Positional Determinacy of Infinite Games *

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Abstract. We survey results on determinacy of games and on the existence of positional winning strategies for parity games and Rabin games. We will then discuss some new developments concerning positional determinacy for path games and for Muller games with infinitely many priorities.

1 Introduction

1.1 Games and strategies

We study infinite two-player games with complete information, specified by an *arena* (or game graph) and a *winning condition*. An arena $\mathcal{G} = (V, V_0, V_1, E, \Omega)$, consists of a directed graph (V, E), equipped with a partioning $V = V_0 \cup V_1$ of the nodes into positions of Player 0 and positions of Player 1, and a function $\Omega : V \to C$ that assigns to each position a *priority* (or colour) from a set C. Although the set V of positions may be infinite, it is usually assumed that C is finite. (We will drop this assumption in Section 4 where we discuss games with infinitely many priorities.)

In case $(v, w) \in E$ we call w a successor of v and we denote the set of all successors of v by vE. To avoid tedious case distinctions, we assume that every position has at least one successor. A *play* of $\mathcal{G}$ is an infinite path $v_0v_1\dots$ formed by the two players starting from a given initial position v_0. Whenever the current position v_i belongs to V_0, then Player 0 chooses a successor $v_{i+1} \in v_iE$, if $v_i \in V_1$, then $v_{i+1} \in v_iE$ is selected by Player 1. The *winning condition* describes which of the infinite plays $v_0v_1\dots$ are won by Player 0, in terms of the sequence $\Omega(v_0)\Omega(v_1)\dots$ of priorities appearing in the play. Thus, a winning condition is given by a set $W \subseteq C^\omega$ of infinite sequences of priorities.

Winning conditions can be specified in several ways. In the theory of Gale-Stewart games as developed in descriptive set theory, the winning condition is just an abstract set $W \subseteq \{0,1\}^\omega$. In computer science applications winning conditions are often specified by formulae from a logic on infinite paths, such as LTL (linear time temporal logic), FO (first-order logic), or MSO (monadic second-order logic) over a vocabulary that uses the linear order $<$ and monadic predicates P_c for each priority $c \in C$. Of special importance are also *Muller conditions*, where the winner of a play depends on the set of priorities that have been seen infinitely often.

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A *(deterministic) strategy* for Player σ is a partial function $f : V^*V_\sigma \to V$ that maps initial segments $v_0 v_1 \dots v_m$ of plays ending in a position $v_m \in V_\sigma$ to a successor $f(v_0 \dots v_m) \in v_m E$. A play $v_0 v_1 \dots \in V^\omega$ is *consistent with* f, if Player σ always moves according to f, i.e., if $v_{m+1} = f(v_0 \dots v_m)$ for every m with $v_m \in V_\sigma$. We say that such a strategy f is winning from position v_0, if every play that starts at v_0 and that is consistent with f, is won by Player σ. The *winning region* of Player σ, denoted W_σ, is the set of positions from which Player σ has a winning strategy.

1.2 Determinacy

A game $\mathcal{G}$ is *determined* if $W_0 \cup W_1 = V$, i.e., if from each position one of the two players has a winning strategy. On the basis of the axiom of choice it is not difficult to prove that there exist nondetermined games. The classical theory of infinite games in descriptive set theory links determinacy of games with topological properties of the winning conditions. Usually the format of Gale-Stewart games is used where the two players strictly alternate, and in each move a player selects an element of $\{0,1\}$; thus the outcome of a play is an infinite string $\pi \in \{0,1\}^\omega$. Gale-Stewart games can be viewed as graph game, for instance on the infinite binary tree, or on a bipartite graph with four nodes. Zermelo [21] proved already in 1913 that if in each play of a game, the winner is determined already after a finite number of moves, then one of the two players has a winning strategy. In topological terms the winning sets in such a game are clopen (open and closed). Before we mention further results, let us briefly recall some basic topological notions.

Topology. We consider the space B^ω of infinite sequences over a set B, endowed with the topology whose basic open sets are $O(x) := x \cdot B^\omega$, for $x \in B^*$. A set $L \subseteq B^\omega$ is *open* if it is a union of sets $O(x)$, i.e., if $L = W \cdot B^\omega$ for some $W \subseteq B^*$. A tree $T \subseteq B^*$ is a set of finite words that is closed under prefixes. It is easily seen that $L \subseteq B^\omega$ is *closed* (i.e., the complement of an open set) if L is the set of infinite branches of some tree $T \subseteq B^*$, denoted $L = [T]$. This topological space is called *Cantor space* in case $B = \{0,1\}$, and *Baire space* in case $B = \omega$.

The class of *Borel sets* is the closure of the open sets under countable union and complementation. Borel sets form a natural hierarchy of classes $\boldsymbol{\Sigma}^0_\eta$ for $1 \leq \eta < \omega_1$, whose first levels are

$\boldsymbol{\Sigma}^0_1$	(or G) :	the open sets
$\boldsymbol{\Pi}^0_1$	(or F) :	the closed sets
$\boldsymbol{\Sigma}^0_2$	(or F_σ) :	countable unions of closed sets
$\boldsymbol{\Pi}^0_2$	(or G_δ) :	countable intersections of open sets

In general, $\boldsymbol{\Pi}^0_\eta$ contains the complements of the $\boldsymbol{\Sigma}^0_\eta$-sets, $\boldsymbol{\Sigma}^0_{\eta+1}$ is the class of countable unions of $\boldsymbol{\Pi}^0_\eta$-sets, and $\boldsymbol{\Sigma}^0_\lambda = \bigcup_{\eta<\lambda} \boldsymbol{\Sigma}^0_\eta$ for limit ordinals λ.

In the 1950s, Gale and Stewart showed that all open games and all closed games are determined. This was then extended in several papers to higher levels of the Borel hierarchy, until Martin [14] proved in 1975 that in fact all games with Borel winning conditions are determined. The theory of infinite games from there branched in several directions. In descriptive set theory one aims to prove determinacy result for stronger, non-Borel games. For game theory that relates to computer science, determinacy is just a first step in the analysis of a game. Rather than in the mere existence of winning strategies, one is interested in reasonably simple winning strategies that can be effectively constructed, are computationally efficient and do not require too much memory. We will focus on this last aspect here.

1.3 Positional Determinacy

In general, winning strategies can be very complicated. It is of interest to determine which games admit simple strategies, in particular *finite memory strategies* and *positional strategies*. While positional strategies only depend on the current position, not on the history of the play, finite memory strategies have access to bounded amount of information on the past. Finite memory strategies can be defined as strategies that are realisable by finite automata.

More formally, a strategy with memory M for Player σ is given by a triple (m_0, U, F) with initial memory state $m_0 \in M$, a memory update function $U : M \times V \to M$ and a next-move function $F : V_\sigma \times M \to V$. Initially, the memory is in state m_0 and after the play has gone through the sequence $v_0 v_1 \dots v_m$ the memory state is $u(v_0 \dots v_m)$, defined inductively by $u(v_0 \dots v_m v_{m+1}) = U(u(v_0 \dots v_m), v_{m+1})$. In case $v_m \in V_\sigma$, the next move from $v_1 \dots v_m$, according to the strategy, leads to $F(v_m, u(v_0 \dots, v_m))$. In case $M = \{m_0\}$, the strategy is positional; it can be described by a function $F : V_\sigma \to V$.

We will say that a game is *positionally determined* if it is determined and both players have positional winning strategies on their winning regions.

2 Muller games, Streett-Rabin games, and parity games

Parity games are graph games with priority labeling $\Omega : V \to \{0, \dots, d\}$ for some $d \in \mathbb{N}$ and parity winning condition: Player 0 wins a play π if the least priority occurring infinitely often in π is even. Parity games are of importance for several reasons [19, 20].

- Parity games are positionally determined. This has been first established by Mostowski [16] and by Emerson and Jutla [5]. An immediate consequence of positional determinacy is that winning regions of parity games can be decided in NP $\cap$ Co-NP.
- Many complicated games can be reduced to parity games (over larger game graphs).

– Parity games arise as the model checking games for fixed point logics. In particular the model checking problem for the modal μ-calculus can be solved in polynomial time if, and only if, winning regions for parity games can be decided in polynomial time.

Parity games are a special case of Muller games.

Definition 1 A *Muller condition* over a finite set C of priorities is written in the form $(\mathcal{F}_0, \mathcal{F}_1)$ where $\mathcal{F}_0 \subseteq \mathcal{P}(C)$ and $\mathcal{F}_1 = \mathcal{P}(C) - \mathcal{F}_0$. A play π in a game with Muller winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ is won by Player σ if, and only if, $\mathrm{Inf}(\pi)$, the set of priorities occurring infinitely in π, belongs to $\mathcal{F}_\sigma$.

The *Zielonka tree* for a Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ over C is a tree $Z(\mathcal{F}_0, \mathcal{F}_1)$ whose nodes are labelled with pairs (X, σ) such that $X \in \mathcal{F}_\sigma$. We define $Z(\mathcal{F}_0, \mathcal{F}_1)$ inductively as follows. Let $C \in \mathcal{F}_\sigma$ and $C_0, \ldots, C_{k-1}$ be the maximal sets in $\{X \subseteq C : X \in \mathcal{F}_{1-\sigma}\}$. Then $Z(\mathcal{F}_0, \mathcal{F}_1)$ consists of a root, labeled by (C, σ), to which we attach as subtrees the Zielonka trees $Z(\mathcal{F}_0 \cap \mathcal{P}(C_i), \mathcal{F}_1 \cap \mathcal{P}(C_i))$, for $i = 0, \ldots, k-1$.

It has been proved by Gurevich and Harrington [8] that Muller games are determined via finite memory strategies. However, Muller games need not be positionally determined, not even for solitaire games (where only one player moves). Consider, for instance, the game with three positions $1, 2, 3$, all belonging to Player 0, with possible moves $(1,2), (2,1), (1,3), (3,1)$, and winning condition $\mathcal{F}_0 = \{\{1,2,3\}\}$ (i.e., all three positions must be seen infinitely often). Clearly Player 0 can win this game, but not with a positional strategy.

Besides parity games there are other important special cases of Muller games. Of special relevance for us are games with Rabin and Street conditions because these are positionally determined for one player [11].

Definition 2 A *Streett-Rabin condition* is a Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ such that $\mathcal{F}_0$ is closed under union.

In the Zielonka tree for a Streett-Rabin condition, the nodes labeled $(X, 1)$ have only one successor. We remark that in the literature, Streett and Rabin conditions are often defined in a different manner, based on a collection $\{(E_i, F_i) : i = 1, \ldots k\}$ of pairs of sets. However, it is not difficult to see that the definitions are equivalent [22]. Further, it is also easy to show that if both $\mathcal{F}_0$ and $\mathcal{F}_1$ are closed under union, then $(\mathcal{F}_0, \mathcal{F}_1)$ is equivalent to a parity condition. The Zielonka tree for a parity condition is just a finite path.

In a Streett-Rabin game, Player 1 has a positional wining strategy on his winning region. On the other hand, Player 0 can win, on his winning region, via a finite memory strategy, and the size of the memory can be directly read of from the Zielonka tree. We present an elementary proof of this result. The exposition is inspired by [4]. In the proof we use the notion of an attractor.

Definition 3 Let $\mathcal{G} = (V, V_0, V_1, E, \Omega)$ be an arena and let $X, Y \subseteq V$, such that X induces a subarena of $\mathcal{G}$ (i.e., every position in X has a successor in X). The *attractor* of Player σ of Y in X is the set $\mathrm{Attr}_\sigma^X(Y)$ of those positions $v \in X$ from which Player σ has a strategy to force the play into Y. More formally $\mathrm{Attr}_\sigma^X(Y) = \bigcup_\alpha Z^\alpha$ where

$$\begin{aligned} Z^0 &= X \cap Y, \\ Z^{\alpha+1} &= Z^\alpha \cup \{v \in V_\sigma \cap X : vE \cap Z^\alpha \neq \emptyset\} \cup \{v \in V_{1-\sigma} \cap X : vE \subseteq Z^\alpha\} \\ Z^\lambda &= \bigcup_{\alpha<\lambda} Z^\alpha \quad \text{for limit ordinals } \lambda \end{aligned}$$

On $\mathrm{Attr}_\sigma^X(Y)$, Player σ has a *positional attractor strategy* to bring the play into Y. Moreover $X \setminus \mathrm{Attr}_\sigma^X(Y)$ is again a subarena.

Theorem 4 *Let $\mathcal{G} = (V, V_0, V_1, E, \Omega)$ be game with Streett-Rabin winning condition $(\mathcal{F}_0, \mathcal{F}_1)$. Then $\mathcal{G}$ is determined, i.e. $V = W_0 \cup W_1$, with a finite memory winning strategy of Player 0 on W_0, and a positional winning strategy of Player 1 on W_1. The size of the memory required by the winning strategy for Player 0 is bounded by the number of leaves of the Zielonka tree for $(\mathcal{F}_0, \mathcal{F}_1)$.*

Proof. We proceed by induction on the number of priorities in C or, equivalently, the depth of the Zielonka tree $Z(\mathcal{F}_0, \mathcal{F}_1)$. Let ℓ be number of leaves of $Z(\mathcal{F}_0, \mathcal{F}_1)$. We distinguish two cases.

We first assume that $C \in \mathcal{F}_1$.

$$X_0 := \{v : \text{Player 0 has a winning strategy with memory of size} \leq \ell \text{ from } v\},$$

and $X_1 = V \setminus X_0$. It suffices to prove that Player 1 has a positional winning strategy on X_1. To construct this strategy, we combine three positional strategies of Player 1, a trap strategy, an attractor strategy, and a winning strategy on a subgame with fewer priorities.

First, we observe that X_1 is a trap for Player 0; this means that Player 1 has a positional trap-strategy t on X_1 to enforce that the play stays within X_1.

Since $\mathcal{F}_0$ is closed under union, there is a unique maximal subset $C' \subseteq C$ with $C' \in \mathcal{F}_0$. Let $Y := X_1 \cap \Omega^{-1}(C \setminus C')$ and let $Z = \mathrm{Attr}_1^{X_1}(Y) \setminus Y$. Observe that Player 1 has a positional attractor strategy a, by which he forces from any position $z \in Z$ that the play reaches Y.

Finally, let $V' = X_1 \setminus (Y \cup Z)$ and let $\mathcal{G}'$ be the subgame of $\mathcal{G}$ induced by V', with winning condition $(\mathcal{F}_0 \cap \mathcal{P}(C'), \mathcal{F}_1 \cap \mathcal{P}(C'))$. Since this game has fewer priorities, the induction hypothesis applies, i.e. $V' = W_0' \cup W_1'$, Player 0 has a winning strategy with memory $\leq \ell$ on W_0' and Player 1 has a positional winning strategy g' on W_1'. However, $W_0' = \emptyset$; otherwise we could combine the strategies of Player 0 to obtain a winning strategy with memory $\leq \ell$ on $X_0 \cup W_0' \supsetneq X_0$ contradicting the definition of X_0. Hence $W_1' = V'$.

We can now define a positional strategy g for Player 1 on X_1 by

$$g(x) = \begin{cases} g'(x) & \text{if } x \in V' \\ a(x) & \text{if } x \in Z \\ t(x) & \text{if } x \in Y \end{cases}$$

Consider any play π that starts at a position $v \in X_1$ and is consistent with g. Obviously π stays within X_1. If it hits $Y \cup Z$ only finitely often, then from some point onward, it stays within V_1 and coincides with a play consistent with g'. It is therefore won by Player 1. Otherwise π hits $Y \cup Z$, and hence also Y, infinitely often. Thus, $\mathrm{Inf}(\pi) \cap (C \setminus C') \neq \emptyset$ and therefore $\mathrm{Inf}(\pi) \in \mathcal{F}_1$.

We now consider the second case, $C \in \mathcal{F}_0$. There exist maximal subsets $C_0, \ldots, C_{k-1} \subseteq C$ with $C_i \in \mathcal{F}_1$. Observe that for every set $D \subseteq C$, we have that if $D \cap (C \setminus C_i) \neq \emptyset$ for all $i < k$, then $D \in \mathcal{F}_0$. Let

$$X_1 := \{v : \text{Player 1 has a positional winning strategy from } v\},$$

and $X_0 = V \setminus X_1$. We claim that Player 0 has a finite memory winning strategy of size $\leq \ell$ on X_0. To construct this strategy, we proceed in a similar way as above, for each of the sets $C \setminus C_i$. We will obtain strategies $f_0, \ldots, f_{k-1}$ for Player 0, such that f_i has finite memory M_i, and we will use these strategies to build a winning strategy f on X_0 with memory $M_0 \cup \cdots \cup M_{k-1}$.

For $i = 0, \ldots, k-1$, let $Y_i = X_0 \cap \Omega^{-1}(C \setminus C_i)$ let $Z_i = \mathrm{Attr}_0^{X_0}(Y_i) \setminus Y_i$, and let a_i be a positional attractor strategy, by which Player 0 can force a play from any position in Z_i to Y_i. Further, let $U_i = X_0 \setminus (Y_i \cup Z_i)$ and let $\mathcal{G}_i$ be the subgame of $\mathcal{G}$ induced by U_i, with winning condition $(\mathcal{F}_0 \cap \mathcal{P}(C_i), \mathcal{F}_1 \cap \mathcal{P}(C_i))$. The winning region of Player 1 in $\mathcal{G}_i$ is empty; indeed, if Player 1 could win $\mathcal{G}_i$ from v, then, by induction hypothesis, he could win with a positional winning strategy. By combining this strategy with the positional winning strategy of Player 1 on X_1, this would imply that $v \in X_1$; but $v \in U_i \subseteq V \setminus X_1$.

Hence, by induction hypothesis, Player 0 has a winning strategy f_i with finite memory M_i on U_i. Let $(f_i + a_i)$ be the combination of f_i with the attractor strategy a_i. From any position $v \in U_i \cup Z_i$ this strategy ensures that the play either remains inside U_i and is winning for Player 1, or it eventually reaches a position in Y_i.

We now combine the finite-memory strategies $(f_0 + a_0), \ldots, (f_{k-1} + a_{k-1})$ to a winning strategy f on X_0, which ensures that either the play ultimately remains within one of the regions U_i and coincides with a play according to f_i, or that it cycles infinitely often through all the regions $Y_0, \ldots, Y_{k-1}$.

At positions in $\bigcap_{i<k} Y_i$, Player 0 just plays with a (positional) trap strategy ensuring that the play remains in X_0. At the first position $v \notin \bigcap_{i<k} Y_i$, Player 0 takes the minimal i such that $v \notin Y_i$, i.e. $v \in U_i \cup Z_i$, and uses the strategy $(f_i + a_i)$ until a position in $w \in Y_i$ is reached. At this point, Player 0 switches from i to $j = i + \ell \pmod{k}$ for the minimal ℓ such that $w \notin Y_j$. Hence $w \in U_j \cup Z_j$; Player 0 now plays with strategy $(f_j + a_j)$ until a position in Y_j is reached. There Player 0 again switches to the appropriate next strategy, and so on.

Assuming that $M_i \cap M_j = \emptyset$ for $i \neq j$ it is not difficult to see that f can be implemented with memory $M = M_0 \cup \cdots \cup M_{k-1}$. We leave a formal definition of f to the reader.

It remains to prove that f is winning on X_0. Let π be a play that starts in X_0 and is consistent with f. If π eventually remains inside some U_i then it coincides, from some point onwards, with a play that is consistent with f_i, and therefore won by Player 0. Otherwise it hits each of the sets $Y_0, \ldots, Y_{k-1}$ infinitely often. But this means that $\mathrm{Inf}(\pi) \cap (C \setminus C_i) \neq \emptyset$ for all $i \leq k$; as observed above this implies that $\mathrm{Inf}(\pi) \in \mathcal{F}_0$.

Note that, by induction hypothesis, the size of the memory M_i is bounded by the number of leaves of the Zielonka subtrees $Z(\mathcal{F}_0 \cap \mathcal{P}(C_i), \mathcal{F}_1 \cap \mathcal{P}(C_i))$. Consequently the size of M is bounded by the number of leaves of $Z(\mathcal{F}_0, \mathcal{F}_1)$.

□

Of course it also follows from this Theorem that parity games are positionally determined.

3 Path games

Another interesting variant of two-player games on graphs are *path games* where in each move a player can select a path of arbitrary finite length rather than just an edge. Such games arise in several contexts.

3.1 Banach-Mazur games

In descriptive set theory, path games have been studied in the form of *Banach-Mazur games* (see [9, Chapter 6] or [10, Chapter 8.H]). In their original variant (see [15, pp. 113–117], the winning condition is a set W of real numbers; in the first move, one of the players selects an interval d_1 on the real line, then his opponent chooses an interval $d_2 \subset d_1$, then the first player selects a further refinement $d_3 \subset d_2$ and so on. The first player wins if the intersection $\bigcap_{n\in\omega} d_n$ of all intervals contains a point of W, otherwise his opponent wins.

By identifying real numbers with infinite sequences of natural numbers, this game is essentially equivalent to a path game on the ω-branching tree T^ω with winning condition $W \subseteq \omega^\omega$. Player 0 starts by selecting a finite path $a_0 a_1 \ldots a_m$ from the root, and in each further move, the player extend the path by another finite sequence of numbers. The outcome of the play is an infinite path π through $\mathcal{T}^\omega$; Player 0 has won if $\pi \in W$, otherwise Player 1 has won. This game is usually denoted $G^{**}(W)$.

A classical result due to Banach and Mazur characterises, in terms of topological properties, the winning conditions W such that one of the two players has a winning strategy for the game $G^{**}(W)$. We recall that a set X in a topological space is *nowhere dense* if its closure does not contain a non-empty open set. A set is *meager* if it is a union of countably many nowhere dense sets and it has the *Baire property* if its symmetric difference with some open set is meager. In particular, every Borel set has the Baire property.

Theorem 5 (Banach-Mazur) (1) *Player 1 has a winning strategy for the game $G^{**}(W)$ if, and only if, W is meager.*

(2) *Player 0 has a winning strategy for $G^{**}(W)$ if, and only if, there exists a finite word $x \in \omega^*$ such that $x \cdot \omega^\omega \setminus W$ is meager (i.e., W is co-meager in some basic open set).*

As a consequence, it can be shown that for any class $\Gamma \subseteq \mathcal{P}(\omega^\omega)$ that is closed under complement and under union with open sets, all games $G^{**}(W)$ with $W \in \Gamma$ are determined if, and only if, all sets in Γ have the Baire property. Since Borel sets have the Baire property, it follows that Banach-Mazur games are determined for Borel winning conditions. (Via a coding argument, this can also been easily derived form Martin's Theorem.)

3.2 Path games in computer science

Pistore and Vardi [17] use path games as a tool for task planning in nondeterministic domains. In their scenario, the desired infinite behaviour of a system is specified by formulae in LTL, and it is assumed that the outcome of actions may be nondeterministic; hence a plan has not only one possible execution path, but an execution tree. Between weak planning (some possible execution path satisfies the specification) which is of course not very useful, and strong planning (all possible outcomes are consistent with the specification) which is often unrealistic, there is a spectrum of intermediate cases such as for instance strong cyclic planning: every possible partial execution of the plan can be extended to an execution reaching the desired goal. In this context, planning can be modelled by a game between a friendly player E and a hostile player A selecting the outcomes of nondeterministic actions. This game is a path game on the execution tree of the plan, and the question is whether the friendly player E has a strategy to ensure that the outcome (an infinite path through the execution tree) satisfies the given LTL-specification.

In [1] we have studied path games in a quite different scenario: once upon a time in the west, two players set out on an infinite ride. More often than not, they had quite different ideas on where to go, but for reasons that have by now been forgotten they were forced to stay together – as long as they were both alive. They agreed on the rule that each player can determine on every second day, where the ride should go. Hence, one of the players began by choosing the first day's ride: he indicated a finite, non-empty path p_1 from the starting point v; on the second day his opponent selected the next stretch of way, extending p_1 to a finite path $p_1 q_1$; then it was again the turn of the first player to extend the path to $p_1 q_1 p_2$ and so on. After ω days, an infinite ride is completed and it is time for payoff. The scenario is quite useful to capture the interest of the audience at at conference, and provides good motivation to study general issues like positional determinacy, algorithmic complexity and logical definability of path games.

In the Banach-Mazur games $G^{**}(W)$, the players strictly alternate. But it is also interesting to study cases, where after a few alternations, one of the players is eliminated, and the games then becomes a solitaire game. In the planning application studied in [17] these are in fact the most relevant cases. For instance, strong cyclic planning corresponds to what we call an AE^ω-game: a single move by A is followed by actions of E. In the scenario of the wild west as investigated in [1] it is of also quite realistic that one of players may not make to the end (see e.g. [12, 13]). To describe the alternation pattern between the players we call the players Ego and Alter, and denote a move where Ego selects a finite path by E and an ω-sequence of such moves by E^ω; for Alter we use corresponding notation A and A^ω. For any fixed triple $\mathcal{G} = (G, W, v)$ consisting of an arena G, a winning condition W and an initial position v, we then have the following games.

- $(EA)^\omega(\mathcal{G})$ and $(AE)^\omega(\mathcal{G})$ are the path games with infinite alternation of finite path moves.
- $(EA)^k E^\omega(\mathcal{G})$ and $A(EA)^k E^\omega(\mathcal{G})$, for arbitrary $k \in \mathbb{N}$, are the games ending with an infinite path extension by Ego.
- $(AE)^k A^\omega(\mathcal{G})$ and $E(AE)^k A^\omega(\mathcal{G})$ are the games where Alter chooses the final infinite lonesome ride.

This infinite collection of games collapses to a finite lattice of just eight different games, a result that has been proved independently in [17] and [1].

Theorem 6 *For every triple* $\mathcal{G} = (G, W, v)$

$$\begin{array}{ccccccc}
E^\omega(\mathcal{G}) & \succeq & EAE^\omega(\mathcal{G}) & \succeq & AE^\omega(\mathcal{G}) & & \\
 & & \rotatebox[origin=c]{-90}{$\succeq$} & & \rotatebox[origin=c]{-90}{$\succeq$} & & \\
 & & (EA)^\omega(\mathcal{G}) & \succeq & (AE)^\omega(\mathcal{G}) & & \\
 & & \rotatebox[origin=c]{-90}{$\succeq$} & & \rotatebox[origin=c]{-90}{$\succeq$} & & \\
 & & EA^\omega(\mathcal{G}) & \succeq & AEA^\omega(\mathcal{G}) & \succeq & A^\omega(\mathcal{G})
\end{array}$$

Further, every path game on $\mathcal{G}$ *is equivalent to one of these eight games.*

For each comparison $\succeq$ in the diagram there are simple games for which it is strict.

3.3 Positional determinacy of path games

Path games with only finitely many alternations between the two players are trivially determined, for whatever winning condition, and for path games with infinite alternations, the Banach-Mazur Theorem establishes determinacy for a very large class of winning conditions, covering almost all path games that are likely to appear in computer science applications.

As for the usual graph games, the question arises, which path games admit positional winning stratgies. Observe that a positional strategy for a path game on the graph $G = (V, F)$ has the form $f : V \to V^*$ assigning to every node v a finite path from v through G.

We first look at path games with infinite alternations.

Proposition 7 *If Ego has a winning strategy for a path game* $(EA)^\omega(G, W, v)$ *with* $W \in \boldsymbol{\Sigma}^0_2$*, then he also has a positional winning strategy.*

Proof. Let $G = (V, F)$ be the game graph. Since W is a countable union of closed sets, we have $W = \bigcup_{n<\omega}[T_n]$ where each $T_n \subseteq V^*$ is a tree. Further, let f be any (non-positional) winning strategy for Ego. We claim that, in fact, Ego can win with one move.

We construct this move by induction. Let x_1 be the initial path chosen by Ego according to f. Let $i \geq 1$ and suppose that we have already constructed a finite path $x_i \notin \bigcup_{n<i} T_n$. If $x_i y \in T_i$ for all finite y, then all infinite plays extending x_i remain in W, hence Ego wins with the initial move $w = x_i$. Otherwise choose some y_i such that $x_i y_i \notin T_i$, and suppose that Alter prolongs the play from x_i to $x_i y_i$. Let $x_{i+1} := f(x_i y_i)$ the result of the next move of Ego, according to his winning strategy f.

If this process did not terminate, then it would produce an infinite play that is consistent with f and won by Alter. Since f is a winning strategy for Ego, this is impossible. Hence there exists some $m < \omega$ such that $x_m y \in T_m$ for all y. Thus, if Ego moves to x_m in his opening move, then he wins, no matter how the play proceeds afterwards. In particular, Ego wins with a positional strategy. □

We cannot extend this observation beyond the $\boldsymbol{\Sigma_2}$-level of the Borel hierarchy. Consider the path game on the completely connected directed graph with nodes 0 and 1, with a $\boldsymbol{\Pi}_2$-winning condition for Ego, consisting of those infinite plays that have infinitely many initial segments containing more ones than zeros. Clearly, Ego has a winning strategy for $(EA)^\omega(G, W, v)$, but not a positional one.

Muller and S1S winning conditions. We recall that there are very simple examples of Muller games that do not admit positional winning strategies. For path games the situation is different.

Proposition 8 *All path games $(EA)^\omega(\mathcal{G})$ with a Muller winning condition admit positional winning strategies.*

Proof. We will write $v \leq w$ to denote that position w is reachable from v in the arena G. For every position v, let $C(v)$ be the set of priorities reachable from v, that is, $C(v) := \{\Omega(w) : v \leq w\}$. Obviously, $C(w) \subseteq C(v)$ whenever $v \leq w$. We call v a *stable* position if $C(w) = C(v)$ for all w that are reachable from v. Note that from every u some stable position is reachable. Further, if v is stable, then every reachable position $w \geq v$ is stable as well.

Let the set of winning plays in $\mathcal{G}$ be defines by a Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$. We claim that Ego has a winning strategy in $(EA)^\omega(\mathcal{G})$ iff there is a stable position v that is reachable from the initial position v_0, so that $C(v) \in \mathcal{F}_0$. To see this, let us assume that there is such a stable position v with $C(v) \in \mathcal{F}_0$. Then, for every $u \geq v$, we choose a path p from u so that, when moving along p, each colour of $C(u) = C(v)$ is visited at least once, and set $f(u) := p$. In case v_0 is not reachable from v, we assign $f(v_0)$ to some path that leads from v_0 to v. Now f is a positional winning strategy for Ego in $(EA)^\omega(\mathcal{G})$, because, after the first move, no colours other then those in $C(v)$ are seen. Moreover, every colour in $C(v)$ is visited at each move of Ego, hence, infinitely often.

Conversely, if for every stable position v reachable from v_0 we have $C(v) \in \mathcal{F}_1$, then we can construct a winning strategy for Alter in a similar way. □

Note that in a finite arena all positions of a strongly connected component that is terminal, i.e., with no outgoing edges, are stable. Thus, the above characterisation translates as follows: Ego wins the game iff there is a terminal component whose set of colours belongs to $\mathcal{F}_0$. Obviously this can be established in linear time w.r.t. the size of the arena and the description of the winning condition.

Corollary 9 *On a finite arena G, path games with a Muller winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ can be solved in time $O(|G| \cdot |\mathcal{F}_\sigma|)$.*

In fact, this result can be extended to a very large class of path games, with general winning conditions definable in S1S. For path games with finite alternations there are, however, also some cases where memory is required. The situation is summarized by the following result.

Theorem 10 *Let $\gamma(G, \varphi)$ denote the path game on the arena G with alternation pattern γ and winning condition defined by the S1S-formula φ.*

(1) *All S1S path games $\gamma(G, \varphi)$ are determined via finite memory strategies.*
(2) *All S1S path games $(EA)^\omega(G, \varphi)$ and $(AE)^\omega(G, \varphi)$ are positionally determined.*
(3) *For future conditions $\varphi \in$ S1S, all path games $\gamma(G, \varphi)$ are positionally determined.*
(4) *For any game prefix γ with finite alternations there exist games $\gamma(G, \varphi)$ with $\varphi \in$ S1S that do* not *admit positional winning strategies.*

Here, a future condition is a formula that does not depend on initial segments: for any ω-word π and any pair of finite words x and y, we have $x\pi \models \psi$ if, and only if, $y\pi \models \psi$. We just sketch the proof. First of all, it is not difficult to prove that path games with parity winning condition are positionally determined for any game prefix. (By Theorem 6 it suffices to consider the eight prefixes $E^\omega, A^\omega, AE^\omega, EA^\omega$, EAE^ω, AEA^ω, $(EA)^\omega$, and $(AE)^\omega$.)

One can then use parity games as an instrument to investigate path games with winning conditions specified in by arbitrary S1S-formulae. It is well known that every S1S-formula φ is equivalent to a deterministic parity automaton $\mathcal{A}$ (see e.g. [6]). To prove (1), we analyse a path game on G with winning condition φ by considering two games on the product arena $G \times \mathcal{A}$, one, denoted $\mathcal{H}[G]$ with the priority labelling inherited from G and winning condition φ, the other one, denoted $\mathcal{H}[\mathcal{A}]$ with priorities inherited from $\mathcal{A}$ and the parity winning condition.

- A play through $G \times \mathcal{A}$ is winning for Ego in $\mathcal{H}[G]$ if and only if it winning for Ego in $\mathcal{H}[\mathcal{A}]$.
- The two arenas G and $\mathcal{H}[G]$ are bisimilar.

The positional determinacy of parity path games then implies positional determinacy of $\mathcal{H}[G]$ which in turn implies obtains finite memory determinacy for the original game; the value $f(v,q)$ of a winning strategy depends on the current position v in G and a state q of the automaton $\mathcal{A}$.

To prove (2) we have to unify, for each position $v \in G$, the values $f(v,q)$ for those pairs (v,q) that are reachable in a play of according to f. Let us assume that Ego wins the game $(EA)^\omega(G,\varphi)$ starting from position v_0. We will base our argumentation on the assiciated game $\mathcal{H}[G]$ for which Ego has a positional winning strategy f.

For any v, we denote by $Q_f(v)$ the set of states q so that the position (v,q) can be reached from position (v_0,q_0) in a play according to f. Let $\{q_1,q_2,\ldots,q_n\}$ be an enumeration of $Q_f(v)$, in which the initial state q_0 is taken first, in case it belongs to $Q_f(v)$. We construct a path associated to v along the following steps. First, set $p_1 := f(v,q_1)$; for $1 < i \leq n$, let (v',q') be the node reached after playing the path $p_1 \cdot p_2 \cdot \cdots \cdot p_{i-1}$ from position (v,q_i) and set $p_i := f(v',q')$. Finally, let $f'(v)$ be the concatenation of $p_1,p_2,\ldots,p_n$.

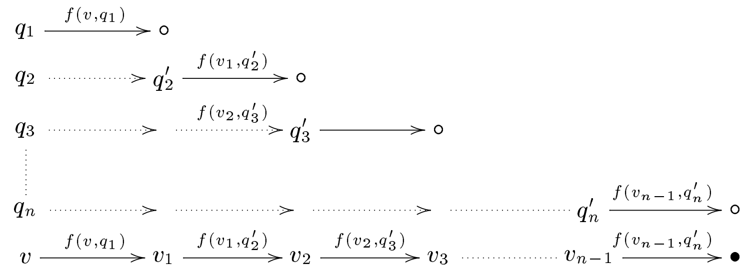

Fig. 1. Merging strategies at node v

Now, consider a play in $\mathcal{H}[G]$ in which Ego chooses the path $f'(v)$ at any node $(v,q) \in V \times Q$. This way, the play will start with $f(q_0,v_0)$. Further, at any position (v,q) at which Ego moves, $f'(v)$ contains some segment of the form $(v',q') \cdot f(v',q')$. In other words, every move of Ego has some "good part" which would also have been produced by f at the position (v',q'). But this means that the play coincides with a play where Ego always moves according to the strategy f while all the "bad parts" were produced by Alter. This proves that f' is a positional strategy for Ego in the game $\mathcal{H}[G]$. Since the values do not depend on the second component, f' induces a positional strategy for Ego in $(EA)^\omega(G,\psi)$. The same construction works for the case $(AE)^\omega(G,\psi)$, if we take instead of $Q_f(v)$ the set $Q(v) := \{\delta(q_0,s) : s \text{ is a path from } v_0 \text{ to } v\}$.

The argument for (2) relies on players always taking turns. If we consider games where the players alternate only finitely many times, the situation changes. Consider, for instance, the winning condition $\psi \in$ S1S that requires the number of zeroes occurring in a play to be odd on the completely connected arenea with two positions $0, 1$. When starting from position 1, Ego obviously has winning strategies for each of the games $E^\omega(G, \psi)$, $AE^\omega(G, \psi)$, and $EAE^\omega(G, \psi)$, but no positional ones. Nevertheless, such games are always positionally determined for one of the players. Indeed, if a player wins a game $\gamma(G, \psi)$ finally controlled by his opponent, he always has a positional winning strategy. This is trivial when $\gamma \in \{E^\omega, A^\omega, AE^\omega, EA^\omega\}$; for the remaining cases EAE^ω and AEA^ω a positional strategy can be constructed as above. For a proof of (3) the reader is refered to [1].

4 Games with infinitely many priorities

Muller games and parity games have been studied extensively both for the cases of finite and for infinite game graphs. However, even in the case of infinite game graphs, it has always been assumed that the positions are labelled with a finite set of priorities, and this is essential for the proofs of positional determinacy exhibitted above, that proceed by induction on the number of priorities or on the depth of the Zielonka tree.

We find it interesting to generalise the theory of infinite games to the case of infinitely many priorities. Besides the theoretical interest, such games arise in several contexts. For instance, pushdown games with winning conditions depending on stack contents as considered in [2, 3] can be viewed as special cases of Muller or parity games with infinitely many priorities. We have started to and report here on some of the results. For more information the reader is referred to [7].

The definition of Muller games (Definition 1) directly generalises to countable sets C of priorities[1]. However, a representation of a Muller condition by a Zielonka tree is not always possible, since we may have sets $D \in \mathcal{F}_\sigma$ that have subsets in $\mathcal{F}_{1-\sigma}$ but no maximal ones. Further, it turns out that the condition that $\mathcal{F}_0$ and $\mathcal{F}_1$ are both closed under finite unions is no longer sufficient for positional determinacy. To see this let us discuss the possible generalisations of parity games to the case of priority assigments $\Omega : V \to \omega$. For parity games with finitely many priorities it is of course purely a matter of taste whether we let the winner be determined by the least priority seen infinitely often or by the greatest one. Here this is no longer the case.

Parity games are games where Player 0 wins the plays in which the least priority seen infinitely often is even, or where no priority appears infinitely often. Thus,

$$\mathcal{F}_0 = \{X \subseteq \omega : \min(X) \text{ is even}\} \cup \{\emptyset\}$$
$$\mathcal{F}_1 = \{X \subseteq \omega : \min(X) \text{ is odd}\}$$

[1] With minor modifications, it can also be generalised to uncountable sets C. See [7] for a discussion of this.

Max-parity games are games where Player 0 wins if the maximal priority occurring infinitely often is even, or does not exist, i.e.

$$\mathcal{F}_0 = \{X \subseteq \omega : \text{ if } X \text{ is finite and non-empty, then } \max(X) \text{ is even}\}$$
$$\mathcal{F}_1 = \{X \subseteq \omega : X \text{ is finite, non-empty, and } \max(X) \text{ is odd}\}$$

Note that for both definitions, $\mathcal{F}_0$ and $\mathcal{F}_1$ are closed under finite unions. Nevertheless the two conditions behave quite differently. The parity condition has a very simple Zielonka tree, namely just a Zielonka path

$$\omega \longrightarrow \omega \setminus \{0\} \longrightarrow \omega \setminus \{0,1\} \longrightarrow \omega \setminus \{0,1,2\} \longrightarrow \cdots$$

whereas there is no Zielonka tree for the max-parity condition since $\omega \in \mathcal{F}_0$ has no maximal subset in $\mathcal{F}_1$ (and $\mathcal{F}_1$ is not closed under unions of chains). This is in fact related to a much more important difference concerning the memory needed for winning strategies. Indeed, consider the max-parity game with positions $V_0 = \{0\}$ and $V_1 = \{2n+1 : n \in \mathbb{N}\}$ (where the name of a position is also its priority), such that Player 0 can move from 0 to any position $2n+1$ and Player 1 can move back from $2n+1$ to 0. Clearly Player 0 has a winning strategy from each position but no winning strategy with finite memory.

Hence positional determinacy, and even finite-memory determinacy fails for max-parity games with infinitely many priorities. On the other hand we prove in [7] that parity games with priorities in ω do admit positional winning strategies for both players. In fact, parity games over ω turn out to be the only Muller games with this property.

Theorem 11 *A Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ over a countable set C of priorities admits positional winning strategies if, and only if, it is isomorphic to a parity condition over $n \leq \omega$ priorities.*

This discrepancy between (min-)parity games and max-parity games has an interesting application to a classical problem posed in [18]. The curious reader is refered to [7],

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