; \gamma_{i+1})^*; \Omega_{<i} ?; \gamma_{i-1}; \gamma_{i-2}; \cdots \gamma_1 \rangle \top.$$

Let us consider the strategy τ_* for player $\Diamond$ in $\mathcal{G}_*$ which works like σ (and τ) on structure choices while preventing him on formula choices ($*$ or $\cap$) to loose whithin the next two steps.

Clearly, τ_* carries precisely the same information as τ. In fact, both strategies mirror the winning strategy σ on $\mathcal{A}, v$. It is easy to verify that the priorities are assigned in a compatible way in $\mathcal{G}$ and $\mathcal{G}_*$, such that the set of games obtained with these strategies are essentially the same for both model checking games, consequently, all wins for player $\Diamond$.

By the same token, we can also show conversely, that a winning strategy for player $\Diamond$ in $\mathcal{G}_*$ can be transferred via projection onto $\mathcal{A}, v$ to a winning strategy in $\mathcal{G}$ which concludes the proof. $\square$

Since the formula W^n is strict for the n-th level of the L_μ alternation hierarchy, we can immediately draw the following consequence.

Theorem 19. *No finite level of the μ-calculus alternation hierarchy captures the expressive power of* GL.

Obviously, W^n_* is contained in Σ_n. Since the translation of GL-formulae into L_μ preserves the alternation level, that is, the number of alternated nestings of * and $^\circ$ translates into the same number of nested least and greatest fixed point operators, and the L_μ alternation hierarchy is strict, no GL-formula $\psi \in \Sigma_{n-1}$ can be equivalent to W^n_*.

Theorem 20. *The alternation hierarchy of Game Logic is strict.*

Finally, observe that the length of W_*^n is at most quadratical in n. Knowing that model checking of an L_μ-formula ψ of alternation level n in a structure $\mathcal{A}, v$ can be reduced to the problem of establishing whether player $\Diamond$ has a winning strategy in $\mathcal{G}(\mathcal{A}, \mathrm{W}^n, v)$, or equivalently, to the model checking problem for Game Logic $\mathcal{G}(\mathcal{A}, \psi, v) \models \mathrm{W}_*^n$, we can state:

Theorem 21. *Model Checking for the μ-calculus can be performed efficiently iff this is the case for Game Logic.*

Although the above results show that we can define classes in GL which are arbitrarily hard for L_μ, a hint not to underestimate its expressive power, the question whether Game Logic attaines the full power of L_μ remains open. A possible research direction is the investigation of the variable hierarchy of L_μ. Even if it seems unlikely that this hierarchy collapses on the second level, we have no evidence yet to separate L_μ from its two-variable fragment.

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