Optimization Problems: Expressibility, Approximation Properties and Expected Asymptotic Growth of Optimal Solutions

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1 Introduction

Although the notion of NP-completeness was defined in terms of *decision problems*, the prime motivation for its study and development was the apparent intractability of a large family of combinatorial *optimization problems*. NP-completeness of a decision problem rules out the possibility of finding an optimal solution of the corresponding optimization problem in polynomial time unless P = NP. It does not exclude, however, the possibility that there are efficient algorithms which produce *approximate* solutions. In fact, for many optimization problems with NP-complete decision problems, there are simple and efficient algorithms that produce solutions differing from optimal solutions by at most a constant factor. For some problems, there even exist so-called *polynomial-time approximation schemes* (PTAS), which produce approximate solutions to any desired degree of accuracy. For other problems, notably the Traveling Salesperson Problem, there do not exist efficient approximations unless P = NP (see [7]). Until now the "structural" reasons for the different approximation properties of NP optimization problems have not been sufficiently understood.

Papadimitriou and Yannakakis [17] provided a new perspective by relating the approximation properties of optimization problems to their logical representation. Exploiting Fagin's characterization of NP by existential second order logic [6], they defined two classes of optimization problems, MAX SNP and MAX NP, and showed that all problems in these classes are approximable in polynomial time up to a constant factor. They also identified a host of problems which are complete for MAX SNP with respect to reductions that preserve polynomial-time approximation schemes. Very recently, the classes MAX SNP and MAX NP have received a lot of attention due to results by Arora et al. [2] showing that problems which are hard for MAX SNP cannot have a PTAS, unless P = NP.

We present the syntactic criterion of Papadimitriou and Yannakakis in the more general form and notation provided by Kolaitis and Thakur [11].

Definition 1.1 Recall that Σ_n (respectively Π_n) are prefix classes in first order logic, consisting of formulae in prefix normal form with n alternating blocks of quantifiers

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beginning with $\exists$ (respectively $\forall$). The classes MAX Σ_n (respectively MAX Π_n) consist of maximization problems Q whose input instances are finite structures A of a fixed signature σ, such that the cost of an optimal solution of Q on input A is definable by an expression

$$opt_Q(A) = \max_{\bar{S}} |\{\bar{x} : A \models \psi(\bar{x}, \bar{S})\}|$$

where $\psi(\bar{x}, \bar{S})$ is a Σ_n-formula (respectively a Π_n-formula) and where $\bar{S}$ are predicate variables not contained in σ.

Examples.

- MAX CUT (MC) is the problem of decomposing the vertex set of a given graph G into two subsets such that the number of edges between them is maximal. It is in MAX Σ_0:

$$opt_{\mathrm{MC}}(G) = \max_{U} |\{(x, y) : G \models Exy \wedge (Ux \leftrightarrow \neg Uy)\}|.$$

- MAX SAT is the problem of finding an assignment that satisfies the maximal number of clauses in a given propositional formula in CNF. Such a formula can be represented by a structure $F = (U; P, N)$ with universe U consisting of the clauses and the variables, and with binary predicates P and N where Pxy and Nxy say that variable y occurs positively, respectively negatively in clause x. MAX SAT is in MAX Σ_1 with the defining expression

$$opt_Q(F) = \max_{S} |\{x : F \models (\exists y)((Sy \wedge Pxy) \vee (\neg Sy \wedge Nxy))\}|.$$

  Kolaitis and Thakur proved that MAX SAT $\notin$ MAX Σ_0.

- MAX CLIQUE is the problem of finding a clique of maximal size in a graph. The size of such a clique in G is usually denoted by $\omega(G)$. MAX CLIQUE is in MAX Π_1 because

$$\omega(G) = \max_{C} |\{x : G \models Cx \wedge (\forall y)(\forall z)(Cy \wedge Cz) \rightarrow (y = z \vee Eyz)\}|$$

  Simple monotonicity arguments [16] show that MAX CLIQUE is not in MAX Σ_1. By very recent results in [2], there exists an $\varepsilon > 0$ such that MAX CLIQUE cannot be approximated in polynomial-time within a factor of n^{ε}.

The syntactic criteria for MAX SNP and MAX NP used by Papadimitriou and Yannakakis are those for MAX Σ_0 (where $\psi(\bar{x}, \bar{S})$ is quantifier-free) and for MAX Σ_1 (where $\psi(\bar{x}, \bar{S})$ is existential). However, two remarks about the definitions of these classes should be made. First, the definition of MAX Σ_n as given above is not really sufficient to establish that all problems in MAX Σ_1 are approximable up to a constant factor, at least if approximability means — as usually understood — that we can actually find in polynomial time a nearly optimal solution. The criterion as given by Definition 1.1 only allows us to determine the *cost* of an optimal solution up to a constant factor. We will therefore propose a modified notion for the logical representation of an optimization problem which requires that the formula models (in some sense to be made precise later) *all feasible solutions* of the problem, and not just the cost of an optimal one.

Second, it should be noted that in most papers the definitions of the classes MAX SNP and MAX NP have been interpreted differently than what was originally intended in [17]. While most authors (see [8, 12, 15, 16]) understood MAX SNP, respectively MAX NP to be *precisely* MAX Σ_0 and MAX Σ_1, Papadimitriou and Yannakakis actually had in mind their closures under the appropriate reductions (although they did not really make this clear; but see the remark in [18]). In particular, these extended versions of MAX SNP and MAX NP can also contain minimization problems. Kann [10] defines yet another, intermediate version of MAX SNP. We think that these different classes all have their merits, but it is important not to confuse them. The "pure" syntactic classes are interesting because they provide a logical criterion for approximability, and provide an opportunity to prove results about optimization problems using tools from logic (or, more precisely, finite model theory). In logic we have lower bound techniques that have no counterpart in computational complexity theory. In many cases these techniques (e.g. monotonicity arguments, Ehrenfeucht-Fraïssé games and limit laws) show that a problem does not satisfy a certain syntactic criterion, and thus establish separation and hierarchy results among the syntactic classes (without referring to unproved hypotheses from complexity theory). On the other hand, the closure classes may be appropriate if one is interested in pure complexity results. But closing syntactic classes under a class of reductions that are defined in terms of computational complexity, rather than logical definability, precludes the use of the logical techniques.

This paper is about syntactic classes. One of our goals was to find a more general syntactic criterion for approximability than the one provided by Papadimitriou and Yannakakis. This is achieved using other results from finite model theory than just Fagin's theorem, in particular the close connection between *fixpoint logic* and polynomial-time computability. To avoid confusion, we use the names MAX Σ_0 and MAX Σ_1, introduced by Kolaitis and Thakur, rather than MAX SNP and MAX NP.

Kolaitis and Thakur [11, 12] systematically investigated the logical expressibility of optimization problems. They proved that the class MAX $\mathcal{PB}$, consisting of all polynomially bounded maximization problems, coincides with MAX Π_2 and that there is proper hierarchy of four levels.

Theorem 1.2 MAX $\Sigma_0 \subsetneq$ MAX $\Sigma_1 \subsetneq$ MAX $\Pi_1 \subsetneq$ MAX $\Pi_2 =$ MAX $\mathcal{PB}$.

It is interesting that the classes MAX Π_2 and MAX Π_1 are separated by MAXIMUM CONNECTED COMPONENT (MCC), the problem of finding a connected component of maximal cardinality in a graph. This optimization problem is clearly solvable in polynomial time.

New expressibility classes of optimization problems. In this paper we extend the classes MAX Σ_i and MAX Π_i in several ways. Most importantly we allow the formulae to contain relations definable in least fixpoint logic. In addition, we maximize not only over relations but also over constants. We call the extended classes MAX Σ_i^{FP} and MAX Π_i^{FP}. The proof of [17] can be extended to MAX Σ_1^{FP} to show that all problems in this class are approximable. Some problems, such as MCC, descend from the highest level MAX Π_2 in the original hierarchy to the lowest level MAX Σ_0^{FP} in the new hierarchy. *Thus our extended class* MAX Σ_1^{FP} *provides a more powerful sufficient criterion for* *approximability than the original class* MAX Σ_1 *of* [17]. However, we also prove that even MAX Σ_1^{FP} does not contain all approximable problems; in fact, it does not even contain all polynomial time optimization problems. We discuss the question of how far the class MAX Σ_1 can be extended while preserving approximability.

Separation of the extended classes by the probabilistic method. We separate also the extended classes; e.g. we prove that

$$\text{MAX } \Sigma_0^{\mathrm{FP}} \subsetneq \text{MAX } \Sigma_1^{\mathrm{FP}} \subsetneq \text{MAX } \Pi_1^{\mathrm{FP}} \subsetneq \text{MAX } \Pi_2^{\mathrm{FP}} = \text{MAX } \mathcal{PB}.$$

Also, we prove that a number of important problems do not belong to MAX Σ_1^{FP}. These include MAX CLIQUE, MAX INDEPENDENT SET, V-C DIMENSION and MAX COMMON INDUCED SUBGRAPH. To do this we have to use more sophisticated methods than the techniques of [16, 11] which break down in the presence of fixpoint definitions. We use two alternative methods.

The first method, introduced in the present paper, characterizes rates of growth of average optimal solution sizes. For instance, it is known [4] that the expected size of a maximal clique in a random graph of cardinality n grows asymptotically like $2 \log n$. We show that no problem in MAX Σ_1^{FP} can have this property, thus proving that MAX CLIQUE is not in MAX Σ_1^{FP}. This technique is related to *limit laws for various logics* [5, 13, 14] and to the *probabilistic method* from combinatorics [1]. We believe that this method may be of independent interest.

The second method uses special classes of structures where fixpoint logic has no more expressive power than quantifier-free formulae. On such classes we can apply monotonicity arguments that break down on arbitrary finite structures. With this technique we give an alternative proof that MAX CLIQUE is not in MAX Σ_1^{FP}. We also show that MAX MATCHING is not expressible by existential sentences with fixpoint definitions.

In contrast to the recent results on the non-approximability of many maximization problems, among them MAX CLIQUE, our results do not depend on any unproved hypothesis from complexity theory, such as P $\neq$ NP.

2 Preliminaries

Definition 2.1 An NP *optimization problem* is a quadruple $Q = (I_Q, \mathcal{F}_Q, f_Q, opt)$ such that

- I_Q is the set of *input instances* for Q.
- $\mathcal{F}_Q(I)$ is the set of *feasible solutions* for input I. Here, "feasible" means that the size of the elements $S \in \mathcal{F}_Q(I)$ is polynomially bounded in the size of I and that the set $\{(I, S) : S \in \mathcal{F}_Q(I)\}$ is recognizable in polynomial time.
- $f_Q : \{(I, S) : S \in \mathcal{F}_Q(I)\} \to \mathbb{N}$ is a polynomial-time computable function, called the *cost function*.
- $\mathrm{opt} \in \{\max, \min\}$.

For every NP optimization problem Q, the following decision problem is in NP: given an instance I of Q and a natural number k, is there a solution $S \in \mathcal{F}_Q(I)$ such that $f_Q(I, S) \geq k$ when $\mathrm{opt} = \max$, (or $f_Q(I, S) \leq k$, when $\mathrm{opt} = \min$).

Let $opt_Q(I) := \mathrm{opt}_{S \in \mathcal{F}_Q(I)} f_Q(I,S)$. An NP optimization problem is said to be *polynomially bounded* if there exists a polynomial p such that $opt_Q(I) \leq p(|I|)$ for all instances I. We denote by Max $\mathcal{PB}$ (Min $\mathcal{PB}$) the set of all polynomially bounded maximization (minimization) problems.

Approximation. The *performance ratio* of a feasible solution S for an instance I of Q is defined as $R(I,S) := opt_Q(I)/f_Q(I,S)$ if Q is a maximization problem and as $R(I,S) := f_Q(I,S)/opt_Q(I)$ if Q is a minimization problem.

Definition 2.2 We say that an NP optimization problem Q is *approximable up to a constant factor* if there exists a constant $c > 0$ and a polynomial-time algorithm Π which produces, for every instance I of Q, a feasible solution $\Pi(I)$ with performance ratio $R(I, \Pi(I)) \leq c$. APX is the class of all NP optimization problems that are approximable up to a constant factor.

A weaker notion of approximability that is sometimes used requires only that the *cost* of an optimal solution can be approximated; for instance, in the case of Max Clique it would only be required that the algorithms approximates the clique number $\omega(G)$, not that it actually finds a nearly optimal clique.

Logical representation of optimization problems. Let Q be an optimization problem whose input instances are finite structures of fixed vocabulary σ. The definition of the classes Max Σ_n and Max Π_n as given by Kolaitis and Thakur requires only that there is an appropriate logical definition of $opt_Q(A)$, the cost of an optimal solution. However, optimization problems can be modelled by logical formulae in a much closer way.

Definition 2.3 A formula $\psi(\bar{x}, \bar{S})$ of vocabulary $\sigma \cup \{S_1, \ldots, S_r\}$ *represents* Q if and only if the following holds.

(i) For every instance A and every feasible solution $S_0 \in \mathcal{F}_Q(A)$, there exists an expansion $B = (A, S_1, \ldots, S_r)$ of A such that $f_Q(A, S_0) = |\{\bar{x} : B \models \psi(\bar{x}, \bar{S})\}|$.

(ii) Conversely, every expansion $B = (A, S_1, \ldots, S_r)$, for which the set $L = \{\bar{x} : B \models \psi(\bar{x}, \bar{S})\}$ is non-empty defines a feasible solution S_0 for A with $f_Q(A, S_0) = |L|$; moreover, this solution can be computed in polynomial time from B.

In particular, $opt_Q(A) = \max_{\bar{S}} |\{\bar{x} : A \models \psi(\bar{x}, \bar{S})\}|$.

In all examples that we consider, the feasible solution defined by $(A, S_1, \ldots, S_r)$ will either be one of the S_i or the set $\{\bar{x} : A \models \psi(\bar{x}, \bar{S})\}$ itself. Lautemann [15] has independently considered this more detailed logical representation of optimization problems. A more constructive alternative to Definition 1.1 might then be the following.

Definition 2.4 Max Σ_n is the class of all maximization problems that can be represented by Σ_n-formulae. The classes Max Π_n, Min Σ_n and Min Π_n are defined analogously.

Clearly this definition is more restrictive than the one used by Kolaitis and Thakur. We think that it is justified by the following observations. First, the more restrictive definition is necessary to establish the result of Papadimitriou and Yannakakis that

$\text{Max } \Sigma_1 \subseteq \text{APX}$. Second, on all natural examples in the literature, the two definitions make no difference. Third, the results of Kolaitis and Thakur, in particular the fact that

$$\text{Max } \Sigma_0 \subsetneq \text{Max } \Sigma_1 \subsetneq \text{Max } \Pi_1 \subsetneq \text{Max } \Pi_2 = \text{Max } \mathcal{PB}$$

remain true with the more restrictive definition. (However, the proof that $\text{Max } \Pi_2 = \text{Max } \mathcal{PB}$ needs some modification.) All results presented in this paper are true for both possible choices of logical representation of optimization problems. However, if the more liberal one (modelling only the cost of the optimal solution) is chosen, then the more liberal definition of approximability must also be adopted.

Fixpoint logic. It is well-known that the expressive power of first-order logic is limited by the lack of a mechanism for unbounded iteration or recursion. The most notable example of a query that is not first-order expressible is the transitive closure (TC). This has motivated the study of more powerful languages that add recursion in one way or another to first-order logic. The most prominent of these are the various forms of *fixpoint logics*.

Let σ be a signature, P an r-ary predicate not in σ and $\psi(\bar{x})$ be a formula of the signature $\sigma \cup \{P\}$ with only positive occurrences of P and with free variables $\bar{x} = x_1, \ldots, x_r$. Then ψ defines for every finite σ-structure A with universe $|A|$ an operator ψ^A on the class of r-ary relations over $|A|$ by

$$\psi^A : P \longmapsto \{\bar{a} : (A, P) \models \psi(\bar{a})\}.$$

Since P occurs only positively in ψ, this operator is monotone, i.e. $Q \subseteq P$ implies that $\psi^A(Q) \subseteq \psi^A(P)$. Therefore this operator has a *least fixed point* which may be constructed inductively beginning with the empty relation. Set $\Psi^0 := \emptyset$ and $\Psi^{j+1} := \psi^A(\Psi^j)$. At some stage i, this process reaches a stable predicate $\Psi^i = \Psi^{i+1}$, which is the *least fixed point* of ψ on A, and denoted by Ψ^∞. Since $\Psi^i \subseteq \Psi^{i+1}$, the least fixed point is reached in a polynomial number of iterations, with respect to the cardinality of A.

The fixed point logic (FO + LFP) is defined by adding to the syntax of first order logic the *least fixed point formation rule:* if $\psi(\bar{x})$ is a formula of the signature $\sigma \cup \{P\}$ with the properties stated above and $\bar{u}$ is an r-tuple of terms, then

$$[\text{LFP}_{P,\bar{x}}\ \psi](\bar{u})$$

is a formula of vocabulary σ (to be interpreted as $\Psi^\infty(\bar{u})$).

Example. Here is a fixpoint formula that defines the transitive closure of the binary predicate E:

$$[\text{TC } E](u, v) \equiv [\text{LFP}_{T,x,y}\ (x = y) \vee (\exists z)(Exz \wedge Tzy)](u, v).$$

On the class of all finite structures, (FO + LFP) has strictly more expressive power than first-order logic — it can express the transitive closure — but is strictly weaker than PTIME-computability. However, Immerman [9] and Vardi [19] proved that on *ordered structures* the situation is different. There (FO + LFP) characterizes precisely the queries that are computable in polynomial time. On the other hand, on very simple classes of structures, such as structures with empty signatures (i.e. sets), (FO + LFP) collapses to first-order logic.

3 Optimization problems definable by fixpoint logic

The fact that the problem Maximum Connected Component (MCC) appears only in the highest level Max Π_2 of the expressibility hierarchy suggests that we do not yet have the "right" definitions. After all, MCC is computationally a very simple problem, and it appears high in the expressibility hierarchy just because first-order logic cannot express the transitive closure.

It is possible that there will always remain a certain "mismatch" between computational complexity and logical expressibility. But this mismatch is certainly not as big as the difference between first-order logic and Ptime. If we base our definitions on fixpoint logic (or other logical systems that allow recursion) rather than first-order logic, we obtain a closer relationship between logical and computational complexity.

Definition 3.1 Let Q be a maximization problem whose instances are finite structures over a fixed vocabulary σ. We say that Q belongs to the class Max Σ_i^{FP} if there exists a Σ_i-formula $\psi(\bar{x}, \bar{c}, \bar{S}, \bar{P})$ of vocabulary $\sigma \cup \{\bar{S}, \bar{c}\}$ (where $\bar{S}$ and $\bar{c}$ are tuples of predicate symbols and constants that do not occur in σ) such that

- $\bar{P} = P_1, \ldots, P_r$ are global predicates on σ-structures that are definable in fixpoint logic.
- the formula $\psi(\bar{x}, \bar{c}, \bar{S}, \bar{P})$ represents Q (in the sense of Definition 2.3, with the obvious modifications). In particular, $opt_Q(A) = \max_{\bar{S},\bar{c}} |\{\bar{x} : A \models \psi(\bar{x}, \bar{c}, \bar{S}, \bar{P})\}|$.

The classes Max Π_i^{FP}, Min Σ_i^{FP} and Min Π_i^{FP} are defined in an analogous way.

We insist that the fixpoint-predicates must not depend on the relations $\bar{S}$ over which we maximize; we therefore call them *predefined fixpoint predicates*. (We will discuss this condition below).

The results of Kolaitis and Thakur [11] translate to the extended classes and prove that

$$\text{Max } \Sigma_0^{\mathrm{FP}} \subseteq \text{Max } \Sigma_1^{\mathrm{FP}} \subseteq \text{Max } \Pi_1^{\mathrm{FP}} \subseteq \text{Max } \Pi_2^{\mathrm{FP}} = \text{Max } \mathcal{PB}.$$

The increased expressive power provided by the fixpoint predicates has the effect that some problems occur in lower levels in the new hierarchy than they did in the original one.

Example. The problem Maximum Connected Component belongs to Max Σ_0^{FP}. Its optimum on a graph $G = (V, E)$ is definable by

$$opt_{\mathrm{MCC}}(G) = \max_c |\{x : G \models [\mathrm{TC}\ E](c, x)\}|.$$

Thus MCC descends from the highest level (Max Π_2) of the original hierarchy to the lowest level (Max Σ_0^{FP}) of the new hierarchy. This is interesting because, as we will see, also the extended class Max Σ_1^{FP} contains only problems in APX. Thus our approach provides a more powerful syntactic criterion for approximability than the original class Max Σ_1. Let us consider to what extent our definitions are adequate and discuss some alternatives.

Maximization over constants. Does maximization over constants really give more power? We prove that it does up to the level Max Σ_1^{FP}.

Proposition 3.2 MAX CONNECTED COMPONENT *is not expressible in* MAX Σ_1^{FP} *without maximization over constants.*

PROOF. Let G_n be the graph with n vertices and no edges. Obviously, $opt_{\mathrm{MCC}}(G_n) = 1$ for all n. Moreover, for every fixpoint-definable predicate P, there exists a natural number n_0 such that P is in fact Σ_0-definable on $\{G_n : n > n_0\}$. Therefore, if MCC is expressible in MAX Σ_1^{FP} without constants, then there is an existential formula $\psi(\bar{x}, \bar{S})$ (without fixpoint predicates), such that for all $n > n_0$

$$opt_{\mathrm{MCC}}(G_n) = \max_{\bar{S}} |\{\bar{x} : G_n \models \psi(\bar{x}, \bar{S})\}| = 1.$$

Choose a tuple $\bar{u} \in G_n$ and predicates $\bar{S}$ such that $G_n \models \psi(\bar{u}, \bar{S})$. Note that G_{2n} consists of two disjoint copies of G_n; let $\bar{S}^*$ be the union of the two copies of $\bar{S}$. Existential sentences are preserved by extensions, so there exist at least two tuples $\bar{u}$, satisfying $G_{2n} \models \psi(\bar{u}, \bar{S}^*)$. This contradicts the fact that $opt_{\mathrm{MCC}}(G_{2n}) = 1$. ∎

Another simple problem that requires maximization over constants is the *maximal degree* $\Delta(G)$ of a graph, defined by $\Delta(G) = \max_c |\{x : G \models Ecx\}|$. Similar monotonicity arguments as above show that $\Delta(G)$ is not definable in MAX Σ_1^{FP} without maximization over constants.

However, constants can be replaced by monadic predicates at the expense of a universal subformula.

Proposition 3.3 *Every problem in* MAX Π_1^{FP} *or* MAX Π_2^{FP} *can be expressed without maximization over constants.*

Fixpoint definitions over the new predicates. To strengthen our classes we could modify the definition of MAX Σ_i^{FP} so that the fixpoint predicates might depend also on the predicates $\bar{S}$ over which we maximize. In fact, one could propose classes of all maximization problems Q such that

$$opt_Q(A) = \max_{\bar{S}} |\{\bar{x} : A \models \psi(\bar{x}, \bar{S})\}|$$

where $\psi(\bar{x}, \bar{S})$ is a formula of (FO + LFP), possibly with restrictions on the quantifier structure. If we stipulate that $\psi(\bar{x}, \bar{S})$ has the form $[\mathrm{LFP}_{R,\bar{z}}\ \varphi](\bar{u})$ with φ quantifier-free then we will remain inside MAX Σ_0 because fixpoints over quantifier-free formulae are again Σ_0-definable. The next stronger possible class, motivated by the existential nature of the class MAX Σ_1 is the class MAX EFP defined as above with the condition that $\psi(\bar{x}, \bar{S}) \equiv [\mathrm{LFP}_{R,\bar{z}}\ \varphi](\bar{u})$ where φ is existential. In particular $\psi(\bar{x}, \bar{S})$ is a formula in *existential fixpoint logic* [3]. However, this class is already too expressive.

Proposition 3.4 *If* P $\neq$ NP*, then* MAX EFP *contains non-approximable problems.*

PROOF. We consider the following variant of circuit satisfiability. A circuit is described by a finite structure $C = (V, E, I, out)$ where (V, E) is a directed acyclic graph, $I \subseteq V$ is the set of sources (vertices with no incoming edges) describing the input nodes, every node in $V - I$ has fan-in two, and *out* is a sink (no outgoing edges). Every non-input node is considered as a NAND-gate and *out* is the output node. Every subset $S \subset I$ defines an assignment to the input nodes, and therefore a value $f_C(S) \in \{0, 1\}$, the value computed by C for input S.

Now the circuit-satisfiability problem is

$$\textsc{Circuit-Sat} := \{C : (\exists S \subseteq I) f_C(S) = 1\}.$$

Since a Boolean formula is a special case of a circuit, it is clear that CIRCUIT-SAT generalizes SAT and is therefore NP-complete.

On the other hand, it is not difficult to construct a formula $\psi(S)$ in existential fixpoint logic such that

$$C \models \psi(S) \Longleftrightarrow f_C(S) = 1.$$

We now can define a problem $Q \in$ MAX EFP by

$$opt_Q(C) = \max_S |\{x : C \models \psi(S)\}|.$$

Note that x does not occur freely in ψ, so $opt_Q(C) = |V|$ if $C \in$ CIRCUIT-SAT and $opt_Q(C) = 0$ otherwise. Therefore, if Q were approximable up to any constant $\varepsilon > 0$, then the corresponding approximation algorithm would solve the CIRCUIT-SAT problem, and it would follow that P = NP. ∎

Maximization over total orderings. Papadimitriou and Yannakakis [17] proposed another direction for generalization: to maximize over total orderings. A natural problem expressible in this way is MAX SUBDAG: given a digraph G, find an acyclic subgraph of G with maximal number of edges. The expression defining the optimum for this problem is

$$opt_Q(G) = \max_< |\{(x, y) : G \models Exy \wedge x < y\}|.$$

This suggests the following definition.

Definition 3.5 For every class M, as defined in Definitions 2.3 or 3.1, let $M(<)$ defined in the same way as M, except that some of relations over which we optimize are binary predicates $<_1, <_2, \ldots$, which do not run over all binary predicates, but only over total orderings of the given structure. We write the defining expression of an optimization problem in $M(<)$ on an input structure A as

$$opt_Q(A) = \operatorname*{opt}_{\bar{<}, \bar{S}, \bar{c}} |\{\bar{x} : A \models \psi(\bar{x}, \bar{c}, \bar{<}, \bar{S}, \bar{P})\}|.$$

As remarked in [17] this feature does not destroy approximability: MAX $\Sigma_1^{\mathrm{FP}} \subseteq$ APX. (We prove a more general fact below.)

Note that for classes MAX Π_1 and above, maximization over orderings does not increase the expressive power, because total orderings are axiomatizable by Π_1-formulae. Thus MAX $\Pi_1(<) =$ MAX Π_1 and MAX $\Pi_1^{\mathrm{FP}}(<) =$ MAX Π_1^{FP}, etc.

Maximization over general classes of relations. How far can we generalize the idea of the previous paragraph? Instead of just maximizing over orderings, we could maximize over any specified class of predicates. Let $\bar{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_q$ where each $\mathcal{C}_i$ is a class of relations of some fixed arity r_i. We now maximize over tuples $\bar{S} = S_1, \ldots, S_q$ of relations, subject to the condition that $S_i \in \mathcal{C}_i$. The expression defining the cost of an optimal solution then has the form

$$opt_Q(A) = \operatorname*{opt}_{\bar{S} \in \bar{\mathcal{C}}, \bar{c}} |\{\bar{x} : A \models \psi(\bar{x}, \bar{c}, \bar{S}, \bar{P})\}|.$$

For any class M of Definitions 2.3 or 3.1, and any $\bar{\mathcal{C}}$, this defines a new class $M(\bar{\mathcal{C}})$. The classes $M(<)$ are special cases.

In view of our goal to find a good criterion for approximability, we ask what condition $\bar{\mathcal{C}}$ must satisfy so that problems in $\text{Max } \Sigma_1^{\text{FP}}(\bar{\mathcal{C}})$ be approximable. We give such a condition here, and prove a general form of the Theorem of Papadimitriou and Yannakakis.

Definition 3.6 For every $n \in \mathbb{N}$, let $\mathcal{C}(n)$ be a class of relations of fixed arity r over n and let $\mathcal{C} = \bigcup_n \mathcal{C}(n)$. We say that $\mathcal{C}$ is *well-behaved* if for every $k \in \mathbb{N}$ there exists a number $\varepsilon(k) > 0$ such that for all n and for any set of conditions $\alpha_1, \ldots, \alpha_k$ of the form $S(\bar{u})$ or $\neg(S\bar{u})$, (where $\bar{u}$ are r-tuples over n) the probability that $\alpha_1 \wedge \cdots \wedge \alpha_k$ is satisfied by a randomly chosen relation $S \in \mathcal{C}(n)$, is computable in polynomial time, and is either 0 or at least $\varepsilon(k)$.

Examples.

- The class of all r-ary relations is well-behaved with $\varepsilon(k) = 2^{-k}$.
- If we fix, in a consistent manner, an ordering on k pairs of elements then there are at least $n!/(2k)!$ ways to extend this to a total ordering. Thus, the class of all total orderings is also well-behaved with $\varepsilon(k) = ((2k)!)^{-1}$.
- The class of unary functions, represented by binary relations, is not well-behaved. If we fix one value $f(u) = v$, then the probability that a function on n satisfies this condition is $1/n$.

A tuple $\bar{\mathcal{C}}$ of well-behaved classes is again well-behaved, in the sense that the probability that any k atomic formulae $S_i(\bar{u})$ or $\neg S_i(\bar{u})$ are satisfied has the properties required by Definition 3.6.

Theorem 3.7 *If $\bar{\mathcal{C}}$ is well-behaved then every maximization problem in* $\text{Max } \Sigma_1^{\text{FP}}(\bar{\mathcal{C}})$ *is approximable up to a constant factor.*

Proof. Let Q be a maximization problem, such that for all input structures $A \in I_Q$

$$opt_Q(A) = \max_{\bar{S} \in \bar{\mathcal{C}}, \bar{c}} |\{\bar{x} : A \models (\exists \bar{y}) \psi(\bar{x}, \bar{y}, \bar{c}, \bar{S}, \bar{P})\}|$$

where ψ is a quantifier-free formula with predefined fixpoint predicates $\bar{P}$. Fix an input structure A of cardinality n. The fixpoint predicates can be evaluated in polynomial time, so we may assume that they are part of the input, and not worry about them anymore. Moreover, there are only polynomially many possible tuples $\bar{c}$, so we can compute the optimum for each of them separately; thus we can assume that the value of $\bar{c}$ is fixed. To enhance readability we drop $\bar{P}$ and $\bar{c}$ and write $\psi(\bar{x}, \bar{y}, \bar{S})$ in the sequel.

We consider the class $\bar{\mathcal{C}}(n)$ as a probability space Ω with uniform distribution. Define the random variable X^A on Ω by

$$X^A(\bar{S}) = |\{\bar{u} : A \models (\exists \bar{y}) \psi(\bar{u}, \bar{y}, \bar{S})\}|.$$

Obviously $opt_Q(A) = \max(X^A) \geq E(X^A)$ where $E(X^A)$ is the expected value of X^A. Note that X^A can be written as the sum of the indicator random variables

$$X^A_{\bar u}(\bar S) = \begin{cases} 1 & \text{if } A \models (\exists \bar y)\psi(\bar u, \bar y, \bar S) \\ 0 & \text{otherwise} \end{cases}$$

By linearity of expectation $E(X^A) = \sum_{\bar u} E(X^A_{\bar u})$. In this sum we can discard those $X^A_{\bar u}$ which are identically 0, so let

$$B^A := \{\bar u : X^A_{\bar u} \neq 0\} = \{\bar u : A \models (\exists \bar y)\psi(\bar u, \bar y, \bar S) \text{ for some } \bar S \in \bar{\mathcal{C}}(n)\}$$

Now

$$E(X^A) = \sum_{\bar u \in B^A} E(X^A_{\bar u}) \leq |B^A|.$$

Fix $\bar u \in B^A$. There exist predicates $\bar S^* \in \Omega$ and a tuple $\bar v$ such that $A \models \psi(\bar u, \bar v, \bar S^*)$. This formula depends only a fixed number $\alpha_1, \ldots, \alpha_k$ of $\bar S$-atoms, and every tuple $\bar S \in \Omega$ respecting the values of $\alpha_1, \ldots, \alpha_k$ on $\bar S^*$ will also satisfy $\psi(\bar u, \bar v, \bar S)$. Since $\bar{\mathcal{C}}$ is well-behaved, the probability that a randomly chosen $\bar S$ has this property is at least $\varepsilon(k)$. Thus, $E(X^A_{\bar u}) \geq \varepsilon(k)$ for every $\bar u \in B^A$. This implies

$$|B^A| \geq \max(X^A) \geq E(X^A) \geq \varepsilon(k)|B^A|$$

and, in particular, $E(X^A) \geq \varepsilon(k)\,\mathrm{opt}_Q(A)$.

It remains to prove that an assignment $\bar S^* \in \Omega$ with $X^A(S^*) \geq E(X^A)$ can be found in polynomial time. Enumerate all atoms $S_i(\bar u)$ as $\alpha_1, \ldots, \alpha_m$. Clearly m is a polynomial in n. We determine truth-values for $\alpha_1, \ldots, \alpha_m$ as follows.

Suppose values for $\alpha_1, \ldots, \alpha_i$ are already computed, and let β_i be the conjunction of those α_j and $\neg\alpha_j$ (for $j \leq i$) that have been set to TRUE. Now we define α_{i+1} to be TRUE if

$$E(X^A|(\beta_i \wedge \alpha_{i+1})) \geq E(X^A|(\beta_i \wedge \neg\alpha_{i+1}))$$

and FALSE otherwise. Note that these conditional expectations can be computed in polynomial time, by the same arguments as in the first part of this proof and because $\bar{\mathcal{C}}$ is well-behaved. At the end, β_m determines relations $\bar S^*$, so $E(X^A|\beta_m) = X^A(S^*)$. Note that

$$\begin{aligned} E(X^A|\beta_i) &= P(\alpha_{i+1}|\beta_i)E(X^A|(\beta_i \wedge \alpha_{i+1})) + P(\neg\alpha_{i+1}|\beta_i)E(X^A|(\beta_i \wedge \neg\alpha_{i+1})) \\ &\leq E(X^A|\beta_{i+1}) \end{aligned}$$

where $P(\alpha_{i+1}|\beta_i)$ is the conditional probability that α_{i+1} holds, given that β_i is TRUE. In particular

$$E(X^A) = E(X^A|\beta_0) \leq E(X^A|\beta_m) = X^A(\bar S^*).$$

This technique for "derandomizing" a probabilistic argument is well known. Alon and Spencer [1] call it *the method of conditional expectations*. ∎

4 Probabilistic methods

Let Q be an optimization problem whose input instances are finite structures over a fixed vocabulary σ. We now consider the behaviour of $opt_Q(A)$ on a *randomly chosen* σ-structures A. Fix n and let Ω be the probability space of all σ-structures over universe n, with uniform distribution; then opt_Q is a random variable on Ω whose expected value is denoted by $E(opt_Q)$.

We will establish a probabilistic (necessary) criterion for membership in $\text{MAX } \Sigma_1^{\text{FP}}$. In fact it holds for any class $\text{MAX } \Sigma_1^{\text{FP}}(\bar{\mathcal{C}})$ provided that $\bar{\mathcal{C}}$ is Σ_2-axiomatizable. In particular, our criterion applies to $\text{MAX } \Sigma_1^{\text{FP}}(<)$, since linear orderings are in fact Π_1-axiomatizable.

Theorem 4.1 (Probabilistic criterion for $\text{MAX } \Sigma_1^{\text{FP}}$**)** *Suppose that Q is a problem in $\text{MAX } \Sigma_1^{\text{FP}}(\bar{\mathcal{C}})$ where $\bar{\mathcal{C}}$ is Σ_2-axiomatizable. Then there exists a polynomial $p(n)$ and a constant $\varepsilon > 0$ such that*

$$p(n) \geq E(opt_Q) \geq \varepsilon p(n)$$

or $E(opt_Q)$ decreases to 0 exponentially fast as n goes to infinity.

Before we prove Theorem 4.1, we assemble some results from the theory of asymptotic probabilities that we need. As usual, we denote by $\mu_n(\psi)$ the probability that the sentence ψ is true in a random structure with universe n.

Fact 4.2 *For every formula $\varphi(\bar{x})$ in fixpoint logic, there exists a quantifier-free first-order formula $\alpha(\bar{x})$ and a constant $c > 0$ such that*

$$\mu_n((\forall \bar{x})[\varphi(\bar{x}) \leftrightarrow \alpha(\bar{x})]) > 1 - c^n$$

for large enough n.

In fact this result is true even for stronger logics than fixpoint logic, e.g. the infinitary logic $L^\omega_{\infty\omega}$. It is essentially Theorem 3.13 in [14]. The second fact that we need is a generalization of the 0-1 law for strict Σ^1_1-sentences, due to Kolaitis and Vardi [13]. Strict Σ^1_1-formulae have the form $(\exists \bar{S})(\exists \bar{y})(\forall \bar{z})\varphi$ where φ is quantifier-free. A dyadic rational is a rational number whose denominator is a power of two.

Fact 4.3 *Let $\psi(\bar{x})$ be a strict Σ^1_1-formula with free variables $\bar{x} = x_1, \ldots, x_k$. For every k-tuple $\bar{u} \in \mathbb{N}^k$, there exists a dyadic rational $p_{\bar{u}}$, such that $\mu_n(\psi(\bar{u}))$ tends to $p_{\bar{u}}$ exponentially fast. Moreover, $p_{\bar{u}}$ only depends on the equality type of $u_1, \ldots, u_k$ (not on $\bar{u}$ itself).*

Finally we will need a Lemma about binomial distributions $b(n,k,p) := \binom{n}{k} p^k q^{n-k}$ where $0 \leq p \leq 1$ and $q = 1 - p$. For a proof, see [4, p. 10] or [1, Appendix A].

Lemma 4.4 *If $\varepsilon > 0$ and $k \geq (1+\varepsilon)pn$, then $b(n,k,p)$ tends to 0 exponentially fast as n goes to infinity.*

PROOF OF THEOREM 4.1. Let $Q \in \text{MAX } \Sigma_1^{\text{FP}}(\bar{\mathcal{C}})$. We first assume that opt_Q can be expressed without maximization over constants, i.e. that there exists a Σ_1-formula $\psi(\bar{x}, \bar{S})$ (with predefined fixpoint predicates) such that

$$opt_Q(A) = \max_{\bar{S} \in \bar{\mathcal{C}}} |\{\bar{x} : A \models \psi(\bar{x}, \bar{S})\}|.$$

The proof of Theorem 3.7 shows that for some constant $\varepsilon > 0$, $|B^A| \geq opt_Q(A) \geq \varepsilon |B^A|$ where

$$B^A = \{\bar{u} : A \models (\exists \bar{S} \in \bar{\mathcal{C}})\psi(\bar{u}, \bar{S})\}.$$

On Ω, we define the random variable $X(A) := |B^A|$. It follows that $E(X) \geq E(opt_Q) \geq \varepsilon E(X)$. It suffices to prove that $E(X)$ converges to a polynomial $F(n)$. We write X as the sum of the indicator random variables

$$X_{\bar{u}}(A) := \begin{cases} 1 & \text{if } \bar{u} \in B^A \\ 0 & \text{otherwise} \end{cases}$$

By linearity of expectation, $E(X) = \sum_{\bar{u}} E(X_{\bar{u}})$. Let $\alpha(\bar{S})$ be a Σ_2-axiom for $\bar{\mathcal{C}}$. Then $E(X_{\bar{u}})$ is the probability that the formula

$$\eta(\bar{u}) \equiv (\exists \bar{S})(\alpha(\bar{S}) \wedge \psi(\bar{u}, \bar{S}))$$

holds on a random structure with universe n. Fact 4.2 tells us that except on a exponentially decreasing fraction of structures, the predefined fixpoint predicates are definable by quantifier-free formulae. If we substitute them into $\eta(\bar{u})$, then we obtain a strict Σ^1_1-formula $\varphi(\bar{u})$ such that, for some constant $c > 0$,

$$|E(X_{\bar{u}}) - \mu_n(\varphi(\bar{u}))| < c^n.$$

Now, by Fact 4.3, the probability $\mu_n(\varphi(\bar{u}))$ converges exponentially fast to a dyadic rational $p_{\bar{u}}$ which only depends on the equality type of $\bar{u}$. If k is fixed then the number of equality types of k-tuples is also fixed; moreover, the number of k-tuples of equality type e over n is a polynomial $f_e(n)$. Let p_e be the asymptotic probability of $\varphi(\bar{u})$ for tuples of equality type e. It follows that $E(X)$ converges exponentially fast to the polynomial

$$F(n) = \sum_e p_e f_e(n).$$

With maximization over constants, the situation becomes more complicated. We now have

$$opt_Q(A) = \max_{\bar{c}, \bar{S}} |\{\bar{x} : A \models \psi(\bar{x}, \bar{c}, \bar{S})\}|.$$

To establish Theorem 3.7 we fixed for every input structure A an optimal tuple $\bar{c}$ which then was considered as part of the input. Since $\bar{c}$ depends on A this no longer works when A is a random input. Therefore, let

$$B^A(\bar{c}) := \{\bar{x} : A \models (\exists \bar{S} \in \bar{\mathcal{C}})\psi(\bar{u}, \bar{c}, \bar{S})\}.$$

On every input structure A we then have

$$\max_{\bar{c}} |B^A(\bar{c})| \geq opt_Q(A) \geq \varepsilon \max_{\bar{c}} |B^A(\bar{c})|$$

for a fixed constant $\varepsilon > 0$. Let $X := \max_{\bar{c}} |B^A(\bar{c})|$; it suffices to prove that there exists a polynomial $F(n)$ such that $E(X) \sim F(n)$.

As above, we find a strict Σ^1_1-formula $\varphi(\bar{c}, \bar{x})$ such that, for any fixed $(\bar{c}, \bar{u})$, the expectation that $\bar{u} \in B^A(\bar{c})$ is exponentially close to the asymptotic probability of $\varphi(\bar{c}, \bar{u})$. Again, the asymptotic probability of $\varphi(\bar{c}, \bar{u})$ is a dyadic rational that depends only on the equality type of $(\bar{c}, \bar{u})$.

Let D be the set of equality types of $\bar{c}$ in n; clearly, the size of D is bounded (independently of n) and the cardinality of every $d \in D$ is a polynomial $f_d(n)$. Each equality type e of tuples $(\bar{c}, \bar{u})$ is an extension of an equality type $d \in D$; we write $d \prec e$ when this occurs. If $\bar{c} \in d \prec e$, let $U_e(\bar{c}) = \{\bar{u} : (\bar{c}, \bar{u}) \in e\}$. The cardinality of $U_e(\bar{c})$ is described by a polynomial $g_e(n)$ (which depends only on e). We denote the asymptotic probability of $\varphi(\bar{c}, \bar{u})$ (for $(\bar{c}, \bar{u}) \in e$) by p_e. If $\bar{c} \in d$ is fixed, then the arguments in the first part of this proof show that $E(|B^A(\bar{c})|) = \sum_{d \prec e} E(|B^A(\bar{c}) \cap U_e(\bar{c})|)$ converges exponentially fast to $G_d(n) := \sum_{d \prec e} p_e g_e(n)$ which is a polynomial. Eventually one of the $G_d(n)$ will dominate all the other ones, so asymptotically $F(n) := \max_{d \in D} G_d(n)$ is a polynomial. This implies that

$$E(X) = E(\max_{\bar{c}} |B^A(\bar{c})|) \geq \max_{\bar{c}} E(|B^A(\bar{c})|) \sim \max_{d \in D} \sum_{d \prec e} p_e g_e(n) = F(n).$$

It remains to prove that asymptotically $E(X)/F(n) < 1 + \varepsilon$ for every $\varepsilon > 0$. We first prove a Lemma.

Lemma 4.5 *Let $d \in D$ and $d \prec e$. Then, for every $\varepsilon > 0$, the probability that there exists a tuple $\bar{c} \in D$ such that*

$$|B^A(\bar{c}) \cap U_e(\bar{c})| \geq (1 + \varepsilon) p_e g_e(n)$$

tends to 0 exponentially fast.

PROOF. Fix $k = k(n)$ and define the random variable $Y(A)$ to be the number of tuples $\bar{c} \in d$ such that $B^A(\bar{c}) \cap U_e(\bar{c})$ has cardinality k. We can write Y as the sum of the indicator random variables

$$Y_{\bar{c},U}(A) = \begin{cases} 1 & \text{if } U = B^A(\bar{c}) \cap U_e(\bar{c}) \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{c}$ has equality type d and U is a subset of $U_e(\bar{c})$ of cardinality k. Let $m := g_e(n)$, $p := p_e$ and $q := 1 - p$. Markov's inequality and linearity of expectation give

$$P(Y \geq 1) \leq E(Y) = \sum_{\bar{c},U} E(Y_{\bar{c},U}) = f_d(n) \binom{m}{k} p^k q^{m-k} = f_d(n) b(m, k, p).$$

By Lemma 4.4, if $k \geq (1 + \varepsilon) pm = (1 + \varepsilon) p_e g_e(n)$ then $b(m, k, p)$ converges to 0 exponentially fast. Thus the same holds for the probability that there exists a tuple $\bar{c}$ for which $|B^A(\bar{c}) \cap U_e(\bar{c})|$ exceeds $(1 + \varepsilon) p_e g_e(n)$. ■

Suppose that $E(X) \geq (1 + \varepsilon) F(n)$. Then there is a constant $\varepsilon > 0$ such that there exists with non-negligible probability at least one tuple $\bar{c}$ (of equality type, say, d) with $|B^A(\bar{c})| \geq (1 + \varepsilon) G_d(n)$. But then there must exist an extension e of d such that with non-negligible probability there is a $\bar{c}$ with $|B^A(\bar{c}) \cap U_e(\bar{c})| \geq (1 + \varepsilon) p_e g_e(n)$.

The Lemma just proved shows that this is not the case. This proves the theorem. ■

Applications. As usual in graph theory, let $\omega(G)$, $\alpha(G)$ and $\chi(G)$ denote the size of a maximum clique, the size of a maximum independent set and the chromatic number of a graph G. We use the following results from the theory of random graphs (see [1, 4]).

Fact 4.6 *(i)* $E(\omega) = E(\alpha) \sim 2 \log n$,

(ii) $E(\chi) \sim n/(2 \log n)$.

Together with our probabilistic criterion, this implies that Max Clique and Max Independent Set are not in Max $\Sigma_1^{\mathrm{FP}}(<)$.

There are other important maximization problems Q for which $E(opt_Q)$ does not grow like $\Theta(p(n))$ for any polynomial $p(n)$, and which therefore are not in Max $\Sigma_1^{\mathrm{FP}}(<)$. Examples include the following.

V-C Dimension. Given a collection $S_1, \ldots, S_m$ of subsets of a finite set M, find a set $T \subseteq M$ of maximal cardinality which is shattered by $S_1, \ldots, S_m$ (this means that every subset of T occurs as $T \cap S_i$ for some i). The cardinality of T is called the *Vapnik-Chervonenkis dimension* of $S_1, \ldots, S_m$; it plays an important rôle e.g. in learning theory. We can represent a collection $S_1, \ldots, S_m$ of subsets of n by a binary predicate S over $\max(n, m)$ such that $S_i = \{j : (i, j) \in S\}$.

Max Common Induced Subgraph (MCIS). Given two graphs G and H, find a graph of maximal cardinality which is an induced subgraph of both G and H.

Longest Chordless Path (LCP). Given a graph G, find a set V of nodes, as large as possible, such that $G|_V$ is a simple path.

Note that to apply the probabilistic criterion to these (and other) problems, it is not necessary to determine $E(opt_Q)$ explicitly. Theorem 4.1 implies the following proposition.

Proposition 4.7 *Let Q be a maximization problem. If, over the probability space Ω of all σ-structures with universe n,*

$$1 \leq E(opt_Q) = o(n)$$

then $Q \notin$ Max $\Sigma_1^{\mathrm{FP}}(<)$.

In fact, if we can show that for all $\varepsilon > 0$, the probability that $opt_Q(A) \geq \varepsilon |A|$ tends to 0 exponentially fast then $E(opt_Q) = o(n)$ follows because opt_Q is polynomially bounded.

There is a large class of graph problems, for which this can be established by the following method. For any property P of graphs, let $c_n(P)$ be the number of graphs with vertex set n that satisfy P. Let Max Induced Subgraph with Property P (MIS(P)) be the problem of maximizing the cardinality of a set of nodes V in a given graph G, such that the induced subgraph $G|_V$ has property P. The problems Max Clique, Max Independent Set, Longest Chordless Path are special cases of MIS(P); another example is Max Induced k-Colourable Subgraph.

Theorem 4.8 *If* $0 \leq \log c_n(P) \leq n^2/2 - n^{1+\delta}$ *for some* $\delta > 0$, *then* MIS(P) $\notin$ Max $\Sigma_1^{\mathrm{FP}}(<)$.

PROOF. For $r = r(n)$, let $Y_r(G)$ be the random variable whose value is the number of induced subgraphs in G of cardinality r with property P. Clearly $\text{opt}_{\text{MIS}(P)}(G) = \max\{r : Y_r(G) > 0\}$. By linearity of expectation

$$E(Y_r) = \binom{n}{r} c_r(P) 2^{-\binom{r}{2}}.$$

Stirling's formula implies that $\binom{n}{r} = 2^{O(r)}$ for $n = O(r)$. It follows that

$$P(Y_r \geq 1) \leq E(Y_r) = 2^{O(r)+\log c_r(P) - \binom{r}{2}} \leq 2^{-r^{1+\delta}+O(r)}$$

which tends to 0 exponentially fast as n, and hence r, goes to infinity. Hence, $1 \leq E(\text{opt}_{\text{MIS}(P)}) = o(n)$ and by Proposition 4.7 it follows that $\text{MIS}(P) \notin \text{MAX } \Sigma_1^{\text{FP}}(<)$.
■

The arguments showing that MCIS and V-C-DIMENSION are not in MAX $\Sigma_1^{\text{FP}}(<)$ are very similar.

We also obtain results for minimization problems.

Theorem 4.9 MIN COLOURING *is not in* MIN Σ_1^{FP}.

PROOF. Recall that MIN Σ_1^{FP} = MIN Σ_0^{FP}. Suppose that the chromatic number could be defined by

$$\chi(G) = \min_{\bar{S},\bar{c}} |\{\bar{x} : G \models \psi(\bar{x}, \bar{c}, \bar{S})\}|$$

where ψ is quantifier-free (with predefined fixpoint predicates) and $\bar{x} = x_1, \ldots, x_k$. It follows that $\chi(G)$ can be defined as

$$\chi(G) = n^k - \max_{\bar{S},\bar{c}} |\{\bar{x} : G \models \neg\psi(\bar{x}, \bar{c}, \bar{S})\}| = n^k - opt_Q(G)$$

with a maximization problem $Q \in$ MAX Σ_0^{FP}. But this implies that $E(\chi) = n^k - \Theta(p(n))$ for some polynomial p, which is the not the case, since $E(\chi) \sim n/(2\log n)$.
■

There also is a probabilistic criterion for membership in MAX Π_1^{FP}, which will allow us to separate MAX Π_2^{FP} from MAX Π_1^{FP}.

Theorem 4.10 (Probabilistic criterion for MAX Π_1^{FP}**)** *For every optimization problem $Q \in$ MAX Π_1^{FP} and every natural number $k \in \mathbb{N}$, the property that $opt_Q(A) > k$ satisfies a 0-1 law.*

PROOF. We again use the fact that except on a exponentially decreasing fraction of structures, fixpoint predicates are definable by quantifier-free formulae. Thus, the optimum for $Q \in$ MAX Π_1^{FP}, is defined on almost all structures by an expression

$$opt_Q(A) = \max_{\bar{S},\bar{c}} |\{\bar{x} : A \models \psi(\bar{x}, \bar{c}, \bar{S})\}|$$

where $\psi(\bar{x}, \bar{c}, \bar{S})$ is a Π_1-formula.

Then, the property that $opt_Q(A) > k$ is expressed by the strict Σ_1^1-formula

$$(\exists \bar{S})(\exists \bar{c})(\exists \bar{x}_0)\cdots(\exists \bar{x}_k) \bigwedge_{0\le i<j\le k} (\bar{x}_i \neq \bar{x}_j) \wedge \bigwedge_{0\le i\le k} \psi(\bar{x}_i, \bar{c}, \bar{S}).$$

The theorem now follows from the 0-1 law for strict Σ^1_1-formulae. ■

We did not find natural optimization problems that do not satisfy this criterion. However, we can cook up artificial ones such as Maximum Connected Component with Perfect Matching (MCCPM) which, given a graph, asks for a maximum connected component that admits a perfect matching. Note that MCCPM is solvable in polynomial time.

Proposition 4.11 MCCPM $\notin$ Max Π_1^{FP}.

Proof. With probability tending to 1, a random graph is connected. Obviously, a graph of odd cardinality cannot have a perfect matching, but almost all graphs of even cardinality admit a perfect matching [4]. Thus, $opt_{\mathrm{MCCPM}}(G)$ tends to 0 on graphs with odd cardinality, and to $|G|$ on graphs of even cardinality. ■

Corollary 4.12 Max $\Pi_1^{\mathrm{FP}} \subsetneq$ Max $\Pi_2^{\mathrm{FP}} =$ Max $\mathcal{PB}$.

5 Monotonicity properties

The simple monotonicity properties that were used in [16, 11] to separate expressibility classes of optimization problems do not survive in the presence of fixpoint definitions. Nevertheless we can make use of them to prove inexpressibility results for Max Σ_1^{FP} and Min Σ_1^{FP} by looking at classes of structures where fixpoint logic collapses to Σ_0-formulae. Although such classes cannot contain very interesting structures, they somewhat surprisingly suffice to show that prominent problems such as Max Clique, Max Matching and Min Colouring are not in Max Σ_1^{FP} and Min Σ_1^{FP} respectively.

As usual, we identify n with the set $\{0, \ldots, n-1\}$ and denote the group of all permutations on n by S_n.

Definition 5.1 For all natural numbers n, m, we define the graphs $K_{n;m} = (n\times m, E_{n,m})$ where $E_{n,m} = \{((x,y)(x',y')) : x \neq x'\}$. $K_{n;m}$ is the complete n-partite graph where each partition class contains exactly m vertices.

Let $\sigma \in S_n$ and $\pi_1, \ldots, \pi_n \in S_m$. Then the bijection $(i,j) \longmapsto (\sigma i, \pi_i j)$ on $n \times m$ is an automorphism of $K_{n;m}$, and every automorphism of $K_{n;m}$ can be described in this way.

Definition 5.2 For k-tuples $\bar{u}, \bar{v}$ in $K_{n;m}$ we write $\bar{u} \equiv_k \bar{v}$ if there is an automorphism f of $K_{n;m}$ with $fu_i = v_i$ for $i = 1, \ldots, k$. The equivalence classes with respect to $\equiv_k$ are called $\equiv_k$-types.

The following lemma is easy to prove.

Lemma 5.3 *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that the number of $\equiv_k$-types in $K_{n;m}$ is bounded by $f(k)$ (independently of n and m). Furthermore every $\equiv_k$-type is uniformly definable by a quantifier-free formula. This means that there exist quantifier-free formulae $e_1(\bar{x}), \ldots, e_{f(k)}(\bar{x})$ such that every $\equiv_k$-type in any $K_{n;m}$ is defined by precisely one formula $e_i(\bar{x})$.*

Note that for small n, m, some of the $\equiv_k$-types may not occur in $K_{n;m}$. However, for every k, there exists a n_0 such that for $n, m > n_0$, the number of $\equiv_k$-types in $K_{n;m}$ is precisely $f(k)$.

Proposition 5.4 *For any formula $\varphi(\bar{x}) \in L^{\omega}_{\infty\omega}$ there exists a Σ_0-formula $\alpha(\bar{x})$ and a number n_0 such that for all $n, m > n_0$*

$$K_{n;m} \models (\forall \bar{x})(\varphi(\bar{x}) \leftrightarrow \alpha(\bar{x})).$$

PROOF. Take n, m large enough such that every $\equiv_k$-type e_i is realized by some k-tuple $\bar{u}_i$ in $K_{n;m}$. Let $I(\varphi) = \{i \leq f(k) : K_{n;m} \models \varphi(\bar{u}_i)\}$ and set

$$\alpha(\bar{x}) \equiv \bigvee_{i \in I(\varphi)} e_i(\bar{x}).$$

∎

By proposition 5.4 there exists for every optimization problem Q in MAX $\Sigma_1^{\mathrm{FP}}(<)$ or MIN $\Sigma_1^{\mathrm{FP}}(<)$ an existential first-order formula $\psi(\bar{x}, \bar{c}, \bar{<}, \bar{S})$ (without fixpoint predicates) such that

$$opt_Q(K_{n;m}) = \operatorname*{opt}_{\bar{S}, \bar{<}, \bar{c}} |\{\bar{x} : K_{n;m} \models \psi(\bar{x}, \bar{c}, \bar{<}, \bar{S})\}|$$

for all large enough n, m. Using the fact that existential formulae are closed under extensions, we can then establish a useful monotonicity criterion to prove inexpressibility results even for MAX $\Sigma_1^{\mathrm{FP}}(<)$ and MIN $\Sigma_1^{\mathrm{FP}}(<)$:

Theorem 5.5 *Let Q be an optimization problem on graphs in* MAX $\Sigma_1^{\mathrm{FP}}(<)$ *or in* MIN $\Sigma_1^{\mathrm{FP}}(<)$. *Then either $opt_Q(K_{n;m}) = O(1)$ or there exists a constant n_0 such that $opt_Q(K_{n;m}) < opt_Q(K_{n;m+1})$ for all all $n, m > n_0$.*

Obviously, $\omega(K_{n;m}) = \chi(K_{n;m}) = n$, for all m. Thus Theorem 5.5 implies that MAX CLIQUE $\notin$ MAX $\Sigma_1^{\mathrm{FP}}(<)$ and MIN COLOURING $\notin$ MIN $\Sigma_1^{\mathrm{FP}}(<)$.

A very similar criterion applies to the structures $K_n^d = (n, R^d)$ with the d-ary predicate $R^d = \{(a_1, \ldots, a_d) : a_i \neq a_j$ for all $i \neq j\}$.

Theorem 5.6 *Let Q be an optimization problem on d-ary relations in* MAX $\Sigma_1^{\mathrm{FP}}(<)$ *or* MIN $\Sigma_1^{\mathrm{FP}}(<)$. *Then either $opt_Q(K_n^d) = O(1)$ or there exists a constant n_0 such that $opt_Q(K_n^d) < opt_Q(K_{n+1}^d)$ for all $n > n_0$.*

As applications, we present the problems to find a maximum matching in a graph and to find a maximum (disjoint) covering in a d-dimensional predicate.

MAX MATCHING(MM) is the problem of finding a set of independent edges of maximal size in a given graph. It is well-known that this is solvable in polynomial-time. A *covering* of a d-dimensional predicate R is a subset $M \subseteq R$ of mutually disjoint d-tuples (i.e. if $\bar{u} \in M$ and $\bar{v} \in M$ then $u_i \neq v_j$ for all $i, j \leq d$). MAX d-COVER (MdC) is the problem of finding a maximum covering of a given d-dimensional predicate. Note that MAX 2-COVER is MAX MATCHING. For all d, MAX d-COVER is in APX. Panconesi and Ranjan [16] proved that MAX d-COVER is not in MAX Σ_1. Since $opt_{MdC}(K_{dn}^d) = opt_{MdC}(K_{dn+1}^d) = n$ we can extend this to the following result.

Theorem 5.7 Max Matching *and* Max d-Cover *are not in* Max $\Sigma_1^{\mathrm{FP}}(<)$.

To complete the picture we note that the problem Max Sat separates Max Σ_1 from Max Σ_0^{FP}.

Theorem 5.8 Max Sat $\notin$ Max Σ_0^{FP}.

Corollary 5.9 Max $\Sigma_0^{\mathrm{FP}} \subsetneq$ Max $\Sigma_1^{\mathrm{FP}} \subsetneq$ Max $\Pi_1^{\mathrm{FP}} \subsetneq$ Max $\Pi_2^{\mathrm{FP}} =$ Max $\mathcal{PB}$.

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