Two-variable descriptions of regularity

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Abstract

We prove that the class of all languages that are definable in $\Sigma_1^1(\mathrm{FO}^2)$, that is, in (non-monadic) existential second-order logic with only two first-order variables, coincides with the regular languages. This provides an alternative logical description of regularity to both the traditional one in terms of monadic second-order logic, due to Büchi and Trakhtenbrot, and the more recent ones in terms of prefix fragments of Σ_1^1, due to Eiter, Gottlob and Gurevich.

Our result extends to more general settings than words. Indeed, definability in $\Sigma_1^1(\mathrm{FO}^2)$ coincides with recognizability by appropriate notions of automata on a large class of objects, including ω-words, trees, pictures and, more generally, all weakly deterministic, triangle-free transition systems.

1 Logical characterizations of the regular languages

An important connection between automata theory and mathematical logic was established in the early 1960's by Büchi and Trakhtenbrot [5, 24], who showed that a language is regular if and only if it is definable in monadic second-order logic MSO. This result has been extended in many directions. For example, McNaughton and Papert [16] characterized the regular languages that admit a first-order definition, and Büchi [6], McNaughton [15], and Rabin [18] established equivalences between monadic second-order logic and finite automata also on ω-words and on trees. Recently, Thomas and others [9, 10, 20, 23, 21, 22] have developed an automata theory on more general objects (for instance two-dimensional pictures, partial orders, or arbitrary graphs of bounded degree) and established a close relationship between recognizability by automata and definability in *existential* monadic second-order logic.

Let us briefly explain the connection between regularity and monadic second-order logic. A word $w_0 \cdots w_{n-1} \in A^*$ (for some finite alphabet A) can be viewed as a structure $W = (n, S, \min, \max, (P_a)_{a\in A})$ where the universe $n = \{0, \dots, n-1\}$ is the set of letter positions in the word, S is the usual successor relation on n, $\min$ and $\max$ are constants for the first and the last element of n, and the P_a are monadic predicates indicating the set of positions at which the letter a occurs, i.e. $P_a = \{i < n : w_i = a\}$. We write τ_A for the vocabulary $\{S, \min, \max, (P_a)_{a\in A}\}$ of word structures for the alphabet A. A sentence ψ of vocabulary τ_A defines the language $L(\psi) \subseteq A^*$ consisting of all words W such that $W \models \psi$. The Büchi-Trakhtenbrot Theorem says that a language is regular if and only if it is definable by a sentence ψ in monadic second-order logic.

We write FO for first-order logic and $\Sigma_1^1 := \{\exists R_1 \cdots \exists R_k \varphi : \varphi \in \mathrm{FO}\}$ for existential second-order logic. The proof of the Büchi-Trakhtenbrot Theorem shows actually that the existential fragment monΣ_1^1 of MSO suffices to define all regular languages and that therefore, over word structures, MSO collapses to monΣ_1^1.

Recall that, by Fagin's Theorem, existential second-order logic captures the complexity class NP and is hence much more powerful than finite automata. Recently, Eiter, Gottlob and Gurevich gave new characterizations of the regular languages by fragments of non-monadic existential second-order logic. Instead of restricting second-order quantification to monadic predicates (as in the Büchi-Trakhtenbrot Theorem), they admit quantification over arbitrary relations but restrict the quantifier prefix of the first-order part of the formulae.

Definition 1.1. Let $X \subseteq$ FO be a fragment of first-order logic. We denote by $\Sigma_1^1(X)$ the set of all sentences of the form $\exists R_1 \cdots \exists R_k \varphi$ such that φ is a sentence in X.

A *prefix class* in FO is given by a set of words $\mathcal{Q}$ over the alphabet $\{\exists, \forall\}$ that is closed under taking subwords (i.e. if $w \in \mathcal{Q}$ and w' is obtained by deleting some of the letters in w, then also $w' \in \mathcal{Q}$). We identify such a set of prefixes with the set of first-order formulae of form $Q_1x_1 \cdots Q_rx_r\varphi$ such that φ is quantifier-free and the word $Q_1 \cdots Q_r$ belongs to $\mathcal{Q}$. Fragments of first-order logic defined by quantifier prefixes have been studied very intensively in the context of the classical decision problem (see [4]). Eiter, Gottlob and Gurevich gave a complete classification of the prefix classes $\mathcal{Q}$ such that $\Sigma^1_1(\mathcal{Q})$ over words characterizes the regular languages.

Theorem 1.2 (Eiter, Gottlob, Gurevich). *Let $\mathcal{Q} \subseteq \mathrm{FO}$ be any prefix class in first-order logic. Then, over word structures, either*

(i) $\mathcal{Q} \subseteq \exists^*\forall\exists^* \cup \exists^*\forall^2 = \exists^*\forall(\forall \cup \exists^*)$ and $\Sigma^1_1(\mathcal{Q})$ defines only regular languages, or

(ii) $\mathcal{Q}$ contains at least one of the prefix classes $\forall\forall\forall$, $\forall\forall\exists$, $\forall\exists\forall$ and $\Sigma^1_1(\mathcal{Q})$ defines some NP-complete language.

Hence there is an interesting gap phenomenon. For any logic $\Sigma^1_1(\mathcal{Q})$ (where $\mathcal{Q}$ is a prefix class in FO) the data complexity of model checking on words is either very simple (it can be done by a finite automaton) or NP-complete. There are no intermediate levels of complexity.

Further, in the case of *(i)* the reduction from a given sentence $\psi \in \Sigma^1_1(\mathcal{Q})$ to an automaton recognizing the language defined by ψ is effective. Hence the satisfiability problem for $\Sigma^1_1(\mathcal{Q})$ over strings (or equivalently, the satisfiability problem for $\mathcal{Q}$ over finite successor structures) is decidable. It was observed by the present authors that in the case of *(ii)* the satisfiability problem for $\Sigma^1_1(\mathcal{Q})$ over finite successor structures is undecidable (see [7, p. 24-25] for a sketch of the proof). As a consequence one obtains another dichotomy theorem.

Corollary 1.3. *Let $\mathcal{Q}$ be any prefix class in first-order logic. Then, over words, either*

(i) the satisfiability problem for $\mathcal{Q}$ is decidable and $\Sigma^1_1(\mathcal{Q})$ defines only regular languages, or

(ii) the satisfiability problem for $\mathcal{Q}$ is undecidable and $\Sigma^1_1(\mathcal{Q})$ defines some NP-complete language.

There are of course other fragments of first-order logic which are known to be 'well-behaved' in a certain sense, and which may possibly lead to interesting fragments of Σ^1_1. The *two-variable fragment* of first-order logic is one example which has recently been studied quite intensively ([8, 12, 13]).

Definition 1.4. Two-variable first-order logic FO^2 is the set of all first-order formulae whose vocabulary contains only relation symbols and constants (but no function symbols of positive arity), and which contain only two variables x and y.

It is known that FO^2 has the finite model property [17] and the satisfiability problem for FO^2 is complete for NEXPTIME [12]. Etessami, Vardi and Wilke [8] establish an interesting relationship between FO^2 and regular languages that are definable by a fragment of temporal logic. Recently, the expressive power of $\Sigma^1_1(\mathrm{FO}^2)$ has been investigated by Le Bars [2, 3].

Our main result is the following new characterisation of the regular languages.

Theorem 1.5. *$\Sigma^1_1(\mathrm{FO}^2)$ characterizes the regular languages.*

The fact that every regular language can be defined by a sentence in $\Sigma^1_1(\mathrm{FO}^2)$ is not difficult to prove and well-known. Indeed, the straightforward description of the behaviour of a finite automaton in monadic second-order logic only requires the use of monadic Σ^1_1-sentences whose first-order parts have prefix $\forall\forall$ or $\forall\exists$ (see e.g. [7]). Hence

$$\begin{aligned}\mathrm{REG} = \mathrm{mon}\Sigma^1_1 = \mathrm{mon}\Sigma^1_1(\forall\forall) = \mathrm{mon}\Sigma^1_1(\forall\exists) \\ = \mathrm{mon}\Sigma^1_1(\mathrm{FO}^2) \subseteq \Sigma^1_1(\mathrm{FO}^2).\end{aligned}$$

The other direction is not obvious, and the proof will be given in Sect. 2. The result may be useful for verification purposes since interesting properties of transition systems can sometimes be described more naturally by non-monadic existential second-order formulae than by monadic ones. If such a description requires only two variables, then our main result immediately implies that the property can be checked by a finite automaton.

In Sect. 5 we extend our main result to more general settings than words. Indeed, we can show that definability in $\Sigma^1_1(\mathrm{FO}^2)$ coincides with recognizability by appropriate notions of automata on a large class of objects, including ω-words, trees, pictures and, more generally, all weakly deterministic, triangle-free transition systems.

In Sect. 6 we discuss the power of $\Sigma^1_1(\mathrm{GF})$ on words, where GF is the *guarded fragment*, another 'well-behaved' portion of first-order logic that has received a lot of attention recently. However, it will turn out that $\Sigma^1_1(\mathrm{GF})$ has the same expressive power as Σ^1_1 on words and hence captures all of NP.

2 $\Sigma^1_1(\mathrm{FO}^2)$ characterizes the regular languages

In this section we prove our main result, that every sentence in $\Sigma^1_1(\mathrm{FO}^2)$ defines a regular language. To this end we will show that, over words, every $\Sigma^1_1(\mathrm{FO}^2)$-sentence is equivalent to an $\Sigma^1_1(\exists^*\forall\forall)$-sentence. We can then use the result of Eiter, Gottlob and Gurevich that, on words, every sentence in $\Sigma^1_1(\exists^*\forall\forall)$ is equivalent to a monadic Σ^1_1-sentence and hence defines a regular language.

Theorem 2.1. *Over words, every $\Sigma^1_1(\mathrm{FO}^2)$-sentence is equivalent to a sentence in $\Sigma^1_1(\exists^*\forall\forall)$.*

Given a sentence $\psi \in \Sigma_1^1(\mathrm{FO}^2)$ we first make use of some well-known techniques to simplify the sentence. First, we eliminate all existentially quantified relation symbols of arity greater than two. Second, we replace the constants min and max for the first and last elements by unary singleton relations F and L, respectively. This is no loss of generality since F and L are axiomatizable by FO^2-sentences of vocabulary $\{S\}$. Third we make use of the Scott normal form for FO^2 (see [19, 4, 12]).

Lemma 2.2. *For every sentence $\psi \in \Sigma_1^1(\mathrm{FO}^2)$ there is an equivalent sentence of the form*

$$\psi_1 := (\exists \bar{R})(\forall x \forall y \alpha \wedge \bigwedge_{i=1}^{k} \forall x \exists y \beta_i)$$

where all relation symbols in $\bar{R}$ are unary or binary and the formulae α and β_i are quantifier-free.

A *binary atomic type* of vocabulary τ is a maximally consistent set t of τ-atoms and negated τ-atoms in the two variables x and y. By slight abuse of notation, we write $t(x,y)$ for the conjunction over all formulae in t. Similarly, a *unary atomic type* is a maximally consistent set of τ-atoms and negated τ-atoms in a single variable x.

In the rest of this proof, we only consider atomic types that are consistent with the axioms for word structures for the given alphabet A (i.e. with the requirements that S is a successor relation, F and L are singleton relations that contain the first, resp. last element with respect to S and that at each position precisely one of the predicates P_a is true). We call these (unary or binary) *word types*. A binary word-type $t(x,y)$ is *local* if it contains one of the atoms $x = y$, Sxy or Syx (called the *local atom* of t).

Recall that every quantifier-free formulae $\beta(x,y)$ is equivalent to the disjunction over the binary atomic types implying it: $\beta(x,y) \equiv \bigvee_{t \models \beta} t(x,y)$. Hence we can translate the sentence ψ_1 into an equivalent sentence

$$\psi_2 := (\exists \bar{R})(\forall x \forall y \alpha \wedge \forall x \bigwedge_{i=1}^{k} \bigvee_{t \in T(i)} \exists y t(x,y))$$

where $T(i)$ is the set of binary atomic types that imply $\beta_i(x,y)$.

Note that ψ_2 is satisfiable over words only if each of the sets $T(i)$ includes at least one word type that contains the atom Fx and at least one word type that contains the atom Lx. If this is not the case, we can replace ψ_2 by *false*, and we are done. Otherwise we proceed as follows. We add monadic predicates U_{it}, for each $i \leq k$ and $t \in T(i)$, which pick for each element x and each i one particular disjunct. Let δ be a sentence expressing that for each $i \leq k$, the sets U_{it} ($t \in T(i)$) partition the universe. Notice that δ is a universal sentence with just one variable. Now, ψ_2 is equivalent to the sentence

$$\psi_3 := (\exists \bar{U})(\exists \bar{R})(\delta \wedge \forall x \forall y \alpha \wedge \forall x \bigwedge_{i=1}^{k} \bigwedge_{t \in T(i)} (U_{it}x \rightarrow \exists y t(x,y))).$$

Each of the sets $T(i)$ can be decomposed into two sets $L(i)$ and $N(i)$, where $L(i)$ is the set of all *local* binary types in $T(i)$ and $N(i) = T(i) - L(i)$. We want to remove the local types from the conjunction $\bigwedge_{t \in T(i)} \cdots$ in ψ_3 by incorporating them into the purely universal part of the formula. This is not difficult. Indeed, let $\gamma_t(x,y)$ be the local atom of t (i.e. either $x = y$ or Sxy or Syx) and let

$$\alpha^* := \delta \wedge \forall x \forall y (\alpha \wedge \bigwedge_{\substack{1 \leq i \leq k \\ t \in L(i)}} (U_{it}x \wedge \gamma_t(x,y) \rightarrow t(x,y))).$$

Then ψ_3 is equivalent to

$$\psi_4 := (\exists \bar{U})(\exists \bar{R})(\alpha^* \wedge \forall x \bigwedge_{i=1}^{k} \bigwedge_{t \in N(i)} (U_{it}x \rightarrow \exists y t(x,y))).$$

Notice that $W \models \psi_4$ if and only if there exists an expansion of W by relations $\bar{U}, \bar{R}$ and Skolem functions $\bar{f} = f_1, \ldots, f_k$ such that

(i) $(W, \bar{U}, \bar{R}) \models \alpha^*$;

(ii) for all $i \leq k$, $t \in N(i)$ and $p \in U_{it}$, $(W, \bar{R}) \models t(p, f_i(p))$.

We call such a $(W, \bar{U}, \bar{R}, \bar{f})$ an *appropriate expansion* of W. The *remote witnesses* of an appropriate expansions are the elements of the set $\bigcup_{i \leq k} \bigcup_{t \in N(i)} f_i(U_{it})$. The crucial step in the proof of our main theorem is to show that we can uniformly bound the number of remote witnesses.

Lemma 2.3. *Given ψ_4 we can effectively determine a natural number m such that for every model $W \models \psi_4$ there exists an appropriate expansion of W with at most m remote witnesses.*

Proof. Suppose that $(W, \bar{U}, \bar{R}, \bar{f})$ is an appropriate expansion of W, and let τ be the vocabulary of $(W, \bar{R})$. An element $q = f_i(p)$ is a witness for $p \in U_{it}$, if $(W, \bar{R}) \models t(p,q)$, which only depends on the unary atomic types of the elements p and q and on the $\bar{R}$-connections between p and q. Hence, changing the $\bar{R}$-connections between other elements does not affect at all whether q is a witness for p.

Our goal is to define a set B of bounded size so that by appropriate changes of the $\bar{R}$-connections (and the Skolem functions f) we can define a modified apppropriate expansion of W which has all its remote witnesses inside B.

Notice that each element may need up to k remote witnesses. Let $s(x)$ be a unary atomic type (of vocabulary τ), and let $\ell = 3(k+2)$. We call $s(x)$ *scarce* in $(W, \bar{R})$ if it is realized less than ℓ times; otherwise we call it *abundant*. Let K be the set of all elements realizing a scarce unary type in $(W, \bar{R})$ and let $C = \bigcup_{i<k} f_i(K)$ be the set of remote witnesses for elements in K. We choose B to be some subset of the universe that includes K and C and contains at least ℓ realizations of every abundant unary type. Since the total number of unary atomic types is bounded by $2^{|\tau|}$, we can find such a B of cardinality at most $m := \ell(k+1)2^{|\tau|}$. We can write the set B as the disjoint union of K (the realizations of the scarce types) and three sets A_0, A_1 and A_2 where each A_i, contains at least $k+2$ realizations of every abundant unary type.

We define the modified expansion of W by going through all elements p of the universe and providing remote witnesses for p inside B by changing some $\bar{R}$-connections. We have to be careful to do this in a consistent way, in the sense that we do not perform conflicting changes between a pair (p, q) and its inverse (q, p). Further, we have to take care that we only include binary atomic types into the new expansion that also occur in the given expansion (and hence do not violate the $\forall\forall$-part of the formula). We distinguish several cases:

(1) $p \notin B$. Let $q_1, \dots, q_r$ ($r \le k$) be those remote witnesses for p in the given expansion $(W, \bar{U}, \bar{R}, \bar{f})$ which lie outside of B. Pick elements $q'_1, \dots, q'_r \in B$ which are not adjacent to p such that q'_i has the same unary type as q_i. Change the $\bar{R}$-connections between p and q'_i so that they coincide with the $\bar{R}$-connections between p and q_i.

(2) $p \in K$. Since we have included all the remote witnesses of all $p \in K$ into B, nothing remains to be done in this case.

(3) $p \in A_i$ ($i \in \{0, 1, 2\}$). Let $q_1, \dots, q_r$ ($r \le k$) be the remote witnesses for p in the given expansion whose unary atomic type is abundant, i,e. which lie outside of K. In that case choose the elements $q'_1, \dots, q'_r$ such that

- none of the elements $q'_1, \dots, q'_r$ is adjacent to p;
- $q'_1, \dots, q'_r$ have, respectively, the same unary atomic types as $q_1, \dots, q_r$;
- $q'_1, \dots, q'_r \in A_j$ where $j = i + 1 \pmod 3$.

Since A_j has at least $k+2$ elements of each abundant unary type, we can always find such $q'_1, \dots, q'_r$. Again define the $\bar{R}$-connections between p and q'_i in the modified expansion so that they coincide with the $\bar{R}$-connections between p and q_i in the original expansion.

It is easy to verify that the modified expansion has the required properties. $\square$

The following lemma is an immediate consequence of Lemma 2.3.

Lemma 2.4. *ψ_4 is equivalent to a sentence of the form*

$$\psi_5 := (\exists \bar{U})(\exists \bar{R})(\alpha^* \wedge \exists z_1 \cdots \exists z_m \forall x \bigwedge_{i=1}^{k} \bigwedge_{t \in T(i)} (U_{it}x \to \bigvee_{j=1}^{m} t(x, z_j))).$$

Notice that ψ_5 is equivalent to a sentence in $\Sigma_1^1(\exists^*\forall\forall)$. This completes the proof of Theorem 2.1. By the result of Eiter, Gottlob and Gurevich it follows that ψ_5, and hence ψ, is equivalent to a monadic Σ_1^1-sentence.

Theorem 2.5. *On words, $\Sigma_1^1(\mathrm{FO}^2)$ and monadic second-order logic have the same expressive power.*

The main theorem now follows immediately from Theorem 2.5 and the Büchi-Trakhtenbrot Theorem. Further we get a decidability result for $\Sigma_1^1(\mathrm{FO}^2)$.

Corollary 2.6. *The satisfiability problem for $\Sigma_1^1(\mathrm{FO}^2)$ on finite successor structures is decidable.*

3 $\Sigma_1^1(\exists^*\forall\forall)$ versus monadic $\Sigma_1^1(\mathrm{FO}^2)$ on arbitrary structures

Eiter, Gottlob and Gurevich proved that, on words, every sentence in $\Sigma_1^1(\exists^*\forall\forall)$ is equivalent to a monadic Σ_1^1-sentence. They also observed that the proof does not use the finiteness of word structures, and hence applies to ω-words as well. Actually the arguments can be generalized to hold over arbitrary structures (the presence of a successor relation is not necessary) and to give information also on the number of variables of the equivalent monadic sentence. In particular, two variables suffice in the case where the given $\Sigma_1^1(\exists^*\forall\forall)$-sentence has at most binary predicates. The proof will be given in the full paper.

Theorem 3.1. *Every sentence φ in $\Sigma_1^1(\exists^*\forall\forall)$ is logically equivalent to some monadic Σ_1^1-sentence. Further, if φ contains only unary and binary relation symbols, then φ is equivalent to a sentence in monadic $\Sigma_1^1(\mathrm{FO}^2)$.*

4 Words with a linear order

We have represented a word $w = w_0 \cdots w_{n-1} \in A^*$ by the structure $W = (n, S, \min, \max, (P_a)_{a \in A})$. But there are of course other possibilities. A common choice is to use in addition to (or instead of) the successor relation S the linear order $<$ on n. For definability in monadic second-order logic or monadic Σ^1_1 this is immaterial since successor and order are mutually definable from each other. However, for definability considerations in other logics the presence or absence of a linear order does make a difference. In particular, the Theorem of McNaughton and Papert [16], saying that first-order logic captures precisely the star-free regular languages, requires the presence of $<$.

Hence the question arises whether in the presence of a linear order $\Sigma^1_1(\mathrm{FO}^2)$ can still define only regular languages. We prove that this is indeed the case. As in the case without order it suffices to establish the following fact.

Proposition 4.1. *On the class of word structures with a linear order, of the form* $W = (n, S, <, \min, \max, (P_a)_{a\in A})$, *every sentence in* $\Sigma^1_1(\mathrm{FO}^2)$ *is equivalent to a sentence in* $\Sigma^1_1(\exists^*\forall\forall)$.

Proof. The only change with respect to the situation without order is that a more complicated argument is needed in order to show that the number of remote witnesses can be uniformly bounded. With the same notation as in the previous section, we suppose that we have an appropriate expansion $(W, \bar{U}, \bar{R}, \bar{f})$ of a word W that is a model of

$$\psi_4 := (\exists\bar{U})(\exists\bar{R})(\alpha^* \wedge \forall x \bigwedge_{\substack{1\le i\le k\\ t\in N(i)}} (U_{it}x \rightarrow \exists y t(x,y))).$$

We want to find another appropriate expansion of W such that all the remote witnesses can be found inside a set B whose size is bounded by a number m (which depends on the formula, but is independent from W). The additional difficulty that needs to be resolved now is that the types $t(x, y) \in N(i)$ include the information whether $x < y$ or $y < x$.

We construct the set B as follows. Let K be the set that contains, for each unary atomic type s, the smallest $k + 1$ and the largest $k+1$ elements that realize s in $(W, \bar{U}, \bar{R}, \bar{f})$ if at least $2k+2$ realizations exist, and all elements realizing s otherwise. Let $C = \bigcup_{i\le k} f_i(K)$ be the set of remote witnesses for the elements in K. Further, let D be the set that contains for each unary atomic type s, the smallest $k+1$ and the largest $k+1$ realizations of s outside $K\cup C$ if at least $2k + 2$ such realizations exist, and all realizations outside $K\cup C$ otherwise. Finally, let : $B = K\cup C\cup D$. Obviously, the cardinality of B is bounded by $2^{|\tau|}(2k + 2)(k + 2)$.

As in the case without order, to define the modified expansion of W we go through all elements p of the universe and provide remote witnesses for p inside B by changing some $\bar{R}$-connections. When old remote witnesses $q_1, \dots, q_r$ for p are replaced by new witnesses $q'_1, \dots, q'_r$ (where always $r \le k$) we make sure that the following conditions are satisfied.

(a) none of $q'_1, \dots, q'_r$ is adjacent to p;

(b) if $q_i < p$ then $q'_i < p$ and if $q_i > p$ then $q'_i > p$;

(c) $q'_1, \dots, q'_r$ realize, respectively, the same unary atomic type as $q_1, \dots, q_r$.

We then change the $\bar{R}$ connections between p and q'_i so that the binary atomic type of (p, q'_i) in the the modified expansion coincides with the binary atomic type of (p, q_i) in the old expansion. Since the modifications only concern the $\bar{R}$-connections between two distinct elements they never change the unary atomic type of any element.

Depending on whether p lies in K, in $C - K$, in D, or outside of B there are different conditions on which remote witnesses have to be replaced and in which part of B the new witnesses should be found.

(1) $p \notin B$. Let $q_1, \dots, q_r$ be the remote witnesses for p that lie outside B. Replace them by appropriate $q'_1, \dots, q'_r$ satisfying the conditions (a) – (c).

(2) $p \in K$. Nothing remains to be done in this case.

(3) $p \in C - K$. Let $q_1, \dots, q_r$ be the remote witnesses for p that lie outside B. Choose $q'_1, \dots, q'_r \in D$ satisfying (a) – (c).

(4) $p \in D$. Let $q_1, \dots, q_r$ be the remote witnesses for p that lie outside K. Replace them by $q'_1, \dots, q'_r \in K$ satisfying the conditions (a) – (c).

Observe that in all cases, sufficiently many distinct elements $q'_1, \dots, q'_r$ exist in the indicated part of B, since both K and D contain sufficiently many 'small' and sufficiently many 'large' elements of each unary atomic type. Second, note that the assignments of new witnesses is without conflicts. By a conflict we mean that some element q is assigned to be a new witness for p while p is at the same time also a witness for q. This does not happen, since

- by (3) an element $q \in D$ may become a new witness for elements $p \in C - K$. But elements in $C - K$ are remote witnesses only for elements of K and $D\cap K = \varnothing$.
- by (4) an element $q \in K$ may become a new remote witness for elements $p \in D$, but elements of D are not witnesses for elements of K since all remote witnesses for K are in C, and $C \cap D = \varnothing$.

This completes the proof. $\square$

Using Theorem 3.1 and the Büchi-Trakhtenbrot Theorem we obtain the desired result.

Theorem 4.2. *On words with a linear order, the languages definable in* $\Sigma^1_1(\mathrm{FO}^2)$ *are precisely the regular languages.*

5 Graph automata

Notions of regularity or recognizability by appropriate models of automata play an important role also on objects other than words. Most attention has been given to automata on ω-words, and on finite and infinite trees. More recently, two-dimensional languages, called *pictures* have been studied quite intensively (see [9]). In a sequence of papers, Thomas has developed a theory of automata over more general classes of graphs than words, trees or pictures which is closely related to definability in monadic second-order logic (see [20, 23, 21, 22]).

A transition system is a vertex-labelled and edged-labelled graph or equivalently, a relational structure

$$G = (V, (P_a)_{a\in A}, (E_b)_{b\in B})$$

where V is the set of vertices, the P_a are monadic relations, and the E_b are binary relations. We say that there is an edge labelled by b from u to v if $(u, v) \in E_b$. The set A (of vertex labels) and the set B (of edge labels) are always assumed to be finite. The vocabulary of G is the collection of the relation symbols P_a and E_b.

The Gaifman graph of G is the undirected graph (V, E) where $(u, v) \in E$ if and only if for some $b \in B$ $(u, v) \in E_b$ or $(v, u) \in E_b$. Graph-theoretic notions like the distance between two vertices in G, the degree of a vertex in G, or the girth of G (the length of a shortest cycle) are always meant with respect to the Gaifman graph of G.

For $r \geq 0$, the *r-sphere around a vertex v in the transition system G* is the substructure of G induced by the set of vertices of distance at most r from v. In the sequel we will only consider transition systems whose degree is bounded by some fixed constant d. This ensures that there is only a bounded number of isomorphism types of r-spheres.

Definition 5.1. A *graph automaton* has the form $\mathcal{A} = (Q, A, B, \Delta, Occ)$ where Q is a finite set of states, A is the set of vertex labels, B is the set of edge labels, Δ is, for some $r \geq 0$, a finite set of isomorphism types H of r-spheres with vertex labels in $A \times Q$ and edge labels in B, and *Occ* is a Boolean combinations of constraints of the form "there are at least k occurrences of the sphere H". The elements of Δ are called tiles or transitions.

The graph acceptor $\mathcal{A}$ accepts a transition system $G = (V, (P_a)_{a\in A}, (E_b)_{b\in B})$ if there is an assignment $\rho : V \to Q$ (called a *run*), such that each r-sphere of the expanded graph G_ρ (with vertex labels in $A \times Q$) is isomorphic to one of the spheres $H \in \Delta$, and the occurrences of these spheres satisfy the constraints in *Occ*.

Let $\mathcal{K}$ be a class of transition systems. We say that $\mathcal{C}$ is recognizable in $\mathcal{K}$ if there exists a graph automaton $\mathcal{A}$ that accepts a transition system $G \in \mathcal{K}$ if and only if $G \in \mathcal{C}$. We say that a sentence ψ defines $\mathcal{C}$ within $\mathcal{K}$ if $\mathcal{C} = \{G \in \mathcal{K} : G \models \psi\}$. The following result has been proved in [20].

Theorem 5.2 (Thomas). *Let $\mathcal{K}$ be a class of transition systems of bounded degree. A class $\mathcal{C}$ is recognizable in $\mathcal{K}$ by a graph automaton if and only if $\mathcal{C}$ is definable within $\mathcal{K}$ by a monadic Σ^1_1-sentence.*

In view of the results obtained so far the question arises under which conditions recognizability corresponds to definability in $\Sigma^1_1(\mathrm{FO}^2)$. It is not difficult to see that the property of having bounded degree is not sufficient for making $\Sigma^1_1(\mathrm{FO}^2)$ as powerful as monadic Σ^1_1 and hence as powerful as graph automata. On the other side, it is not known whether all properties definable in $\Sigma^1_1(\mathrm{FO}^2)$ are also definable in monadic Σ^1_1. We establish a general and meaningful condition which is sufficient for such a collapse.

Definition 5.3. A transition system G is called *(two-way) deterministic* if for every vertex v of G and every edge label b there is at most one outgoing edge from v and at most one incoming edge to v that are labelled by b.

Clearly, words, ω-words and pictures are two-way deterministic. Actually for what we are going to show a weaker notion of determinism will be sufficient.

Definition 5.4. A binary atomic type $t(x, y)$ of vocabulary σ is called *local* if it contains any one of the atoms $x = y$, E_bxy or E_byx (for some $b \in B$). A transition system G of vocabulary σ is called *weakly deterministic* if for every local type $t(x, y)$ and every vertex v of G there exists at most one vertex w such that $G \models t(v, w)$.

We next introduce the notion of a star.

Definition 5.5. A *σ-star S* is a set of binary atomic types of vocabulary σ. A vertex v realizes the star S in the transition system G if S is the set of those binary atomic σ-types t such that $G \models \exists y t(v, y)$.

Given a transition system G of vocabulary σ, let G^* be the expanded transition system that we obtain by adding for every σ-star S a monadic relation (also denoted S) containing precisely the vertices realizing the star S. We denote the vocabulary of G^* by σ^*.

Note that the number of different σ-stars depends only on σ and that each star is definable in FO^2 by a formula

$$\psi_S(x) := \bigwedge_{t\in S} \exists y t(x, y) \wedge \bigwedge_{t\notin S} \neg\exists y t(x, y).$$

Hence every property definable in FO^2, in $\Sigma^1_1(\mathrm{FO}^2)$ or in monadic Σ^1_1 over these expanded transitions systems is also definable (in the same logic) over the original transition systems. To put it differently, for every sentence ψ^* of vocabulary σ^* we obtain a sentence ψ of vocabulary σ such that $G \models \psi$ if and only if $G^* \models \psi^*$, for all transition systems G of vocabulary σ.

Theorem 5.6. *Let $\mathcal{K}$ be any class of weakly deterministic transition systems. Then every sentence in $\Sigma_1^1(\mathrm{FO}^2)$ is equivalent on $\mathcal{K}$ to a monadic Σ_1^1-sentence.*

Proof. We assume that we have the monadic predicates S describing the σ-stars available and show that, over this expanded vocabulary, every $\Sigma_1^1(\mathrm{FO}^2)$-sentence is equivalent over $\mathcal{K}$ to a sentence in $\Sigma_1^1(\exists^*\forall\forall)$. By Theorem 3.1 the result then follows.

The proof uses the same ideas as in the case of words. We only describe the modifications. Given a sentence $\psi := (\exists\bar{R})\varphi \in \Sigma_1^1(\mathrm{FO}^2)$ of vocabulary σ we first translate it into an equivalent sentence

$$\psi_2 := (\exists\bar{R})(\forall x\forall y\alpha \wedge \forall x \bigwedge_{i=1}^{k} \bigvee_{t\in T(i)} \exists y t(x,y))$$

where each $T(i)$ is a set of binary atomic types of the vocabulary $\tau = \sigma \cup \{\bar{R}\}$.

For every binary τ-type $t(x,y)$, let $s_t(x,y)$ be the unique binary σ-type that is implied by t (clearly s_t is obtained by omitting from t all literals of vocabulary $\tau - \sigma$). As in the case of words, we add unary predicates U_{it} that select for each element of the universe and each i one of the types $t \in T(i)$. However, the sentence δ constraining the properties of these predicates is now slightly more complicated. It expresses that

(i) for each i the sets U_{it} $(t \in T(i))$ partition the universe, and

(ii) if an element is contained in U_{it} then it must realize a star S that is consistent with t. This is expressed by a conjunction of sentences of the form

$$\forall x\Big(U_{it}x \rightarrow \bigvee_{S:s_t\in S} Sx\Big)$$

making use of the star predicates S.

Again, we decompose each of the sets $T(i)$ into the set $L(i)$ of local types and the set $N(i)$ of remote types in $T(i)$. Note that the sentence δ enforces that for each binary type t and each element $x \in U_{it}$ there is indeed a y such that the binary type of (x,y) is s_t. If t is a local type then this element y is always uniquely determined (since the transition systems are weakly two-way deterministic). Hence, for each local type t and each $x \in U_{it}$ we can replace the existential requirement that there exists a y such that (x,y) realizes t by the universal requirement that for all y, if (x,y) realizes s_t then it also realizes t.

Hence the formula ψ_2 above is equivalent to

$$\psi_4 := (\exists\bar{U})(\exists\bar{R})(\alpha^* \wedge \forall x \bigwedge_{\substack{1\le i\le k\\ t\in N(i)}} (U_{it}x \rightarrow \exists y t(x,y)))$$

where

$$\alpha^* := \delta \wedge \forall x\forall y(\alpha \wedge \bigwedge_{\substack{1\le i\le k\\ t\in L(i)}} ((U_{it}x \wedge s_t(x,y)) \rightarrow t(x,y))).$$

The rest of the proof is the same as for words, with the following modification. In the proof of the crucial lemma that bounds the number of remote witnesses, we defined the parameter ℓ to be $3(k+2)$ where k is the maximal number of remote witnesses that each element may need. We used $k+2$ rather than k to make sure that we can always find k witnesses that are indeed remote i.e. distinct from the two neighbours of the given x. Here we have to use the number $k+c$ where c bounds the number of local types. Note that this number is finite and depends only on the vocabulary σ. $\square$

One of the more complicated features of graph automata compared to classical notions of automata is the use of r-spheres for $r \geq 2$ in the set of transitions and the acceptance condition. It is known that automata that are based on 1-spheres only have in general less computational power than arbitrary graph automata (see e.g. [21] for an example due to Seibert). On the other side on certain special classes of graphs, such as for instance words, ω-words and pictures, spheres of higher radius can be eliminated, but there is not yet a good understanding of the conditions under which this is possible.

We next present a general sufficient condition on transition systems under which spheres of radius $r > 1$ can be eliminated and recognizability by automata coincides with definability in $\Sigma_1^1(\mathrm{FO}^2)$.

Theorem 5.7. *Let $\mathcal{K}$ be a class of weakly deterministic transition systems of girth $> 2r+1$. Then for every class $\mathcal{C} \subseteq \mathcal{K}$ the following four conditions are equivalent.*

(1) $\mathcal{C}$ is recognizable in $\mathcal{K}$ by a graph automaton based only on spheres of radius 1.

(2) $\mathcal{C}$ is recognizable in $\mathcal{K}$ by a graph automaton based on spheres of radius r.

(3) $\mathcal{C}$ is definable within $\mathcal{K}$ by a sentence in monadic $\Sigma_1^1(\mathrm{FO}^2)$.

(4) $\mathcal{C}$ is $\Sigma_1^1(\mathrm{FO}^2)$-definable within $\mathcal{K}$.

Proof. The implications from (1) to (2) and from (3) to (4) are trivial, and the implication from (4) to (3) has already been established.

We show that (2) implies (3): Let $\mathcal{A} = (Q, A, B, \Delta, Occ)$ be a graph automaton based on r spheres that recognizes $\mathcal{C}$ in $\mathcal{K}$.

A run ρ of $\mathcal{A}$ on a transition system G is encoded by a sequence $\bar{X} = (X_q)_{q\in Q}$ of unary predicates on G where

X_q represents the set of vertices v with $\rho(v) = q$. Let σ be the vocabulary of G and let $\tau = \sigma \cup \{X_q : q \in Q\}$.

We want to express by appropriate formulae that

(1) all r-spheres occurring in the expanded transition system $(G, \bar{X})$ belong to Δ, and

(2) the occurrence constraints as specified in *Occ* are satisfied.

Note that (1) can be seen as a special case of (2).

In a transition system G of girth $> 2r + 1$ all r-spheres around vertices of G are trees. Hence, for $0 < j \leq r$, the j-sphere around a vertex v is uniquely determined by the list $(t_1, H_1), \ldots, (t_k, H_k)$ of those pairs (t_i, H_i) where t_i is a binary local type and there exists a node w_i such that $G \models t_i(v, w_i)$ and H_i is the $(j-1)$-sphere around w_i. Since G is weakly deterministic each t_i appears at most once in this list.

We show that for each $j \leq r$ and each j-sphere H, there is a formula $\psi_{j,H}(x) \in \mathrm{FO}^2(\tau)$ expressing that the j-sphere around x is isomorphic to H. We proceed by induction on j. For $j = 0$ this is trivial since the 0-sphere around a vertex is just the unary atomic type of that vertex. For $j > 0$, let H be a j-sphere given by the list $(t_1, H_1), \ldots, (t_k, H_k)$. Further let

$$L = \{t : t \text{ is a binary local } \tau\text{-type}, t \models x \neq y\}.$$

The formula $\psi_{j,H}(x)$ lists the existing connections of x via the binary local types t_i to vertices y with appropriate $(j-1)$-neighbourhoods and excludes all other connections via binary local types.

$$\psi_{j,H}(x) = \bigwedge_{i=1}^{k} \exists y \Big(t_i(x, y) \wedge \psi_{j-1,H_i}(y) \Big) \wedge \bigwedge_{t \in L - \{t_1, \ldots, t_k\}} \neg \exists y\, t(x, y).$$

($\psi_{j-1,H_i}(y)$ is the formula obtained from $\psi_{j-1,H_i}(x)$ by interchanging all occurrences of x and y.) Clearly $\psi_{j,H}(x)$ has the desired properties.

Recall that occurrence constraints are Boolean combinations of expressions "there exist at least m occurrences of the r-sphere H in $(G, \bar{X})$". Let n be the maximal number appearing in these expressions.

For each r-sphere H, and for $m = 1, \ldots, n+1$ we introduce new monadic predicates Z_H^m which are axiomatized by an FO^2-sentence α expressing the following requirements:

- The r-sphere around x is isomorphic to H if and only if x is contained in Z_H^m for some m: $\forall x \Big(\psi_{r,H}(x) \leftrightarrow \bigvee_{i=1}^{n+1} Z_H^m x \Big)$.
- For $m \leq n$ the sets Z_H^m are either empty or singletons (however, Z_H^{n+1} can contain more than one element): $\forall x \forall y (Z_H^m x \wedge Z_H^m y \rightarrow x = y)$.
- The predicates Z_H^m are disjoint.
- If Z_H^m is non-empty then so are the Z_H^i for $i < m$: $(\exists x Z_H^m x) \rightarrow (\exists x Z_H^{m-1} x)$.

Once the properties of the Z_H^m are axiomatized in this way an occurrence constraint "there exist at least m occurrences of the r-sphere H in $(G, \bar{X})$" is equivalent to the formula $\exists x Z_H^m x$.

It follows that the acceptance of a transition system G by the automaton $\mathcal{A}$ is expressed by a monadic $\Sigma_1^1(\mathrm{FO}^2)$-sentence of form $(\exists \bar{X})(\exists \bar{Z})\varphi$ (with $\varphi \in \mathrm{FO}^2$).

It remains to prove that (3) implies (1). Suppose that ψ is a sentence of vocabulary σ in monadic $\Sigma_1^1(\mathrm{FO}^2)$ that defines $\mathcal{C}$ in $\mathcal{K}$. Since the transformation to Scott normal form preserves monadic $\Sigma_1^1(\mathrm{FO}^2)$ we can assume that ψ has the form

$$\exists X_1 \cdots X_r (\forall x \forall y \beta_0 \wedge \bigwedge_{i=1}^{k} \forall x \exists y \beta_i)$$

where the X_i are monadic and $\beta_1, \ldots, \beta_k$ quantifier-free.

Let τ be the expanded vocabulary $\sigma \cup \{X_1, \ldots, X_r\}$ and let T be the set of binary atomic τ-types. For $i = 1, \ldots, k$ let $R_i := \{t(x, y) \in T : t(x, y) \models \beta_i(x, y)\}$.

On a triangle-free graph, the 1-sphere around a vertex x is uniquely determined by the unary atomic type $s(x)$ of x in G and by the set of *local* binary types in the star of x. All these binary types are of course consistent with $s(x)$. Hence we can describe 1-spheres by pairs (s, Γ) where s is a unary atomic type and Γ is a set of local τ-types such that $t(x, y) \models s(x)$ for all $t \in \Gamma$.

The states of the desired graph automaton $\mathcal{A} = (Q, A, B, \Delta, \mathit{Occ})$ are words in $\{0, 1\}^r$ to be interpreted as the sequence the truth values of $X_1, \ldots, X_r$ at the given vertex. Hence a run of $\mathcal{A}$ on G defines an expansion $(G, \bar{X})$ of G. The set of transitions is the set of 1-spheres that are compatible with $\forall x \forall y \beta_0$, i.e. $\Delta := \{(s, \Gamma) : \Gamma \subseteq R_0\}$.

So far, this only ensures that all pairs of vertices of distance at most 1 satisfy the $\forall\forall$-part of ψ. We use appropriate occurrence constraints to make sure that for each transition system accepted by $\mathcal{A}$,

(1) also every pair of remote vertices satisfies β_0;

(2) we find for each vertex x (local or remote) witnesses $y_1, \ldots, y_k$ satisfying the formulae $\beta_i(x, y_i)$.

Suppose that x and y are two remote vertices with unary atomic types $s(x)$ and $s'(y)$. Define $\mathrm{nl}[s, s']$ to be the binary atomic type that is the unique non-local completion of $s(x)$

and $s'(y)$, namely $\mathrm{nl}[s, s'] = s(x) \cup s'(y) \cup \{x \neq y\} \cup \{\neg E_b xy : b \in B\}$.

Let (s, Γ) be the 1-sphere around x. Using a Boolean combination of occurrence constraints for spheres (s', Γ') we can say that the total number of occurrences of the atomic type s' is larger than the number of local realizations of s' inside the 1-sphere (s, Γ), which is equivalent to the existence of a remote realization of s', outside the sphere (s, Γ).

Hence, using occurrence constraints of 1-spheres, we can express whether or not a non-local binary type $\mathrm{nl}[s, s']$ appears in the expanded transition system $(G, \bar{X})$.

To satisfy (1), we just have to express that there is no occurrence of a non-local type $\mathrm{nl}[s, s']$ that is incompatible with β_0.

For (2), suppose that the 1-sphere around x is (s, Γ). Clearly Γ tells us for which i's the requirements $\exists y \beta_i(x, y)$ are satisfied locally, i.e. inside the 1-sphere around x. For all other i we need an occurrence constraint saying that, in case there is an occurrence of (s, Γ) then there also exists a remote occurrence of at least one of the atomic types s' such that $\mathrm{nl}[s, s'] \models \beta_i(x, y)$. (In the case that for some i, no such s' exists, we need a constraint saying that there is no occurrence of (s, Γ)).

It is not difficult to verify that the automaton $\mathcal{A}$ constructed in this way acceps G if and only if $G \models \psi$. ☐

In particular, $\Sigma^1_1(\mathrm{FO}^2)$ precisely describes the power of graph automata based on 1-spheres, on the class of weakly deterministic, triangle-free transition systems. We observe that the condition that all $G \in \mathcal{K}$ are triangle-free cannot be removed.

Proposition 5.8. *There exist properties of weakly deterministic graphs that are recognizable by automata based on 1-spheres (and first-order definable with three variables), but not definable in* $\Sigma^1_1(\mathrm{FO}^2)$.

For example, let $\mathcal{C}$ be the class of all directed graphs $G = (V, E)$ that are disjoint unions of directed triangles, within the class $\mathcal{K}$ of all weakly deterministic graphs (or even, if you prefer, within the class of all disjoint unions of directed cycles). It is obvious that $\mathcal{C} \subseteq \mathcal{K}$ is recognizable by a graph automaton using only 1-spheres. By a standard argument based on Ehrenfeucht-Fraïssé games, one can show that this class is not definable in $\Sigma^1_1(\mathrm{FO}^2)$.

Open Problem. Is there *any* class of graphs that is definable in $\Sigma^1_1(\mathrm{FO}^2)$ but not in monadic Σ^1_1?

6 The guarded fragment

Other interesting decidable fragments of first-order logic that have received considerable attention recently are the guarded fragment and its generalizations. The guarded fragment GF was introduced by Andréka, van Benthem and Németi [1]; it consists of relational first-order formulae whose quantifiers are appropriately relativized by atoms. The guarded fragments are interesting because they extend in a natural way many propositional modal logics, because they have useful model-theoretic properties and especially because they are decidable classes that avoid the usual syntactic restrictions (on the arity of relation symbols, the quantifier pattern or the number of variables) of almost all other known decidable fragments of first-order logic as those described in [4]. Indeed the restriction that quantification must always be guarded seems to provide a good explanantion for the convenient algorithmic properties of modal logics (see [14]). We refer to [1, 11] for motivation and results on guarded logics.

Definition 6.1. The *guarded fragment* GF of first-order logic is defined by induction as follows:

(1) Every atomic formula (without function symbols of positive arity) belongs to GF.

(2) GF is closed under Boolean operations.

(3) If $\bar{x}, \bar{y}$ are tuples of variables, $\alpha(\bar{x}, \bar{y})$ is atomic and $\psi(\bar{x}, \bar{y})$ is a formula in GF such that $\mathrm{free}(\psi) \subseteq \mathrm{free}(\alpha) = \{\bar{x}, \bar{y}\}$, then the formulae

$$\exists \bar{y}(\alpha(\bar{x}, \bar{y}) \wedge \psi(\bar{x}, \bar{y}))$$
$$\forall \bar{y}(\alpha(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}))$$

belong to GF.

Here, as usual, $\mathrm{free}(\psi)$ means the set of free variables of ψ. An atom $\alpha(\bar{x}, \bar{y})$ that relativizes a quantifier as in rule (3) is the *guard* of the quantifier. Notice that the guard must contain *all* the free variables of the formula in the scope of the quantifier.

Notation. We use the notation $(\exists \bar{y} \,.\, \alpha)$ and $(\forall \bar{y} \,.\, \alpha)$ for relativized quantifiers, i.e. we write guarded formulae in the form $(\exists \bar{y} \,.\, \alpha)\psi(\bar{x}, \bar{y})$ and $(\forall \bar{y} \,.\, \alpha)\psi(\bar{x}, \bar{y})$. When this notation is used, it is always understood that α is indeed a proper guard as specified by condition (3).

The question arises how GF behaves in the context considered here. Is the satisfiability problem of GF decidable also over successor structures? What is the expressive power of $\Sigma^1_1(\mathrm{GF})$ over word structures. Does it define only regular languages?

We can answer all these questions. Indeed a rather simple argument shows that, on successor structures, $\Sigma^1_1(\mathrm{GF})$ is as powerful as full Σ^1_1. It suffices to show that for every $k \in \mathbb{N}$, the universal k-ary relation can be axiomatized by GF on successor structures. We can then use these relations as guards and thus make guarded quantification as powerful as unguarded quantification.

For every number m, we produce a guarded first-order sentence φ_m which contains, besides, $S, \min, \max$, the predicates $R^1, \ldots, R^m, S^2, \ldots, S^m$ such that a successor structure $\mathfrak{A} = (A, \min, \max, S, R^1, \ldots, R^m, S^2, \ldots, S^m)$ is a model of φ_m if and only if

- R^j is the complete relation of arity j, i.e. $\mathfrak{A} \models \forall x_1 \cdots x_j R^j x_1 \cdots x_j$.
- S^j has arity $j+1$ and $\mathfrak{A} \models \forall x_1 \cdots x_{j-1} yz (S^j \bar{x} yz \leftrightarrow Syz)$.

φ_m is the conjunction of the following sentences (where $j = 1, \ldots, m-1$).

$$\begin{gathered} R^1 \min \wedge (\forall xy \,.\, Sxy)(R^1 x \rightarrow R^1 y) \\ (\forall \bar{x} \,.\, R^j \bar{x}) R^{j+1} \bar{x} \min \\ (\forall \bar{x} y \,.\, R^{j+1} \bar{x} y)(y = \max \vee \exists z S^{j+1} \bar{x} yz) \\ (\forall \bar{x} yz \,.\, S^{j+1} \bar{x} yz)(Syz \wedge R^{j+1} \bar{x} z). \end{gathered}$$

Obviously, φ_m is guarded and it is easy to see that φ_m has the desired properties.

Corollary 6.2. *The satisfiability problem for* GF *over finite successor structures is undecidable.*

Corollary 6.3. *On words,* $\Sigma_1^1(\mathrm{GF})$ *captures* NP.

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