On the Power of Deterministic Transitive Closures*

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Final version

Abstract

We show that transitive closure logic (FO + TC) is strictly more powerful than deterministic transitive closure logic (FO + DTC) on finite (unordered) structures. In fact, on certain classes of graphs, such as hypercubes or regular graphs of large degree and girth, every DTC-query is bounded and therefore first order expressible. On the other hand there are simple (FO + pos TC) queries on these classes that cannot be defined by first order formulae.

1 Introduction and Main Theorem

The transitive closure query, denoted TC, is one of simplest and most fundamental queries that require some kind of recursion. It assigns to a binary relation E its reflexive and transitive closure TC(E). It is well known that TC is not expressible in first order logic (FO), in fact not even in monadic Σ_1^1 (see Fagin (1975)). On the other hand, almost every conceivable logic or database query language with recursive constructs will be able to define some kind of transitive closure. In fact, TC is one of the common examples that are usually given when extensions of first-order logic such as fixpoint logic (FO + LFP) or query languages like Datalog are introduced.

Immerman (1987) has introduced several variants of transitive closure logics. These logics extend first order logic by transitive closure operators which build new formulae from old ones.

Definition 1 Let $\varphi(\bar{x}, \bar{y})$ be a formula containing the $2k$ free variables $\bar{x} = x_1, \ldots, x_k$ and $\bar{y} = y_1, \ldots, y_k$ and let $\bar{u}$ and $\bar{v}$ be two k-tuples of terms. Then

$$[\mathrm{TC}_{\bar{x},\bar{y}}\ \varphi(\bar{x}, \bar{y})](\bar{u}, \bar{v})$$

is also a formula which says that the pair $(\bar{u}, \bar{v})$ is contained in the reflexive, transitive closure of the binary relation on k-tuples that is defined by φ.

We will also consider the *symmetric transitive closure* operator STC, defined by

$$[\mathrm{STC}_{\bar{x},\bar{y}}\ \varphi(\bar{x}, \bar{y})](\bar{u}, \bar{v}) \equiv [\mathrm{TC}_{\bar{x},\bar{y}}\ \varphi(\bar{x}, \bar{y}) \vee \varphi(\bar{y}, \bar{x})](\bar{u}, \bar{v})$$

and the *deterministic transitive closure* operator DTC, defined by

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$$[\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi(\bar{x},\bar{y})](\bar{u},\bar{v}) \equiv [\mathrm{TC}_{\bar{x},\bar{y}}\ \varphi(\bar{x},\bar{y}) \wedge \forall \bar{z}(\varphi(\bar{x},\bar{z}) \rightarrow \bar{y} = \bar{z})](\bar{u},\bar{v}).$$

The logics (FO + TC), (FO + STC) and (FO + DTC) are obtained by increasing the syntax of first order logic by these additional rules for building formulae. Note that transitive closure logics are closed under the usual first order operations. We thus can build Boolean combinations of TC-formulae, we can nest TC-operators, etc. The positive fragments of these logics, (FO + pos TC), (FO + pos STC) and (FO + pos DTC) contain those formula where the transitive closure operators occur only positively.

Immerman was interested in these logics, because they provide characterizations of logspace complexity classes. We say that a logic $\mathcal{L}$ *captures* a complexity class $\mathcal{C}$ on a class of finite structures $\mathcal{O}$, if the queries on $\mathcal{O}$ that are expressible in $\mathcal{L}$ coincide with the queries on $\mathcal{O}$ in the complexity class $\mathcal{C}$. Logical characterizations are known for all major complexity classes, but for PTIME and below they only hold on the class of *ordered finite structures*, i.e. the structures that contain among their relations a total ordering $<$ of the universe (and/or a successor relation). Gurevich (1988) and Immerman (1989) give surveys on this topic. Immerman (1987) proved

Theorem 1 *On the class of all ordered finite structures*

(i) (FO + DTC) *captures* LOGSPACE*;*

(ii) (FO + pos STC) *captures* SYMLOGSPACE*;*

(iii) (FO + TC) *captures* NLOGSPACE*.*

Here, SYMLOGSPACE denotes the class of languages accepted by symmetric Turing machines with logarithmic space. Symmetric Turing machines are nondeterministic Turing machines with symmetric transition relation; if the machine can move from configuration C to configuration C', then also from C' to C. A perhaps more natural definition is that SYMLOGSPACE is the class of all problems that are log-space reducible to the reachability problem on *undirected* graphs. To be precise, Immerman's original proof only showed that (FO + pos TC) captures NLOGSPACE. However, the result of Immerman (1988) and Szelepcsényi (1988) that NLOGSPACE is closed under complementation then implied that, on ordered structures, (FO + pos TC) = (FO + TC).

It is well-known that Theorem 1 fails on unordered structures. For example, (FO + TC) is unable to express: "the cardinality of the universe is even", a query that clearly is in LOGSPACE. In fact this query is not expressible even in much more powerful languages, such as fixpoint logic, partial fixpoint logic (also denoted *while*), and the infinitary logic $L^{\omega}_{\infty\omega}$. For more results on the structure of (FO + TC), we refer to Grädel (1991) and Grädel, McColm (1992).

In this paper we investigate the power of deterministic transitive closure logic on unordered finite structures. We shall prove that on certain classes of finite graphs, (FO + DTC) collapses to first-order logic, but (FO + pos STC) but does not. For this purpose we will discuss different criteria that imply that every path arising from the evaluation of a DTC-formula has bounded length and thus yield the collapse of (FO + DTC) to first-order logic.

The simplest case occurs when every set of nodes can be embedded in a "small" fixed point set of a subgroup of the automorphism group. This means that given any fixed collection of nodes, only a bounded number of the other nodes are uniquely determined. As a consequence, every DTC-path must remain inside an area of bounded size. We call a class of graphs that satisfies this criterion *flexible*. There exists a simple flexible class of graphs, the so-called *double graphs*, due to Immerman (1992), which suffices to separate (FO + DTC) from (FO + TC).

We will then describe a different criterion, the 'short path criterion' which is more complicated to state, but more liberal and therefore more convenient on natural classes of graphs. It exploits the local character of first-order formulae. We will introduce this criterion in section 4 and prove, using Gaifman's Theorem, that every (FO + DTC) query on a class of graphs that satisfies the short path criterion is in fact first-order expressible.

The most important class that satisfies the short path criterion is the class of all *hypercubes*; another example is provided by regular graphs of large degree and girth. On the other side it is easy to find (FO + pos STC) queries on these classes that are not in first-order logic. In particular, we conclude

Theorem 2 (FO + DTC) *is strictly weaker than* (FO + pos STC). *In particular, deterministic transitive closure logic has strictly less expressive power than (nondeterministic) transitive closure logic.*

This result has been independently proved by Immerman (1992). Incidentally, we consider graph queries for expository reasons only. The results of sections 3 and 4 apply to all classes of relational structures, using Gaifman's notion of distance (Gaifman (1982)).

Theorem 1 shows that it is probably difficult to extend the separation of deterministic from nondeterministic transitive closure logic to *ordered* finite structures: such a result would imply that LOGSPACE $\neq$ NLOGSPACE. Actually, ordered structures are rigid, so the automorphism properties we rely on break down on ordered structures.

Remark. This is one of two papers resulting from a joint project investigating expressibility properties of (FO + TC)-like languages on finite (unordered) structures. In the other (see Grädel, McColm (1992)), we use some rather contrived graphs to establish a Hierarchy Theorem on (FO + TC) and $L^{\omega}_{\infty\omega}$ which implies, among other things, that (FO + pos TC) is not closed under negations, and that the hierarchy generated by counting interleavings of TC and $\neg$ does not collapse (thus confirming a conjecture in Immerman (1987)). Since we find here, that (FO + pos DTC) = (FO + DTC) — and hence the hierarchy generating interleavings of DTC and $\neg$ does collapse — the Hierarchy Theorem also implies that (FO + DTC) $\neq$ (FO + TC).

However, the techniques in Grädel, McColm (1992) do not really tell us very much about the expressive power of the DTC operator on structures that are not particularly uniform. In particular they do not give us the collapse of (FO + DTC) to first-order logic on natural (and much-studied) classes like hypercubes.

2 Deterministic and symmetric transitive closures

We first show a few elementary relationships between transitive closure logics:

Theorem 3 *On finite structures*

(i) (FO + pos DTC) = (FO + DTC)*;*

(ii) (FO + DTC) $\subseteq$ (FO + pos STC)*;*

(iii) (FO + pos STC) $\subseteq$ (FO + pos TC).

PROOF. *(i)* A negated DTC-formula $\neg[\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi(\bar{x},\bar{y})](\bar{a},\bar{b})$ can be expressed in (FO + pos DTC) by a formula which says that the deterministic path starting at a that is defined by the formula $\varphi(\bar{x},\bar{y}) \wedge (\bar{y} \neq \bar{b})$ is the same as the path defined by $\varphi(\bar{x},\bar{y})$.

First let

$$\alpha(\bar{u},\bar{z},\bar{b}) \equiv (\bar{u} \neq \bar{b}) \wedge [\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi(\bar{x},\bar{y})](\bar{u},\bar{z}) \wedge [\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi(\bar{x},\bar{y}) \wedge (\bar{y} \neq \bar{b})](\bar{u},\bar{z})$$

The formula $\alpha(\bar{u},\bar{z},\bar{b})$ says that the deterministic φ-path leaving $\bar{u}$ reaches $\bar{z}$ without passing through $\bar{b}$.

We distinguish three possibilitites: The path along φ-arcs leaving $\bar{a}$ will eventually reach a vertex without an outgoing arc, or a vertex with more than one outgoing arc, or it will reach a cycle. Thus

$$\neg[\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi(\bar{x},\bar{y})](\bar{a},\bar{b})$$

is equivalent to

$$(\exists\bar{z})\Big\{\alpha(\bar{a},\bar{z},\bar{b}) \wedge \Big([(\forall\bar{u})(\varphi(\bar{z},\bar{u}) \rightarrow (\exists\bar{v})(\bar{v} \neq \bar{u} \wedge \varphi(\bar{z},\bar{v}))] \vee (\exists\bar{u} \neq \bar{z})(\alpha(\bar{z},\bar{u},\bar{b}) \wedge \alpha(\bar{u},\bar{z},\bar{b}))\Big)\Big\}$$

which is in (FO + pos DTC).

(ii) We claim that a formula $[\mathrm{DTC}_{\bar{x},\bar{y}}\ \alpha(\bar{x},\bar{y})](\bar{a},\bar{b})$ is equivalent to $[\mathrm{STC}_{\bar{x},\bar{y}}\ \beta(\bar{x},\bar{y})](\bar{a},\bar{b})$ where

$$\beta(\bar{x},\bar{y}) \equiv \alpha(\bar{x},\bar{y}) \wedge (\forall\bar{z})(\alpha(\bar{x},\bar{z}) \rightarrow \bar{z} = \bar{y}) \wedge \bar{x} \neq \bar{b}.$$

It is clear that $[\mathrm{DTC}_{\bar{x},\bar{y}}\ \alpha(\bar{x},\bar{y})](\bar{a},\bar{b})$ implies $[\mathrm{STC}_{\bar{x},\bar{y}}\ \beta(\bar{x},\bar{y})](\bar{a},\bar{b})$. Conversely, suppose that $[\mathrm{STC}_{\bar{x},\bar{y}}\ \beta(\bar{x},\bar{y})](\bar{a},\bar{b})$ is true; then there exist elements $\bar{a} = \bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{r-1}, \bar{z}_r = \bar{b}$ such that for all $i < r$, either

$$\alpha(\bar{z}_i,\bar{z}_{i+1}) \wedge (\forall\bar{z})(\alpha(\bar{z}_i,\bar{z}) \rightarrow \bar{z} = \bar{z}_{i+1}) \wedge \bar{z}_i \neq \bar{b} \quad \text{or}$$

$$\alpha(\bar{z}_{i+1},\bar{z}_i) \wedge (\forall\bar{z})(\alpha(\bar{z}_{i+1},z) \rightarrow \bar{z} = \bar{z}_i) \wedge \bar{z}_{i+1} \neq \bar{b}.$$

It suffices to show that for all i, the first of the two formulae is satisfied.

If not, there would exist a minimal $i < r$ such that $\alpha(\bar{z}_{i+1},\bar{z}_i)$ is true. But then $\alpha(\bar{z}_{i+1},\bar{z}_{i+2})$ cannot be true, otherwise $\bar{z}_{i+1}$ would have out-degree greater than one; it follows that $\alpha(\bar{z}_{j+1},\bar{z}_j) \wedge \bar{z}_{j+1} \neq \bar{b}$ must hold for all $j \geq i$, in particular for $j = r-1$. But this cannot be the case because $\bar{z}_r = \bar{b}$.

(iii) The inclusion (FO + pos STC) $\subseteq$ (FO + pos TC) follows immediately from the definitions.
■

3 Flexible structures

To prove that on a class $\mathcal{C}$ of graphs, (FO + DTC) collapses to first-order logic, it suffices to show that all paths arising from the evaluation of a DTC-formula on a graph $G \in \mathcal{C}$ are short, i.e., that their length is bounded by a constant that depends only on the formula, but not on the graph.

In this section we exhibit a simple sufficient criterion for this, and show that it is powerful enough to separate (FO + DTC) from (FO + TC). Intuitively this criterion requires that every set of nodes is contained in a "small" *fixed point set* of a subgroup of the automorphism group.

Definition 2 Let $G = (V, E)$ be a graph. We call a set of nodes $U \subseteq V$ *stable* if for every node $v \notin U$ there exists an automorphism π of G pointwise fixing U with $\pi v \neq v$.

Note that a set is stable if and only if it is the set of fixed points of a subgroup of $\mathrm{Aut}(G)$. Indeed, for every $U \subseteq V$ we denote by A^U the group of all automorphisms of G that fix U pointwise. In turn, we define the set $\mathrm{FP}(U) = \{u \in V : (\forall a \in A^U) au = u\}$ of all fixed points of A^U. Obviously $U \subseteq \mathrm{FP}(U)$ for all U, and U is stable if and only if $U = \mathrm{FP}(U)$. Moreover $\mathrm{FP}(U)$ is the smallest stable set containing U.

We start with the following observation.

Lemma 4 *Let G be a graph and U a stable set of nodes. Then every path that starts inside U and which is defined by a DTC-formula must remain inside U. More precisely, for every formula $\varphi(\bar{x}, \bar{y})$ whose constants are (interpreted by nodes) in U and for all $u_1 \ldots, u_k \in U$ the following holds:*

$$G \models [\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi](\bar{u}, \bar{v}) \implies v_1, \ldots, v_k \in U.$$

Proof. Suppose that $G \models \varphi(\bar{u}, \bar{v})$ where $u_1 \ldots, u_k \in U$ but $v_i \notin U$ for some i. Choose an automorphism π fixing U with $\pi v_i \neq v_i$ for that i. Since π fixes $\bar{u}$ and all constants occurring in φ, it follows that also $G \models \varphi(\bar{u}, \pi\bar{v})$ and therefore $G \models \neg[\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi](\bar{u}, \bar{v})$. ∎

Definition 3 For any function $g : \mathbb{N} \to \mathbb{N}$, we say that a graph $G = (V, E)$ is *g-flexible* if $|\mathrm{FP}(U)| \leq g(|U|)$ for every set $U \subseteq V$. This means that every node set U is contained in a stable set of cardinality at most $g(|U|)$. A class $\mathcal{C}$ of graphs is called *flexible* if all its graphs are g-flexible for some fixed function g.

Theorem 5 *Let $\mathcal{C}$ be flexible. Then every* (FO + DTC)*-query on $\mathcal{C}$ is expressible in first-order logic.*

Proof. Suppose that there exists a function $g(n)$ such that every graph $G \in \mathcal{C}$ is g-flexible. Let φ be a first-order formula with free variables $\bar{x} = x_1, \ldots, x_k$ and $\bar{y} = y_1, \ldots, y_k$ and possibly additional free variables $z_1, \ldots, z_\ell$. We claim that for every graph $G \in \mathcal{C}$ and every interpretation of $z_1 \ldots, z_\ell$ by nodes of G, the length of every path arising from the evaluation of the formula

$$[\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi](\bar{u}, \bar{v})$$

has length at most $r = [g(k + \ell)]^k$. Indeed, the path defined by φ that starts at a k-tuple $\bar{u}$ remains inside the smallest stable set containing $u_1, \ldots, u_k$ and $z_1 \ldots, z_\ell$. This set has at most $g(k + \ell)$ elements; the set of k-tuples over this set and thus the length of the path is therefore bounded by r. Since r depends only on φ, the DTC-formula can be unfolded to a first-order formula.

The collapse of (FO + DTC) to FO now follows by induction on the complexity of (FO + DTC)-formulae. ∎

The notion of flexibility leads to very simple classes of structures on which (FO + DTC) collapses to FO, but (FO + TC) does not. Probably the most simple ones are due to Immerman (1992); they consist of what we call *double graphs*.

Definition 4 Let $G = (V, E)$ be a (directed or undirected) graph. Then we denote by $2G$ the graph with vertex set $V' = V \times \{0, 1\}$ and edge predicate $E' = \{(x, i)(y, j) \mid (x, y) \in E\}$.

Proposition 6 *Let $\mathcal{C}$ be any class of graphs. Then $2\mathcal{C} := \{2G \mid G \in \mathcal{C}\}$ is flexible.*

PROOF. The *partner* of a node $\mathbf{u} = (u, i)$ in a double graph $2G$ is the node $\hat{\mathbf{u}} = (u, 1-i)$. Note, that switching any node with its partner is an automorphism of $2G$. Given any set $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ in $2G$, the set consisting of $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and their partners is stable. Therefore every double graph is $2n$-flexible. ∎

Theorem 7 *On finite graphs,* (FO + DTC) $\subsetneq$ (FO + pos STC).

PROOF. Let $\mathcal{C}$ be the class of all cycles and pairs of cycles. Then easily Connectivity is (FO + pos STC)-expressible but not FO-expressible on $2\mathcal{C}$. ∎

4 The short path criterion

While flexible classes suffice to separate (FO + DTC) from (FO + TC), natural graphs tend not to be flexible. To consider the power of the DTC operator on classes of graphs that occur in practice, it is desirable to have a more liberal requirement than flexibility. We present here such a criterion.

In the previous section we used the fact that no deterministic path can leave a set U with the property, that for every node v outside U there exists another node w which is indiscernible from v. "Indiscernibility" meant that v is mapped to w by an automorphism of G that fixes U pointwise. But given the local character of first-order formulae, a weaker notion of indiscernibility also suffices: If sufficiently large neighbourhoods of v and w are isomorphic via an isomorphism that fixes U, then first-order formulae with constants in U (whose arity and quantifier depth are small compared to the radius of the neighbourhoods) cannot distinguish between v and w, so also in this case, any deterministic path defined by such formulae cannot leave U.

We will use a result by Gaifman (1982) stating that first-order formulae can express only local properties.

Definition 5 Let $G = (V, E)$ be an undirected graph, and let $d(u, v)$ denote the usual edge-distance between nodes u and v. For $r \in \mathbb{N}$ and a tuple $\bar{v} = v_1, \ldots, v_k$ of nodes, the r-neighbourhood around $\bar{v}$ is the subgraph $N^r(\bar{v})$ induced by $\bigcup_{i \leq k}\{u \mid d(u, v_i) \leq r\}$.

Note that r-neighbourhoods are first-order definable for every fixed $r \in \mathbb{N}$, so we can introduce the bounded quantifiers $(\exists x \in N^r(\bar{y}))$ and $(\forall x \in N^r(\bar{y}))$ in first-order logic.

Definition 6 A *r-local formula around $\bar{y}$* is a formula with free variables $\bar{y}$ all whose quantifiers are of the form $(Qx \in N^r(\bar{y}))$.

Obviously, if $\psi(\bar{y})$ is r-local around $\bar{y}$ then the truth of $\psi(\bar{u})$ in G depends only on $N^r(\bar{u})$.

Definition 7 A set of nodes is *d-scattered* if the distance of any two nodes in S exceeds d. A *(d, s)-local sentence* is a sentence asserting the existence of a $2d$-scattered set of s elements each of which satisfies a d-local formula around x.

Theorem 8 (Gaifman) *Every first-order formula of quantifier-depth q and with free variables $\bar{y} = y_1, \ldots, y_m$ is equivalent to a Boolean combination of* *(i) (d,s)-local sentences with $d \leq 7^{q-1}$, $s \leq q+m$, and*

(ii) r-local formulae around $\bar{z}$ where $\bar{z} \subseteq \bar{y}$ and $r \leq (7^q - 1)/2$.

Next, we introduce the notion of an r-stable set, which will play a similar role as the stable sets in the previous section.

Definition 8 Given a set $H \subseteq V$, and nodes $v, w \in V$ we write $N^r(v) \cong_H N^r(w)$ to denote that that $N^r(v) \cap H = N^r(w) \cap H$ and that there exists an isomorphism $\pi : N^r(v) \to N^r(w)$ that maps v to w and fixes the points of H.

We call a set $H \subseteq V$ *r-stable* if for every node $v \notin H$, there exists a $w \neq v$ such that $N^r(v) \cong_H N^r(w)$.

Obviously, every stable set is r-stable for all $r \in \mathbb{N}$.

Lemma 9 *Let $\theta(x_1, \ldots, x_k, y)$ be a first order formula of quantifier depth q and let $r = 7^q \cdot k + 1$. Let H be an r-stable set in the graph G, and let $u_1, \ldots, u_k$ be nodes in H. Then, for every vertex $v \notin H$, there exists another vertex $w \neq v$ such that*

$$G \models \theta(\bar{u}, v) \leftrightarrow \theta(\bar{u}, w).$$

Proof. Given $u_1, \ldots, u_k \in H$, choose w such that $N^r(v) \cong_H N^r(w)$. By Gaifman's Theorem $\theta(\bar{x}, y)$ is equivalent to a Boolean combination of (local) sentences and of R-local formulae around subtuples of $(\bar{x}, y)$, where $R = (7^q - 1)/2$. We want to prove that for every R-local formula φ occuring in the Boolean combination for θ,

$$G \models \varphi(\bar{u}, v) \leftrightarrow \varphi(\bar{u}, w).$$

Since the truth of local formulae depends only on the appropriate neighbourhoods, it suffices to show that $N^R(v, \bar{u})$ is isomorphic to $N^R(w, \bar{u})$.

Note that $r = (2R+1)k + 1$. Thus, given $v \notin H$, there exists t with $R \leq t \leq r$, and a decomposition of $\{u_1 \ldots, u_k\}$ into two subsets U, U' such that $N^R(U) \subseteq N^t(v)$ and $N^R(U') \cap N^t(v) = \varnothing$ (and there are no edges between $N^R(U')$ and $N^t(v)$). Since $N^r(v) \cong_H N^r(w)$, there exists an isomorphism $\pi : N^r(v) \to N^r(w)$ which maps v to w and fixes all u_i.

Now, we can define the isomorphism $\tau : N^R(v, \bar{u}) \to N^R(w, \bar{u})$ by

$$\tau(z) = \begin{cases} \pi(z) & \text{if } z \in N^t(v) \\ z & \text{if } z \in N^R(U'). \end{cases}$$

The local sentences that occur in θ do not depend on the interpretation of the free variables, so the claim follows. ∎

Definition 9 (Short path criterion) We say that a class $\mathcal{C}$ of graphs satisfies the short path criterion if and only if for each pair of natural numbers q, k, there exists a number $s := s(q,k)$ such that cofinitely many of the graphs $G \in \mathcal{C}$ satisfy the following requirement:

Every set of k vertices of G is contained in some $(7^q \cdot k + 1)$-stable set H of cardinality at most s.

The idea behind this definition is demonstrated in the following proposition where we show that no formula $[\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi](\bar{u}, \bar{v})$ can take us out of the appropriate subgraph H.

Proposition 10 *Let $\mathcal{C}$ be a class of graphs that satisfies the short path criterion. Then every* (FO + DTC)*-query on $\mathcal{C}$ is expressible in first order logic.*

Proof. It suffices to show that for every first-order formula φ with the $2k$ free variables $\bar{x} = x_1, \ldots, x_k$ and $\bar{y} = y_1, \ldots, y_k$ and possibly additional free variables $\bar{z} = z_1, \ldots, z_\ell$, there exists another first order formula ψ such that for every graph $G \in \mathcal{C}$ and all nodes $\bar{u}, \bar{v}, \bar{z} \in G$

$$G \models [\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi](\bar{u}, \bar{v}, \bar{z}) \leftrightarrow \psi(\bar{u}, \bar{v}, \bar{z}).$$

The collapse to FO then follows by induction on the complexity of (FO + DTC)-formulae.

Suppose that φ has quantifier depth q. Let G be one of the appropriate cofinitely many graphs in $\mathcal{C}$ and fix nodes $\bar{z} = z_1 \ldots, z_\ell$ and $\bar{u} = u_1 \ldots, u_k$ of G. Let $r = 7^{q+k-1}(2k + \ell - 1) + 1$. By the short path criterion there exists an r-stable set H in G, containing $\bar{z}$ and $\bar{u}$, such that $|H| \leq s(q + k - 1, 2k + \ell - 1)$.

We claim that for all $\bar{v} = v_1, \ldots, v_k$

$$G \models [\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi](\bar{u}, \bar{v}) \implies v_1, \ldots, v_k \in H.$$

Towards a contradiction, suppose that the path $\bar{u} = \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_i, \ldots$ defined by φ leaves H with elements $x_{i,j}$, i.e. up to $\bar{x}_{i-1}, x_{i,1}, \ldots, x_{i,j-1}$ all elements belong to H but $x_{i,j}$ does not.

Fix the elements $x_{i-1,1}, \ldots, x_{i-1,k}, x_{i,1}, \ldots, x_{i,j-1}$. Note, that $x_{i,j}$ is supposed to be the unique element that satisfies the formula

$$(\exists x_{i,j+1}) \cdots (\exists x_{i,k}) \varphi(\bar{x}_{i-1}, \bar{x}_i, \bar{z})$$

which has depth $q + k - j \leq q + k - 1$ and depends on $k + j + \ell \leq 2k + \ell - 1$ parameters. However, by the short path criterion, there exists an element $x'_{i,j}$ such that $N^r(x_{i,j}) \cong_H N^r(x'_{i,j})$, and by Lemma 9 this would imply that $x_{i,j}$ is not unique.

Thus, every deterministic path defined by φ is short, i.e. its length is bounded by a constant that only depends on φ (except for the finitely many exceptional G, which could be listed via some FO formula). This implies that $[\mathrm{DTC}_{\bar{x},\bar{y}}\ \varphi]$ is FO-expressible. ∎

5 Hypercubes

Definition 10 A *N-dimensional hypercube* is a graph $\mathbf{2}^N = \langle 2^N, E \rangle$, where 2^N is the set of all functions from $[N] = \{0, 1, \ldots, N-1\}$ to $\{0, 1\}$, and where

$$E := \{(f, g) \mid (\exists! x)[f(x) \neq g(x)]\}.$$

For $f, g \in 2^N$, let

$$(f \oplus g)(x) = \textbf{if } f(x) = g(x) \textbf{ then } 0 \textbf{ else } 1.$$

For every permutation π on $[N]$ and every $h \in 2^N$ we define the automorphism $T_{\pi,h}$ of $\mathbf{2}^N$ by

$$T_{\pi,h}(f) = (f \circ \pi) \oplus h.$$

As usual, $d(x, y)$ is the distance (along edges) from x to y, and $d(x, Y) = \min\{d(x, y) \colon y \in Y\}$.

Definition 11 Given a collection of vertices $X \subseteq 2^N$, we define the *support* of X as

$$\operatorname{supp}(X) = \{x \in [N] \mid (\exists f, g \in X)[f(x) \neq g(x)]\}.$$

The *closure* of X, denoted $[X]$, is the set

$$\{g \in 2^N \mid (\forall f \in X)(\forall z \notin \operatorname{supp}(X))[f(z) = g(z)]\}.$$

Note that the subgraph induced by $[X]$ is a subhypercube of dimension $M = |\operatorname{supp}(X)|$.

Furthermore, we want to break up collections of vertices X. A *d-distance partition* of X is a partition $X = X_1 \cup \cdots \cup X_m$ such that for all $i \neq j$, if $f \in [X_i]$ and $g \in [X_j]$, then $d(f, g) > d$. Furthermore, we let $[[X]] = \bigcup_i [X_i]$. A d-distance partition is called *reduced* if it can't be refined anymore.

Lemma 11 *If X admits no nontrivial d-distance partition then $|\operatorname{supp}(X)| \leq d(|X| - 1)$ and thus $|[X]| \leq 2^{d(|X|-1)}$. Thus, if $X_1 \cup \cdots \cup X_m$ is a reduced d-distance partition of X, then $[[X]]$ has less than $2^{d|X|}$ elements.*

The proof is an easy induction on the cardinality of X.

Theorem 12 *Every* (FO + DTC)*-query on hypercubes is expressible in first order logic.*

Proof. We show that the class of all hypercubes satisfies the short path criterion.

For any k elements $u_1, \ldots, u_k$ in a hypercube $\mathbf{2}^N$, let $X = \{u_1, \ldots, u_k\}$ and choose a reduced d-distance partion $X = X_1 \cup \cdots \cup X_m$ for $d = 2k \cdot 7^q + 3$. Let $H = [[X]]$ which, by Lemma 11, has less than $s = 2^{kd}$ elements.

We claim that H is r-stable, for $r = (d-1)/2$.

In a hypercube all r-neighbourhoods are isomorphic. Thus, if $d(v, H) > r$, then for every other point w with $d(w, H) > r$, it follows that $N^r(v) \cong_H N^r(w)$.

Otherwise, one component, say $[X_1]$, has distance at most r from v; since $r < d/2$ it follows that $d(v, [X_j]) > r$ for all $j > 1$.

Now choose $y, z \notin \operatorname{supp}(X_1)$ such that for all $u \in [X_1]$, $u(y) = v(y)$ and $u(z) \neq v(z)$; note that z exists because $v \notin [X_1]$, and y exists because v is 'close' to $[X_1]$. There are two possibilities: Either $v(y) \neq v(z)$ (and thus $u(y) = u(z)$ for all $u \in [X_1]$), or $v(y) = v(z)$ (and thus $u(y) \neq u(z)$ for all $u \in [X_1]$). In the first case, let $h(x) = 0$ for all $x \in [N]$, and in the second case, let

$$h(x) = \textbf{if } x \in \{y, z\} \textbf{ then } 1 \textbf{ else } 0.$$

Let π be the permutation that switches y and z.

In both cases the automorphism $T_{\pi,h}$ of $\mathbf{2}^N$ fixes $[X_1]$ pointwise. However $w := T_{\pi,h}(v)$ differs from v at the points y and z. Obviously $N^r(v) \cap H \subseteq [X_1]$; thus, $T_{\pi,h}$ is the desired isomorphism proving that H is r-stable. ■

Our result that (FO + DTC) collapses to FO on hypercubes has been used by Tyszkiewicz (1993) to prove that there exist recursive probability distributions on graphs with the 0-1 law for (FO + DTC) but without the convergence law for (FO + TC).

Theorem 13 *On hypercubes,* (FO + DTC) *is strictly weaker than* (FO + pos STC).

PROOF. We show that there exist (FO + pos STC)-queries on hypercubes which are not first order expressible. Note that on a hypercube two nodes f and g have even distance if and only if every path from f to g has even length. Thus the query "$d(x, y)$ is even" is expressed by the formula

$$[\mathrm{STC}_{u,v}\ (\exists z)(Euz \wedge Ezv)](x, y).$$

But Ehrenfeucht-Fraïssé-games show that no first order formula of depth q can distinguish between the pairs (f, g) and (f, h), provided $d(f, g)$ and $d(f, h)$ are greater than 2^q. It follows that the even distance query is not in FO. ■

6 Regular graphs of large degree and girth

As often happens, proofs on infinite structures are easier than proofs in finite structures. For example, the infinite, acyclic and $\aleph_0$-regular graphs have nice properties that provide an example of, say, non-reachability not being in (FO + pos TC), although the finite case — whether non-reachability *on finite digraphs* is in (FO + pos TC), which was an open problem proposed by Immerman (1987) — is more difficult and was solved in Grädel and McColm (1992) only recently. If we wanted to take advantage of nice properties of these infinite graphs in *finite* model theory, we should find finite graphs that look (at least locally) like they are acyclic and regular:

Definition 12 The *girth* of a graph is the length of its shortest cycle. (Thus, a hypercube has girth four.) We call a class of graphs *spidery* if for all $N \in \mathbb{N}$, cofinitely many of the graphs in the class admit $n, m > N$ and are n-regular and of girth m.

Remark. Erdös and Sachs (1963) have shown that for all sufficiently large natural numbers n and m, there exist n-regular graphs of girth m (see Chartrand and Lesniak (1986) for a textbook where this is proved). The n-regular graphs of girth m with minimal cardinality are called (n, m)-*cages*.

We will show that (FO + DTC) collapses to first order logic also on any spidery class of graphs. First some definitions: Let U be a set of nodes in a graph G. A collection $[U_1], \ldots, [U_m]$ is a *d-decomposition* of U if the sets $U_i = [U_i] \cap U$ form a partition of U such that each $[U_i]$ is a connected subgraph of minimal cardinality containing U_i and such that $d([U_i], [U_j]) > d$ for all $i \neq j$. A d-decomposition is *reduced* if no U_i can be further d-decomposed in a nontrivial way.

Lemma 14 *On a graph of girth and degree $\geq d$, given a reduced d-decomposition, each $[U_i]$ is a tree with no more than $d|U_i| - d + 1$ elements, each of which has degree at most $|U_i|$. Thus $[[U]] := \bigcup_i |[U_i]|$ has less than $d(|U| - 1)$ elements.*

Theorem 15 *On a spidery class of graphs, all* (FO + DTC) *queries are expressible in first order logic.*

PROOF. We show that any spidery class satisfies the short path criterion. Given q and k, let $d = 2k \cdot 7^q + 3$ and consider the graphs in G of degree at least $k + 2$ and of girth at least $2d + 1$. Note that in a n-regular graph of girth m, every r-neighbourhood $N^r(v)$ with $r < m/2$ is a tree with root v of depth r, such that all non-leaf vertices are of degree n.

For any k elements $u_1, \ldots, u_k$ in a graph G, let $U = \{u_1, \ldots, u_k\}$ and choose a reduced d-decomposition $[U_1], \ldots, [U_m]$ of U (note that $m \leq k$). Let $H = [[U]]$, which has less than $d(k - 1)$ elements.

We prove that H is r-stable, for $r = k7^q + 1 = (d-1)/2$. Suppose that $v \notin H$. We have to show that there exists a $w \in G$ such that $N^r(v) \cong_H N^r(w)$. If $d(v, H) > r$ then this is trivial because all r-neighbourhoods are isomorphic. If $d(v, H) \leq r$, then we may assume that $d(v, [U_1]) = e \leq r$, and $d(v, [U_j]) > r$ for all $j > 1$. Construct w as follows. Find the unique $u \in [U_1]$ such that $d(v, u) = e$; since the degree of u is $n > k + 1$, there exists an element w with $d(u, w) = d([U_1], w) = e$. It is clear that $N^r(v) \cong_H N^r(w)$. ∎

For all $n \geq 3$ choose a (n, n)-cage G_n and let H_n consist of two disjoint copies of G_n. Obviously, the class of all H_n is spidery. The usual proof, with Ehrenfeucht-Fraïssé-games, that transitive closure is not in FO, applies also to this class. Thus, there are spidery classes of graphs, where (FO + DTC) is strictly weaker than (FO + pos STC).

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References

G. Chartrand and L. Lesniak (1986), *Graphs and digraphs*, Wadsworth & Brooks, Monterey.

P. Erdös and H. Sachs (1963), *Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl*, Wiss. Z. Univ. Halle, Math.-Nat. **12**, 251–258.

R. Fagin (1975), *Monadic generalized spectra*, Zeitschrift für Math. Logik Grundlagen d. Math. **21**, 89–96.

H. Gaifman (1982), *On Local and Non-local Properties*, Logic Colloquium '81 (Ed. J. Stern), North-Holland 1982.

E. Grädel (1991), *On Transitive Closure Logic*, Proceedings of 5th workshop on Computer Science Logic CSL91, Lecture Notes in Computer Science Nr. 626, Springer Verlag, 149–163.

E. Grädel and G. McColm (1992), *Hierarchies in Transitive Closure Logic, Stratified Datalog and Infinitary Logic*, Proceedings of 33rd Annual IEEE Symposium on Foundations of Computer Science, 167–176.

Y. Gurevich (1988), *Logic and the Challenge of Computer Science*, in: E. Börger (Ed), Trends in Theoretical Computer Science, Computer Science Press, 1–57.

N. Immerman (1987), *Languages that Capture Complexity Classes*, SIAM J. Comput. **16**, 760–778.

N. Immerman (1988), *Nondeterministic space is closed under complementation*, SIAM J. Computing **17**, 935 –939.

N. Immerman (1989), *Descriptive and Computational Complexity*, in: J. Hartmanis (Ed.), Computational Complexity Theory, Proc. of AMS Symposia in Appl. Math. **38**, 75–91.

N. Immerman (1992), *Personal communication.*

J. Tyszkiewicz (1993), *On asymptotic probabilities in logics that capture* DSPACE($\log n$) *in presence of ordering*, Proceedings of Tapsoft '93, Lecture Notes in Computer Science Nr. 668, Springer, 569–583.

R. Szelepcsényi, *The method of forced enumeration for nondeterministic automata*, Acta Informatica **26**, 279 – 284.