Automatic Structures: Twenty Years Later

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Abstract

Automatic structures made their appearance at LICS twenty years ago, at LICS 2000. However, their roots are much older. The idea of automata based decision procedures for logical theories can be traced back to the early days of automata theory and to the work of Büchi, Elgot, Trakhtenbrot and Rabin in the 1960s. The explicit notion of automatic structures has first been proposed in 1976 in the (unfortunately largely unnoticed) PhD thesis of Hodgson, and later been reinvented by Khoussainov and Nerode in 1995.

In this tutorial, we present an introduction into the history and basic definitions of automatic structures, and survey the achievements in the study of different variants of automatic structures. We discuss their most important mathematical and algorithmic properties, their characterisations in terms of logical interpretations, and we present some of the mathematical techniques that are used for the analysis of automatic structures and for proving limitations of these concepts.

CCS Concepts: • **Theory of computation** → **Logic**; **Automata over infinite objects**.

Keywords: Automatic structures, finitely presentable structures, decidable theories, interpretations

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1 What are automatic structures ?

Automatic structures are (in general infinite) structures that admit finite presentations by automata. Roughly speaking, a relational structure $\mathfrak{B} = (B, R_1, \dots, R_m)$ is called automatic if its domain can be represented as a regular set in such a way that its relations become recognisable by synchronous multi-tape automata. More precisely, an automatic presentation of $\mathfrak{B} = (B, R_1, \dots, R_m)$ consists of a language L, which is recognisable by an automaton $\mathcal{A}$, and a surjective function $\nu : L \to B$ that associates with every object in L the element of B that it represents. The function ν must be surjective (every element of B is named by some object of L) but need not be injective (elements may have more than one name). In addition it must be recognisable by automata, reading their inputs synchronously, whether two objects in L name the same element of B, and, for each relation R_i, whether a given tuple of objects from L names a tuple in R_i. Together, the automata $\mathcal{A}$ and the automata that recognise equality and the relations $R_1, \dots, R_m$ provide a finite representation of the structure $\mathfrak{B}$.

In principle we can use automata over finite words, infinite words, finite trees, or infinite trees, and possibly even more general objects to obtain different classes of automatic structures. For most of the purposes pursued in this field, it is essential that these automata models are effectively closed under first-order operations (union, intersection, complementation, and projection) and that their emptiness problem is decidable. Indeed, these properties ensure that

- every automatic structure has a decidable first-order theory and, more specifically,
- given any automatic presentation of $\mathfrak{B}$ and any first-order formula $\varphi(x_1, \dots, x_k)$ one can effectively construct an automaton respresenting the relation $\varphi^{\mathfrak{B}} := \{\bar{b} \in B^k : \mathfrak{B} \models \varphi(\bar{b})\}$.

Thus, all (first-order) definable properties of automatic structures can be algorithmically investigated using automata-theoretic methods based on appropriate finite presentations. This makes automatic structures a domain of considerable interest for computer science.

The best understood case of automatic structures concerns those presented by automata on *finite words*. These are sometimes called *word-automatic structures*, but often, just the term "automatic structures" is used for them, provided that the context makes it clear that we do not mean structures presented by automata on more general objects. Examples of (word-)automatic structures include the standard model of Presburger arithmetic, $(\mathbb{N}, +)$, tree structures, Cayley graphs

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of automatic groups, well-orders of length $< \omega^\omega$, the computation graph of any Turing machine, and so on. By using automata on infinite words for the presentation of structures one obtains the more general class of ω-automatic structures, which may have uncountable cardinality, such as the additive group of real numbers, or more general tree structures that include as elements also their infinite branches.

One of the most prominent and important structures with a decidable first-order theory is certainly the field of reals $(\mathbb{R}, +, \cdot)$. The decidability goes back to Tarski [56] and is based on a quantifier elimination argument. Therefore, it is very natural to ask whether the field of reals admits an automatic presentation of some kind. Of course, such a presentation cannot be based on automata on finite words or finite trees, because languages of finite words and trees are countable. However, it might, a priori, be the case that the field of reals is ω-automatic, i.e., admits a presentation based on automata on infinite words, or that it is ω-tree-automatic, with a presentation based on automata on infinite trees. The question whether this is the case is closely related to classical problems raised by Büchi and Rabin in context with decidable theories such as Presburger arithmetic and the theory of the field of reals. The decidability of Presburger arithmetic, the first-order theory of $(\mathbb{N}, +)$, had originally been proven by quantifier elimination, but Büchi's work on the automata based decision procedure of WS1S (the weak monadic theory of (ω, suc)) carries over to an automata-theoretic decidability argument for Presburger arithmetic. In Rabin's classical paper [52], where he proved the decidability of S2S (the monadic theory of the infinite binary tree) and several other theories, he explicitly raised the question whether also the decidability of the field of reals could be proved by automata-theoretic methods. This is up to now one of the most intriguing open problems in the field of automatic structures.

2 Early History

The history of automatic structures can be traced back to the early days of automata theory, for instance to the automata theoretic decision procedures for Presburger arithmetic and other theories by Büchi [13, 14], Elgot [20], Rabin [52], and others. The first explicit definition of automatic and ω-automatic structures appeared in the PhD thesis of Bernard Hodgson [27] in 1976, and the two articles [28, 29] that are based on it. Hodgson also coined the terms automatic structures and ω-automatic structures (first called macroautomatic structures in [27]). Hodgson's definition is only slightly less general than the one presented above, in the sense that it requires automatic presentations to be injective; this makes no essential difference for word-automatic structures, but (as has been discovered much later) does not provide full generality for ω-automatic ones. Hodgson starts with the notion that a theory $\mathrm{Th}(\mathfrak{A})$ of a τ-structure $\mathfrak{A}$ is *decidable by finite automata* to mean that there is an effective procedure that associates with every formula $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ a finite automaton (on finite or infinite words) that accepts a suitable encoding of $\varphi^{\mathfrak{A}}$, the relation defined by φ on $\mathfrak{A}$. He describes the encoding of relations over finite or infinite words via convolutions, and then identifies automatic and ω-automatic structures as special cases of structures whose theories are decidable by automata, due to the effective closure properties and the decidability of the emptiness problems of automata. He provides a number of examples of automatic structures, including the basic types of dense and discrete linear orders, and the structures of natural numbers, integers, or p-adic numbers with addition. Finally Hodgson proves that finite direct products of ω-automatic structures, as well as weak or strong countable powers of automatic structures have theories that are decidable by finite automata. This includes the theory of any finitely generated Abelian group, Skolem arithmetic $\mathrm{Th}(\mathbb{N}, \cdot)$, and the theory of any free Abelian group of countable rank. Unfortunately, Hodgson's work went largely unnoticed and did not have a major impact at the time.

A different root for the study of automatic structures is the theory of *automatic groups*, developed around 1990 in computational group theory (see [21, 22]). Automatic groups are finitely generated groups, whose Cayley graphs can be presented by finite automata in a specific sense. The importance of automatic groups for computational group theory comes from the fact that an automatic presentation of a group yields efficient algorithmic solutions for computational problems that are undecidable in the general case. However, automatic groups are presented by automata in a specific way, with important differences to more general notions of automatic structures. In particular, it is not the group structure $(G, \cdot)$ itself that is automatically presented, but the Cayley graph, and not even all groups with an automatically presentable Cayley graph are automatic groups in the sense of computational group theory. As a result there has not been that much interaction between the study of automatic groups in computational group theory and the study of automatic structures in logic and computer science. We shall discuss the relationship of automatic groups with automatic structures in Sect. 5.

The notion of an automatic structure has been rediscovered almost twenty years after Hodgsons' work by Khoussainov and Nerode [37]. They were motivated by an earlier approach aiming at a constructive description of infinite structures, namely *recursive model theory*, based on structures presented by Turing machines rather than finite automata (or in a restricted version, based on p-time computability [16]). Recursive structures are mathematically very interesting but the algorithmic content turned out to be rather limited; indeed in general only quantifier-free formulae are decidable on such structures. In contrast, as is nicely pointed out in [37], *"research on automatic structures, unlike research on recursive and p-time structures, concentrates on positive results"*. Khoussainov and Nerode discuss several variants of structures presentable by automata over finite words and also suggest their extension to tree-automatic structures; they provide a number of further examples of automatic structures, such as finitely generated Abelian groups, countable vector spaces over finite fields, permutation structures, and make an important connection to computability theory by observing that the computation graph of any Turing machine is automatic. Khoussainov and Nerode have also proposed a number of mathematical challenges that have been very fruitful for the field, in particular the problem of classifying the automatic structures inside particular algebraic domains such as linear orders, Boolean algebras, etc. They also showed that for word-automatic structures it is no restriction to assume that automatic presentations are injective.

Automatic structures have been brought to LICS in 2000 [10]. One of the motivations for this work came from finite model theory: to *"explore to what extent automatic structures are a suitable framework for extending the methods of finite model theory to infinite structures"*. The results in [10], and in the journal paper [11] extending it, include a detailed complexity analysis of the first-order theory of automatic structures and its low-level fragments, an algorithm for evaluating the quantifier "there exist infinitely many", composition theorems for automatic structures, undecidability results for several important properties that are not first-order expressible, such as isomorphism and connectedness, and methods for establishing that certain structures are not automatic. But the perhaps most important contribution of this work has been the model-theoretic characterisations of automatic structures via *first-order interpretations*. As proved in [10], a structure is automatic if, and only if, it is first-order interpretable in an expansion of Presburger arithmetic by a restricted divisibility relation or, equivalently, in the infinite binary tree with prefix order and equal length predicate (see Sect. 4). Similar results hold for ω-automatic structures and first-order interpretations into appropriate expansions of the real ordered group or into extended tree structures, and also for structures presented by automata on finite or infinite trees. Such results also suggest a very general way for obtaining other interesting classes of infinite structures that admit finite presentations: Fix a structure $\mathfrak{A}$ (or a finitely presentable class of such structures) with good algorithmic and model-theoretic properties, and consider the class of all structures that are first-order interpretable in $\mathfrak{A}$. Obviously each structure in this class is finitely presentable (by an interpretation). Further, since many important properties are preserved under interpretations, every structure in the class inherits them from $\mathfrak{A}$. In particular, every class of formulae that admits effective evaluation on $\mathfrak{A}$ and is closed under first-order operations, also admits effective evaluation on the interpretation-closure of $\mathfrak{A}$.

From 2000 onwards, this field has become very rich, with a large amount of research on different variants of automatic structures and other forms of finitely presented structures. For a survey on such work, up to ten yers ago, we refer to [7], more specific accounts on word-automatic structures had been given earlier in [36, 54]. Interestingly a quite relevant part of all this work has been accomplished by students in a number of very strong theses, including those by Abu Zaid [1], Bárány [6], Blumensath [9], Colcombet [17], Huschenbett [30], Kaiser [31], Kartzow [33], and Rubin [53].

3 Automatic structures and their logical theories

We now formally introduce the notion of an automatic structure, assuming that the reader is familiar with the basic notions of automata theory and regular languages (see [57, 58]). We focus on the basic case of word-automatic structures, but the extension to presentations by automata on other objects is obvious. One slightly nonstandard aspect is a notion of regularity not just for languages $L \subseteq \Sigma^*$, but also for k-ary relations of words where $k > 1$. It can be formulated in terms of synchronous multihead automata that take tuples of words as inputs and work synchronously on components, but instead, we reduce the case of higher arity to the unary one by encoding tuples $\bar{w} \in (\Sigma^*)^k$ by a single word $w_1 \otimes \cdots \otimes w_k$ over the alphabet $(\Sigma \cup \{\Box\})^k$, called the *convolution* of $w_1, \ldots, w_k$. Here $\Box$ is a padding symbol not belonging to Σ. It is appended to some of the words w_i to make sure that all components have the same length. More formally, for $w_1, \ldots, w_k \in \Sigma^*$, with $w_i = w_{i1} \cdots w_{i\ell_i}$ and $\ell = \max\{|w_1|, \ldots, |w_k|\}$,

$$w_1 \otimes \cdots \otimes w_k := \begin{bmatrix} w'_{11} \\ \vdots \\ w'_{k1} \end{bmatrix} \cdots \begin{bmatrix} w'_{1\ell} \\ \vdots \\ w'_{k\ell} \end{bmatrix} \in \left((\Sigma \cup \{\Box\})^k\right)^*$$

where $w'_{ij} = w_{ij}$ for $j \le |w_i|$ and $w'_{ij} = \Box$ otherwise. Now, a relation $R \subseteq (\Sigma^*)^k$ is called *regular*, if $\{w_1 \otimes \cdots \otimes w_k : (w_1, \ldots, w_k) \in R\}$ is a regular language.

Definition 3.1. A relational structure $\mathfrak{A}$ is *automatic* if there exist a regular language $L_\delta \subseteq \Sigma^*$ and a surjective function $\nu : L_\delta \to A$ such that the relation

$$L_\epsilon := \{(w, w') \in L_\delta \times L_\delta : \nu w = \nu w'\} \subseteq \Sigma^* \times \Sigma^*$$

and, for all predicates $R \subseteq A^r$ of $\mathfrak{A}$, the relations

$$L_R := \{\bar{w} \in (L_\delta)^r : (\nu w_1, \ldots, \nu w_r) \in R\} \subseteq (\Sigma^*)^r$$

are regular. A structure with functions is automatic if its relational variant is.

We write AutStr for the class of all automatic structures. Each structure $\mathfrak{A} \in$ AutStr can be represented, up to isomorphism, by a list $\mathfrak{d} = \langle M_\delta,\ M_\varepsilon,\ (M_R)_{R\in\tau}\rangle$ of finite automata that recognise L_δ, L_ϵ, and L_R for all relations R of $\mathfrak{A}$. An automatic presentation $\mathfrak{d}$ is called *injective* if $L_\epsilon = \{(u,u) : u \in L_\delta\}$ (which implies that $\nu : L_\delta \to A$ is injective).

Universal automatic structures. We have already mentioned a number of examples of automatic structures. There are two further important classes of automatic structures, that will turn out to be *universal*, or *complete*, for AutStr. First, let $\mathfrak{N}_p := (\mathbb{N}, +, |_p)$ be the expansion of the standard model of Presburger arithmetic, $(\mathbb{N}, +)$, by the relation

$$x \mid_p y \;:\text{iff}\; x \text{ is a power of } p \text{ dividing } y.$$

Using p-ary encodings (starting with the least significant digit) it is not difficult to construct automata recognising equality, addition and $|_p$.

Secondly, for $p \in \mathbb{N}$ we consider the tree structure

$$\mathrm{Tree}(p) := (\{0, \dots, p-1\}^*, (\sigma_i)_{i<p}, \preceq, \mathrm{el})$$

with the successor functions $\sigma_i(x) := xi$, the prefix order $\preceq$, and the equal level predicate $\mathrm{el}(x,y)$ which holds if $|x| = |y|$. Obviously, this structure is automatic as well.

Decidability and complexity. The standard models of finite automata on finite or infinite words or trees are effectively closed under first-order operations and have a decidable emptiness problem. This implies that the first-order theory $\mathrm{Th}(\mathfrak{A})$ of every automatic structure $\mathfrak{A}$ is decidable, and that given an automatic presentation of $\mathfrak{B}$ and a first-order formula $\varphi(x_1, \dots, x_k)$ one can effectively construct an automaton respresenting the relation $\varphi^{\mathfrak{A}} := \{\bar{a} \in A^k : \mathfrak{A} \models \varphi(\bar{a})\}$ defined by φ on $\mathfrak{A}$.

In general the theory of an automatic structure, while decidable, may have non-elementary complexity, i.e. the time complexity may exceed any fixed number of iterations of the exponential function $n \mapsto 2^n$. In particular, this is the case for the structures $\mathfrak{N}_p$ and $\mathrm{Tree}(p)$ [24]. There have been detailed studies of the complexity of model checking and query evaluation problems for fragments of first-order logic, and for specific classes of automatic structures, for instance in [4, 11, 42, 48, 49]. In particular, the (structure complexity of) model checking and query evaluation problems for fixed quantifier-free, existential, and Σ_2-formulae on general word-automatic structures (presented by deterministic automata) have been classified into complexity levels between Logspace and Pspace. On any fixed automatic structure (possibly with functions), quantifier-free formulae can be evaluated in quadratic time [11]. In particular, this generalises the quadratic time solution for the word problem on automatic groups, which can be formulated via term equations on automatic Cayley graphs (see Sect. 5). Further, it was shown in [42] that for any n, there exists formulae in the Σ_{n+1}-prefix class of FO, whose class of automatic models (presented by automata) is complete for n-Exspace, and there exists a fixed automatic structure such that, for all n, its Σ_{n+1}-theory is complete for n-Exspace. In other words, both the structure and expression complexity of Σ_{n+1}-formulae are n-Exspace complete for automatic structures. Kuske and Lohrey [48, 49] have proved that first-order model checking on word- and tree-automatic structures with Gaifman graphs of bounded degree have elementary complexity, and it is shown in [4] that for every natural number n, there exist automatic structures whose first-order theory is placed in the n-th level of the exponential space hierarchy.

The question arises to what extent the decidability results for the theories of automatic structures can be generalised beyond first-order logic. Positive results have been obtained in a series of papers [10, 32, 39, 46] for extensions by *counting quantifiers* of the form *"there are infinitely many elements"*, *"there are k mod m many elements"*, *"there are at most countably many elements"* and *"there are uncountably many elements"* (which are relevant for ω-automatic structures). Let FOC denote the extension of first-order logic by these counting quantifiers.

Theorem 3.2. *Given an ω-automatic presentation of a structure $\mathfrak{A}$ and an* FOC*-formula φ one can effectively extend the presentation of $\mathfrak{A}$ to one of the expanded structure $(\mathfrak{A}, \varphi^{\mathfrak{A}})$. In particular, the* FOC*-theory of every ω-automatic structure is decidable.*

Another example for a quantifier that is regularity preserving, at least for word-automatic structures, and thus also preserves decidability is the Ramsey quantifier. For any $k \geq 1$, the *k-Ramsey quantifier* $\exists^{\text{k-ram}}$ is defined by $\mathfrak{A} \models \exists^{\text{k-ram}}\bar{x}\varphi(\bar{x}, \bar{c})$ if, and only if, there is an infinite $X \subseteq A$ so that $\mathfrak{A} \models \varphi(a_1, \dots, a_k, \bar{c})$ for all pairwise different $a_1, \dots, a_k \in X$. A similar observation applies to quantifiers saying *"there exists an infinite set X satisfying φ"* provided that X appears only negatively in φ, i.e. φ describes a property of sets that is closed under taking subsets. As a consequence, it has been proved by Kuske and Lohrey [47] that there exist problems which are highly undecidable on recursive graphs but decidable on automatic ones. In particular, it is decidable whether a given automatic graph contains an infinite clique, and if it does, a regular set of representatives of such a clique can be computed.

On the other side, it has already been observed in [37] that any Turing machine M has an automatic computation graph: the nodes are the configurations of M, and there is an edge from C to C' if M reaches C' from C in one step. By the undecidability of the halting problem, this implies that the model checking problem is undecidable on automatic structures for any logic that is powerful enough to express graph reachability, such as for instance transitive closure logics, fixed point logics, or monadic second-order logic.

The isomorphism problem. The fact that computation graphs of Turing machines are automatic implies many further undecidability results on automatic structures, such as, for instance, the connectivity problem for directed or undirected automatic graphs. A further fundamental and extensively studied issue is the *isomorphism problem*: given two automatic presentations, decide whether the structures they present are isomorphic. Already in [11], it has been proved that the isomorphism problem for automatic structures is undecidable, and it has then been shown in [53], that this holds in a very strong sense: it is in fact Σ^1_1-complete, i.e., placed outside the arithmetical hierarchy, and on the first level of the analytical hierarchy. It had been known before that the isomorphism problem is Σ^1_1-complete also for the much richer class of *recursive structures*. Further, Σ^1_1-completeness also holds for a number of interesting specific classes of recursive structures, such as linear orders, trees, undirected graphs, Boolean algebras, Abelian p-groups, and it has been argued that the Σ^1_1-completeness of the isomorphism problem for a class of recursive structures implies that there is no good classification for that class from the point of view of computability theory [15]. This has motivated a considerable amount of research classifying the complexity of the isomorphism problem for specific classes of automatic structures [38, 40, 44, 50, 53]: While isomorphism is decidable for automatic ordinals and automatic Boolean algebras, it is still Σ^1_1-complete for automatic order trees, automatic directed and undirected graphs, automatic commutative monoids, automatic linear orders, automatic lattices of height 4, and automatic unary functions. Classes with intermediate complexity of the isomorphism problem, in specific levels of the arithmetical hierarchy, are automatic equivalence relations (Π^0_1), locally- finite automatic graphs (Π^0_3), and automatic trees of height $n \geq 2$ (Π^0_{2n-3}).

Structures that are not word-automatic. In general, it is not easy to prove that a structure $\mathfrak{A}$ (assuming that it is countable and has a decidable first-order theory) does *not* admit an automatic presentation over words. A recent recursion-theoretic result to this effect [8] shows that it is a Σ^1_1-complete problem to decide, whether a given recursive structure (presented by Turing machines) is in fact automatic, so there is no good recursion-theoretic characterisation of the automatic structures within the recursive ones.

One reason for the difficulty to prove that an individual structure is not automatic is that, a priori, the elements could be named in any way by words from a regular language. In situations where this is not the case, such as for automatic groups, there are more methods available such as the fellow-traveller property (see Sect. 5). However, from some standard and classical results in automata theory, such as the Pumping Lemma and the Theorem by Elgot and Mezei on locally finite regular relations, one can derive counting arguments that give limitations on definable functions and relations in any word-automatic structure. Consider any injective automatic presentation of a structure $\mathfrak{A}$, with the associated bijection $\nu : L \longrightarrow A$ from a regular language L to the universe A. For a finite set $\mathcal{F}$ of first-order definable functions on $\mathfrak{A}$ and a definable set $E = \{e_0, e_1, e_2, \dots\}$ of elements, ordered by the length of their representations in L, the *generations* of E (with respect to $\mathcal{F}$) are defined [37] by setting

$$G_0(E) := \{e_0\},$$

$$G_{n+1}(E) := \{e_{n+1}\} \cup G_n(E) \cup \bigcup_{f \in \mathcal{F}} f(G_n(E) \times \cdots \times G_n(E))$$

It can then be shown that $|\nu^{-1}(a)| = O(n)$ for all $a \in G_n(E)$, so in particular $|G_n| = 2^{O(n)}$. This can be applied to show some interesting non-automaticity results [11]:

Theorem 3.3. *None of the following structures has a word-automatic presentation:*

(i) The free semigroup on $m \geq 2$ generators.
(ii) Any structure $\mathfrak{A}$ in which a pairing function, i.e. a bijection $f : A \times A \to A$, can be defined.
(iii) The divisibility poset $(\mathbb{N}, |)$.
(iv) Skolem arithmetic $(\mathbb{N}, \cdot)$.

Since it is not difficult to see that $(\mathbb{N}, \cdot)$ admits a tree-automatic presentation, this implies that the tree-automatic structures are a strictly richer class than the word-automatic ones.

Somewhat more sophisticated techniques, with a more model-theoretic flavour, have been developed and used in [19] and also in [38]. Given an automatic presentation of $\mathfrak{A}$, with $\nu : L \to A$ as above, for a regular language $L \subseteq \Sigma^*$, the subset $C_n := \nu(L \cap \Sigma^{\leq n}) \subseteq A$ of elements named by words of L of length at most n, can only be *linearly shattered* by formulae with a parameter: For a formula $\varphi(x, y)$, let $\varphi^{\mathfrak{A},b} \cap C_n := \{a \in C_n : \mathfrak{A} \models \varphi(a, b)\}$ be the subset of C_n defined by φ with parameter b.

Proposition 3.4. *For any automatic presentation of a structure $\mathfrak{A}$ and any first-order formula $\varphi(x, y)$, the number of* different *sets $\varphi^{\mathfrak{A},b} \cap C_n$, as the parameter b ranges over A, is linearly bounded in $|C_n|$.*

In particular, this readily implies that the *random graph* is not automatic [19]. Indeed the random graph is characterised by the *extension axioms* saying that for any finite set X of vertices and any subset $U \subseteq X$ there exists a node b with an edge to all nodes in U and to none in $X \setminus U$. Thus, any finite set X of nodes is fully shattered by the adjacency relation Exy, so that the random graph cannot have any word-automatic presentation. By similar arguments, it follows that also the random partial order and the random K_n-free graph are not automatic.

A standard example for word-automatic structures are the ordinals $\alpha < \omega^\omega$. An automatic presentation of ω^n is based on the regular language $(0^*1)^n$ with the lexicographic order $<_{\text{lex}}$, and the function $\nu : (0^*1)^n \to \omega^n$, that represents the ordinal $i_{n-1}\omega^{n-1} + \cdots + i_1\omega + i_0$ by the word $0^{i_{n-1}}1 \cdots 0^{n_1}10^{n_0}1$. Using the methods just explained, Delhommé [19] proved that that the ordinal ω^ω itself and all larger ones do not admit a word-automatic presentation.

Proposition 3.5. *The word-automatic ordinals are precisely those below ω^ω.*

For the tree-automatic ordinals the corresponding bound is ω^{ω^ω} [19].

Finally, an even more sophisticated result, due to Tsankov [59], is based on the study of arithmetic progressions and Freiman's Theorem:

Theorem 3.6. *The additive group of rational numbers,* $(\mathbb{Q}, +)$, *is not (word-)automatic.*

Translations among automatic presentations. In general, an automatic structure may admit different automatic presentations. Methods for comparing these have been developed by Bárány [5, 6].

Definition 3.7. Consider two automatic presentations of a structure $\mathfrak{A}$, based on regular languages L_1, L_2 and associated functions $\nu_1 : L_1 \to A$ and $\nu_2 : L_2 \to A$. We say that the two presentations are equivalent if, for any $R \subseteq A^k$, the relation $\nu_1^{-1}(R)$ is regular if, and only if, the relation $\nu_2^{-1}(R)$ is.

A characterisation of the equivalence of (word-)automatic presentations can be given in terms of *semi-synchronous transducers*, which are automata operating on pairs of words, processing the first in blocks of k letters, and the second in blocks of ℓ letters, for certain fixed $k, \ell \in \mathbb{N}$.

Theorem 3.8. *Two (word-)automatic presentations* (L_1, ν_1) *and* (L_2, ν_2) *are equivalent if, and only if, the translation* $T = \{(x, y) \in L_1 \times L_2 : \nu_1(x) = \nu_2(y)\}$ *is recognisable by a semi-synchronous transducer.*

Bárány also proved that the universal automatic structures $\mathrm{Tree}(p)$ and $\mathfrak{N}_p$ are *rigid*, in the sense that they admit only one automatic presentation, up to equivalence. Automatic structures that are not universal, such as $(\mathbb{N}, +)$, permit more flexibility in choosing automatic presentations. Recall that two natural numbers p, q are called multiplicatively independent if they have no common power $p^k = q^\ell$, for $k, \ell \geq 1$. A celebrated result due to Cobham and Semenov says that a relation $R \subseteq \mathbb{N}^k$, which is regular in base p, is also regular in base q if, and only if, either p and q are multiplicatively dependent, or R is in fact first-order definable in $(\mathbb{N}, +)$. To put it differently, relations that are not definable in $(\mathbb{N}, +)$ but, say, in $(\mathbb{N}, + \,|_p)$, are not definable in $(\mathbb{N}, +, |_q)$ if q is multiplicatively independent from p. An intriguing notion in this context is *intrinsic regularity* [6, 39].

Definition 3.9. A relation $R \subseteq A^k$ is *intrinsically regular* in an automatic structure $\mathfrak{A}$, if $\nu^{-1}(R)$ is regular for all automatic presentations (L, ν) of $\mathfrak{A}$.

Clearly, all relations that are FOC-definable in $\mathfrak{A}$ are intrinsically regular. The Cobham-Semenov Theorem implies that the intrinsically regular relations in $(\mathbb{N}, +)$ are precisely the first-order definable ones, and FOC collapses to FO on $(\mathbb{N}, +)$. A complete logical characterisation of the intrinsically regular relations in automatic structures has not yet been found, but it is known that FOC-definability is not sufficient. This is witnessed for instance by the relations that are order-invariant first-order definable, i.e. definable by a formula that makes use of a linear order, but whose semantics is independent from the particular order that is chosen. These are intrinsically regular, but not necessarily FOC-definable. In fact, no extension of FO with unary generalised quantifiers is capable of capturing intrinsic regularity over all automatic structures [6].

4 Characterisation by interpretations

To explain why the structures $\mathfrak{N}_p$ and $\mathrm{Tree}(p)$ are, in a sense, the most general automatic structures, we recall the notion of a first-order interpretation. Interpretations constitute an important tool in mathematical logic, used to define a copy of a structure inside another one. They thus permit us to transfer definability, decidability, and complexity results among theories.

Definition 4.1. Let $\mathfrak{A} = (A, R_0, \ldots, R_n)$ and $\mathfrak{B}$ be relational structures. A (k-dimensional, first-order) *interpretation* of $\mathfrak{A}$ in $\mathfrak{B}$ is a sequence

$$\mathcal{I} = \left\langle \delta(\bar{x}),\ \varepsilon(\bar{x}, \bar{y}),\ \varphi_{R_0}(\bar{x}_1, \ldots, \bar{x}_r), \ldots,\ \varphi_{R_n}(\bar{x}_1, \ldots, \bar{x}_s) \right\rangle$$

of first-order formulae in the vocabulary of $\mathfrak{B}$ (where each tuple $\bar{x}$, $\bar{y}$, $\bar{x}_i$ consists of k variables), such that

$$\mathfrak{A} \cong \mathcal{I}(\mathfrak{B}) := \left(\delta^{\mathfrak{B}},\ \varphi^{\mathfrak{B}}_{R_0}, \ldots,\ \varphi^{\mathfrak{B}}_{R_n}\right) / \varepsilon^{\mathfrak{B}}.$$

Notice that this requires that $\varepsilon^{\mathfrak{B}}$ is a congruence relation on the structure $\left(\delta^{\mathfrak{B}},\ \varphi^{\mathfrak{B}}_{R_0}, \ldots,\ \varphi^{\mathfrak{B}}_{R_n}\right)$. The map $\delta^{\mathfrak{B}} \to A$ witnessing that $\mathfrak{A} \cong \mathcal{I}(\mathfrak{B})$ is called *coordinate map* and is also denoted by $\mathcal{I}$. An interpretation $\mathcal{I}$ is *injective* if its coordinate map is injective, i.e., if $\varepsilon(\bar{x}, \bar{y}) \equiv \bar{x} = \bar{y}$. We denote the fact that $\mathcal{I}$ is an interpretation of $\mathfrak{A}$ in $\mathfrak{B}$ by $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$. If $\mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$ and $\mathfrak{B} \leq_{\mathrm{FO}} \mathfrak{A}$ we say $\mathfrak{A}$ and $\mathfrak{B}$ are *mutually interpretable*.

If $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$ then every first-order formula φ over the vocabulary of $\mathfrak{A}$ can be translated to a formula $\varphi^{\mathcal{I}}$ over the vocabulary of $\mathfrak{B}$ by replacing every relation symbol R by its definition φ_R, by relativising every quantifier to δ, and by replacing equalities by ε.

Lemma 4.2 (Interpretation Lemma). *If* $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$ *then*

$$\mathfrak{A} \models \varphi(\mathcal{I}(\bar{b})) \quad \leftrightarrow \quad \mathfrak{B} \models \varphi^{\mathcal{I}}(\bar{b})$$

for all $\varphi \in \mathrm{FO}$ *and* $\bar{b} \subseteq \delta^{\mathfrak{B}}$.

By the standard closure properties of languages and relations definable by automata, it readily follows that automatic structures are closed under first-order interpretations.

Proposition 4.3. *If* $\mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$ *and* $\mathfrak{B}$ *is automatic, then so is* $\mathfrak{A}$.

Corollary 4.4. *The class of automatic structures is closed under* (i) *extensions by definable relations,* (ii) *factorisations* *by definable congruences,* (iii) *substructures with definable universe, and* (iv) *finite powers.*

It is not difficult to see that the structures $\mathfrak{N}_p$ and $\mathrm{Tree}(p)$ are mutually interpretable, for each $p \geq 2$ (see e.g. [24]). The same is true if we replace the divisibility predicate $|_p$ by the function $V_p : \mathbb{N} \to \mathbb{N}$ that maps each number to the largest power of p dividing it. Indeed we can define the statement $x = V_p(y)$ in $(\mathbb{N}, +, |_p)$ by the formula $x |_p y \wedge \forall z(z |_p y \to z |_p x)$. In the other direction, $V_p(x) = x \wedge \exists z(x + z = V_p(y))$ is a definition of $x |_p y$. By the Büchi-Bruyère Theorem (see [12]), a relation $R \subseteq \mathbb{N}^k$ is first-order definable in $(\mathbb{N}, +, V_p)$ if, and only if, the set of p-ary encodings of the tuples in R is a regular relation over $\{0, \dots, p-1\}^*$.

Putting this all together, it follows that the class of automatic structures can be characterised via first-order interpretations into the universal structures $\mathfrak{N}_p$ or $\mathrm{Tree}(p)$ [10].

Theorem 4.5. *For every structure $\mathfrak{A}$, the following are equivalent:*

(i) $\mathfrak{A}$ is automatic.
(ii) $\mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{N}_p$ for some (and hence all) $p \geq 2$.
(iii) $\mathfrak{A} \leq_{\mathrm{FO}} \mathrm{Tree}(p)$ for some (and hence all) $p \geq 2$.

There are many variations of such results. First of all, also the structures presented by automata over infinite words, or over finite or infinite trees can be characterised via first-order interpretations into certain universal structures. Examples of universal ω-automatic structures are the expansions $\mathfrak{R}_p := (\mathbb{R}, +, \leq, |_p, 1)$ of the additive group of reals by the relation

$$x |_p y :\text{iff } \exists n, k \in \mathbb{Z}\colon x = p^n \text{ and } y = kx.$$

and the extensions of the tree structures $\mathrm{Tree}(p)$ to

$$\mathrm{Tree}^{\omega}(p) := (\{0, \dots, p-1\}^{\leq\omega}, (\sigma_a)_{a\in\sigma}, \preceq, \mathrm{el})$$

that contain both finite and infinite words. Universal tree-automatic or ω-tree automatic structures instead are structures whose *elements* are finite or infinite trees. For details, see [7, 11].

A further interesting variation has been obtained by Colcombet and Löding [17, 18], on the basis of *set interpretations.* These are defined in the same way as first-order interpretations, but they consist of formulae from monadic second-order logic (MSO) whose free variables are set variables. We write $\mathfrak{A} \leq_{\mathrm{set}} \mathfrak{B}$ if $\mathfrak{A}$ is interpretable in $\mathfrak{B}$ via a set interpretation. Similarly we write $\mathfrak{A} \leq_{\mathrm{fset}} \mathfrak{B}$ to denote that there is a *finite set interpretation* of $\mathfrak{A}$ in $\mathfrak{B}$, which means that the set variables (free or bound) in the interpreting formulae range over finite sets only. A standard example is the finite set interpretation of $(\mathbb{N}, +)$ in $(\mathbb{N}, \mathrm{suc})$.

It turns out that a structure is ω-automatic if, and only if, it is set-interpretable in $(\mathbb{N}, \mathrm{suc})$, and it is (word)-automatic if it is finite set-interpretable in $(\mathbb{N}, \mathrm{suc})$. Analogous results hold for tree automatic and ω-tree automatic structures and (finite) set interpretations into the infinite binary tree. From these characterisations one can get back universality via first-order interpretations by taking the subset envelope $\mathcal{P}(\mathfrak{A})$ and the finite subset envelope $\mathcal{P}_f(\mathfrak{A})$ of relational structures $\mathfrak{A} = (A, R_1, \dots, R_m)$. The universe of $\mathcal{P}(\mathfrak{A})$ is the powerset of A, partially ordered by the subset relation $\subseteq$, and with the relations $R_i' = \{(\{a_1\}, \dots, \{a_r\}) : (a_1, \dots, a_r) \in R_i\}$. The finite subset envelope $\mathcal{P}_f(\mathfrak{A})$ is the substructure of $\mathcal{P}(\mathfrak{A})$ induced by the finite subsets of A. It is not difficult to see that $\mathfrak{A} \leq_{\mathrm{set}} \mathfrak{B}$ if, and only if, $\mathfrak{A} \leq_{\mathrm{FO}} \mathcal{P}(\mathfrak{B})$ and $\mathfrak{A} \leq_{\mathrm{fset}} \mathfrak{B}$ if, and only if, $\mathfrak{A} \leq_{\mathrm{FO}} \mathcal{P}_f(\mathfrak{B})$. As a consequence, the structures $\mathcal{P}_f(\mathbb{N}, \mathrm{suc})$ and $\mathcal{P}(\mathbb{N}, \mathrm{suc})$ are universal for automatic and ω-automatic structures, respectively, via first-order interpretations, and the (finite) subset envelopes of the complete binary tree are universal for tree automatic and ω-tree automatic structures.

Beyond these results and beyond specific classes of automatic structures, interpretations provide a general and powerful method to obtain classes of finitely presented structures with a set of desired properties. One fixes some structure $\mathfrak{B}$ having these properties and chooses a kind of interpretation that preserves them. Then one considers the class of all structures which can be interpreted in $\mathfrak{B}$. Each structure $\mathfrak{A}$ of this class can be represented by an interpretation $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$ which is a finite object, and model checking and query evaluation for such structures can be reduced to the corresponding problem for $\mathfrak{B}$. If $\mathcal{I} : \mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$ then Lemma 4.2 implies that

$$\varphi^{\mathfrak{A}} = \{\bar{a} : \mathfrak{A} \models \varphi(\bar{a})\} = \{\mathcal{I}(\bar{b}) : \mathfrak{B} \models \varphi^{\mathcal{I}}(\bar{b})\}.$$

Hence, the desired representation of $\varphi^{\mathfrak{A}}$ can be constructed by extending the interpretatio to $\langle \mathcal{I}, \varphi^{\mathcal{I}} \rangle : (\mathfrak{A}, \varphi^{\mathfrak{A}}) \leq_{\mathrm{FO}} \mathfrak{B}$.

5 Automatic groups

Consider a group $(G, \cdot)$ with a finite set $S = \{s_1, \dots, s_m\} \subseteq G$ of semigroup generators, so that each group element $g \in G$ can be written as a product $s_{i_1} \cdots s_{i_r}$ of elements of S and hence the canonical homomorphism $\nu : S^* \to G$ is surjective. The *Cayley graph* $\Gamma(G, S)$ of G with respect to S is the edge-labelled graph $(G, S_1, \dots, S_m)$ whose vertices are the group elements and where S_i is the set of pairs (g, h) such that $gs_i = h$. In computational group theory, a group $(G, \cdot)$ is called an *automatic group,* if there is a finite set S of semigroup generators and a regular language $L_\delta \subseteq S^*$ such that the restriction of ν to L_δ is surjective and provides an automatic presentation of $\Gamma(G, S)$. In other words, the inverse image of equality,

$$L_\epsilon = \{(w, w') \in L_\delta \times L_\delta : \nu w = \nu w'\},$$

and the binary relations $\nu^{-1}(S_i)$, for $i = 1, \dots, m$, are regular.

We emphasise that it is not the group structure $(G, \cdot)$ itself that is automatic in the sense of Definition 3.1, but the Cayley graph $\Gamma(G, S)$. Automatic groups have been studied very intensively in computational group theory. There are many natural examples of automatic groups (see [21, 22]) such as Euclidean groups and finitely generated Coxeter groups. The importance of this notion for computational group theory comes from the fact that an automatic presentation of a group yields efficient algorithmic solutions for computational problems that are undecidable in the general case. In particular, the word problem for an automatic group is solvable in quadratic time, and for any word in S^* one can find in quadratic time a representative in L_δ.

Automatic groups can also be characterised via the *fellow traveller property.* For two words $u, v \in S^*$, let $d(u, v)$ be their natural distance in the Cayley graph $\Gamma(G, S)$. Further, let $u|_i$ be the prefix of u with length i, for $i \leq |u|$, and $u|_i = u$, for $i > |u|$. Suppose now that $(G, \cdot)$ is an automatic group, with an automatic presentation based on a regular language $L_\delta \subseteq S^*$. Then there exists some $k \in \mathbb{N}$ such that $\nu : L_\delta \to G$ satisfies the k-fellow traveller property: for any two elements $u, v \in L_\delta$ with $d(u, v) \leq 1$, we have that $d(u|_i, v|_i) \leq k$, for all $i \leq \max(|u|, |v|)$. Indeed, since the pair (u, v) is accepted by one of the synchronously operating automata in the automatic presentation of $\Gamma(G, S)$, a simple pumping argument shows that the prefixes of common length of u and v cannot have large distance in the Cayley graph. It turns out that conversely, the regularity of L_δ and the k-fellow traveller property are in fact sufficient for $(G, \cdot)$ being an automatic group [22].

Theorem 5.1. *A group $(G, \cdot)$ is automatic if, and only if, for some finite $S \subseteq G$ and some $k \in \mathbb{N}$, there exists a regular language $L_\delta \subseteq S^*$ such that the canonical map $\nu : L_\delta \to G$ is surjective and satisfies the k-fellow traveller property.*

The definition of an automatic group requires that the function $\nu : L_\delta \to G$ is the restriction of the canonical homomorphism from S^* to G, so we are not free to change the coordinate map. This has several important consequences, compared to to the general notion of automatic structures. In particular the arguments in Sect. 4 give us a characterisation of automatic groups in terms of definability rather than interpretability [11].

Theorem 5.2. *$(G, \cdot)$ is an automatic group if and only if there exists a finite set $S \subseteq G$ of semigroup generators such that $\Gamma(G, S)$ is FO-definable in $\mathrm{Tree}(S)$.*

By definition, if G is an automatic group, then for some set S of semigroup generators, the Cayley graph $\Gamma(G, S)$ is an automatic structure. The converse does not hold [11, 35]:

Proposition 5.3. *There exist groups G with a set of semigroup generators S such that the Cayley graph $\Gamma(G, S)$ is an automatic structure without G being an automatic group.*

There are many such examples including the Heisenberg groups $\mathcal{H}_n(\mathbb{Z})$ for $n \geq 3$, as well as many nilpotent and metabelian groups. This naturally leads to the more general notion of a *Cayley graph automatic group*, defined by Kharlampovich, Khoussainov, and Miasnikov [35], as the set of finitely generated groups with an automatic Cayley graph $\Gamma(G, S)$. This notion is robust in the sense that if $\Gamma(G, S)$ is an automatic graph for some generating set S, then $\Gamma(G, S')$ is automatic for all generating sets S'. On the positive side, the decidability of the first-order theory of $\Gamma(G, S)$ implies (and is in fact equivalent to) the decidability of the word problem for (G, S) and, just as for automatic groups, the word problem can be solved in quadratic time also for Cayley graph automatic groups. On the negative side, there is no known characterisation of the Cayley graph automatic groups by an analogue of the fellow traveller property. For further results and open problems, we refer to [35].

6 ω-automatic structures and injective presentations

We next discuss the case of ω-automatic structures, presented by automata on infinite words. An important difference to word-automatic structures is that the distinction between injective and non-injective presentations becomes relevant. Injective presentations are much easier to work with, since we do not need an automaton to determine whether two words encode the same element. In the case of word-automatic structures it had already been pointed out in [37] that all such structures admit an injective automatic presentation. However, it had been open for some time whether this is the case for ω-automatic structure, until Hjorth, Khoussainov, Montalbán and Nies [25] described an ω-automatic structure that does not even permit an injective Borel presentation (which is a much more general notion than an injective ω-automatic presentation). Nevertheless, many interesting ω-automatic structures do admit injective presentations such as, for instance, the reducts $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$ of the field of reals.

A central tool in the analysis of ω-automatic presentations is the notion of end-equivalence $\sim_e$ of infinite words: $x \sim_e y$ if x and y are equal from some position onwards. Making this position m explicit we get refined equivalence relations $\sim_e^m$. Clearly, $\sim_e$ is an ω-regular relation, but it does not permit an ω-regular set of representatives, so unlike the finite-word case, injectivity cannot generally be achieved by selecting a regular set of representatives from a given presentation.

However, every ω-regular equivalence relation having only countably many classes does allow to select an ω-regular set of unique representants. Therefore, every countable ω-automatic structure does have an injective presentation. Further, an injective ω-automatic presentation of a countable structure can be"packed" into one over finite words. It follows that for countable structures, ω-automatic presentations are not more powerful than automatic presentations on finite words [32]:

Theorem 6.1. *Let $\mathfrak{A}$ be a countable structure. Then the following statements are equivalent.*

- *$\mathfrak{A}$ is ω-automatic.*
- *$\mathfrak{A}$ has an injective ω-automatic presentation.*

- $\mathfrak{A}$ *is finite word automatic.*

In particular, the results mentioned above that countable structures such as $(\mathbb{N}, \cdot)$, the random graph, the additive group of the rationals, and others are not word-automatic immediately imply that they are not ω-automatic either.

For the analysis of uncountable ω-automatic structures, advanced techniques have been developed in [2, 43, 46] which, as in the case of word-automatic structures, establish limits of the properties of definable functions. However, they are based on more sophisticated mathematical notions and arguments, including the end-equivalence relation $\sim_e$ mentioned above, ω-semigroups and Wilke algebras [51, 60], and Ramsey's Theorem. Some key results concern limitations on the minimal image size of a definable function [2].

Definition 6.2. For every function $f : A^k \to A$ over an infinite set A, the *minimal image size* $\mathrm{MIS}_f : \mathbb{N} \to \mathbb{N}$ is defined by $\mathrm{MIS}_f(n) = \min\{|f(X^k)| : X \subseteq A, |X| = n\}$.

Proposition 6.3. *Let $\mathfrak{A}$ be a structure with an ω-automatic presentation such that there exists an infinite set of elements that are represented by words in the same $\sim_e$-class. Then every* FOC*-definable function f on $\mathfrak{A}$ has the property that* $\mathrm{MIS}_f(n) = O(n)$.

A simple automata-theoretic argument shows that, for any m, one can control the action of a definable function f on $\sim_e^m$-equivalent elements. More precisely, the image of every $\sim_e^m$-equivalent set B of tuples in A can be partitioned into a fixed number of $\sim_e^m$-equivalent sets, and this number only depends on the underlying structure and on f, but not on B. The argument then relies on a analysis of the size of maximal sets of $\sim_e^m$-equivalent elements and their images under f, which leads the assumption that MIS_f grows super-linearly for some definable f to a contradiction. See [2] for details.

The assumption that there exists an infinite collection of end-equivalent elements is clearly satisfied in every structure with an injective ω-automatic presentation, because every ω-regular language has an infinite $\sim_e$-class. This easily follows from the fact that we can write any ω-regular language as a finite union of sets UV^ω where U, V are regular languages of finite words [58].

In non-injectively presented ω-automatic structures, infinite $\sim_e$-equivalent sets need not exist. Indeed, $\sim_e$ is an ω-automatic equivalence and thus the presentation might indeed identify all end-equivalent words. However, this cannot happen for ω-automatic presentations of uncountable structures with a definable linear order. In an interesting analysis of Ramsey theory for ω-automatic graphs, Kuske [43] has shown that any ω-automatic presentation of an uncountable linear order can be restricted to a presentation of $(\{0,1\}^\omega, <_{\mathrm{lex}}) \cong (\mathbb{R}, <)$. This restriction is not ω-automatic, but its domain is the complement of a language $\bigcup_{i \le n} V_i U_i^\omega$ where the V_i are context free and the U_i are regular. In particular his presentation does not contain any two $\sim_e$-equivalent words. A strengthening of Kuske's result, based on techniques from [32], has been obtained in [2].

Proposition 6.4. *Every automatic presentation of an uncountable linear order contains an injective automatic presentation of* $(\{0,1\}^\omega, <_{\mathrm{lex}})$.

Given a presentation of an uncountable linear order over an ω-regular language L, one can find finite words u, v_0, v_1 such that $|v_0| = |v_1|$, $u\{v_0, v_1\}^\omega \subseteq L$ and for any two words $\alpha, \beta \in \{0,1\}^\omega$ it holds that

$$uv_{\alpha[0]}v_{\alpha[1]}v_{\alpha[2]} \ldots < uv_{\beta[0]}v_{\beta[1]}v_{\beta[2]} \ldots$$

if, and only if, $\alpha <_{\mathrm{lex}} \beta$. In other words this shows that if one indentifies v_0 with 0 and v_1 with 1, then the natural encoding of $\{0,1\}^\omega$ by the language $u\{v_0, v_1\}^\omega$ is compatible with the lexicographical ordering $<_{\mathrm{lex}}$. The construction makes use of the characterization of ω-regular languages by morphisms to ω-semigroups and Ramsey's Theorem. Finally, the algebraic structure of the underlying ω-semigroups ensures that the elements encoded by the newly constructed words are ordered as claimed. For details, we again refer to [2].

As a consequence, we can infer that Proposition 6.3 applies also to any ω-automatic structures with a definable linear order. A particular application concerns the definability of *pairing functions*, i.e. bijective functions $f : B \times B \to B$, for an infinite set B. Indeed, $\mathrm{MIS}_f(n) = n^2$ for any such f.

Corollary 6.5. *No ω-automatic structure with a definable linear order or an injective ω-automatic presentation admits an* FOC*-definable pairing function.*

Kuske's result that the real line can be embedded into any uncountable linear order also implies that there are no uncountable ω-automatic ordinals [43]. Hence the ω-automatic ordinals coincide with the automatic ones which, as mentioned above, are those $< \omega^\omega$.

The isomorphism problem. While for many important problems, in particular for the decidability of their FOC-theories, ω-automatic structures have similar algorithmic properties as the (word-)automatic ones, this is not the case for the isomorphism problem. For (word-)automatic structures, the isomorphism problem is Σ_1^1-complete, but Hjorth et al. [25] constructed two ω-automatic structures for which the existence of an isomorphism depends on the axioms of set theory and inferred that isomorphism of ω-automatic structures does not belong to Σ_2^1 (the second level of the analytical hierarchy). Kuske, Liu, and Lohrey [45] then proved that the isomorphism problem for ω-automatic trees of finite height is not even analytical, i.e., is not contained in any of the levels Σ_n^1. More specifically, the isomorphism problem for ω-automatic trees of height $n \geq 4$ is hard for both Σ_{n-3}^1 and Π_{n-3}^1. A more precise analysis reveals at which height the complexity jump occurs: For both automatic and ω-automatic trees of height 2, the isomorphism problem is co-r.e.; for automatic trees of height 3 isomorphism is Π^0_3-complete, but for ω-automatic trees of height 3 it is hard for Π^1_1 (and therefore outside of the arithmetical hierarchy). The lower bounds for ω-automatic trees also hold for those admitting an injective ω-automatic presentation. Further, it has been shown in [23] that the isomorphism problem for ω-tree-automatic Boolean algebras, partial orders, rings, non-commutative groups, and nilpotent groups presented by automata over infinite trees is neither in Σ^1_2 nor Π^1_2.

7 Is there an automatic presentation of the field of reals?

Arguably, the two most important and best-studied mathematical structures with a decidable first-order theory are $(\mathbb{N}, +)$ and the field of reals, $(\mathbb{R}, +, \cdot)$. For both theories, the decidability has originally been showed by quantifier elimination. However, while Büchi's automata based decision procedure for WS1S carries over to Presburger arithmetic, and the automaticity of $(\mathbb{N}, +)$ is obvious, the field of reals has so-far resisted all automata based approaches. Clearly, being uncountable, the field of reals can neither be (word)-automatic nor tree-automatic. However, it could a priori be automatic in a more general sense, based on automata on infinite words or trees, and the question whether this is the case has been explicitly raised already by Rabin [52].

It is quite easy to see that both reducts $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$ of the field of reals admit automatic presentations, in fact even on infinite words, so one might hope that such presentations could be combined to one of the entire field. Today, it is still open whether the field of reals admits an automatic presentation based on automata on infinite trees (or possibly even more general objects), but it has been shown [2] that it is not ω-automatic. This is a consequence of a general analysis of the model-theoretic properties of ω-automatic structures, which implies a number of further limitations of this concept. Indeed one can indicate structural restrictions on the complexity of ω-automatic relations to prove that certain classes of structures (such as infinite fields, or even integral domains) do not have *any* ω-automatic models.

For a more detailed explanation, we make use of the fact [2] that there is no uncountable ω-automatic structure with FOC-definable parameterized functions of unbounded arity such that all pairs of different functions agree on at most countably many arguments.

Lemma 7.1. *Let $\mathfrak{A} = (A, R_1, \dots, R_n)$ be an uncountable ω-automatic structure. Then there is a $k \in \mathbb{N}$ such that for every definable $(k+1)$-ary function $f(\overline{x}, y)$ there exist uncountable sets $M \subseteq A^k$ and $N \subseteq A$ with $f(\overline{a}, b) = f(\overline{a}', b)$ for all $\overline{a}, \overline{a}' \in M, b \in N$. In particular, $M \times N$ is an uncountable set on which f is constant.*

It has been shown in [38] that the word-automatic integral domains are exactly the finite ones. This implies that there exist no countably infinite ω-automatic integral domains. Suppose now that $\mathfrak{A} = (A, +, \cdot)$ is an uncountable ω-automatic integral domain. Fix a presentation of $\mathfrak{A}$ and let k be the constant from Lemma 7.1 with respect to this presentation. Consider the family of polynomials of degree $k-1$, i.e. the family of functions of the form $x \mapsto \sum_{i=0}^{k} a_i x^i$ with k coefficients $a_0, \dots, a_{k-1} \in A$ and input x. It is obvious that this family of functions can be defined in FOC by using the k coefficients $a_0, \dots, a_{k-1}$ as parameters.

On one hand, it is a well-known fact from algebra that, on an integral domain, two different polynomials of degree at most $k-1$ agree on at most $k-1$ inputs. On the other hand, $\mathfrak{A}$ is uncountable and therefore Lemma 7.1 implies that there are $\overline{a} \neq \overline{b} \in A^k$ such that $\sum_{i=0}^{k-1} a_i x^i = \sum_{i=0}^{k-1} b_i x^i$ for even uncountably many $x \in A$. This proves the following result.

Theorem 7.2. *An integral domain is ω-automatic if, and only if, it is finite. In particular, the field of reals is not ω-automatic.*

8 Model checking games

For infinite structures, the standard construction of model checking games leads to infinitely branching game graphs. Thus, the question arises, how a finite presentation via automata can be exploited to construct model checking games with finite game graphs. This problem has been solved by Lukasz Kaiser [31] based on model checking games with hierarchically organised imperfect information. These cover not only first-order logic, but also more expressive formalisms.

Game quantifiers are a classical notion from infinitary model theory. While infinite sequences of quantifiers of the same kind, $\exists x_0 \exists x_1 \exists x_2 \dots$ or $\forall y_0 \forall y_1 \forall y_2 \dots$, do not provide conceptual difficulties, as they can be viewed as a single quantifiers over infinite sequences, it is much more challenging to handle infinite expressions of form $\exists x_0 \forall y_0 \exists x_1 \forall y_1 \dots \psi(\bar{x}, \bar{y})$ where existential and universal quantifiers alternate. Attempts to make precise the sense of such an expression naturally lead to infinite two-person games (Gale-Stewart games) and to the statement that the existential player should have a strategy to guarantee, by appropriate choices of the values for the x_i, that the proposition $\psi(\bar{x}, \bar{y})$ will hold, independently of how the opponent plays. The closure of such expressions under negation is intimately related to the *determinacy* of the associated games, i.e., with the question whether it is always the case that one of the two players has a winning strategy.

Kaiser considered the extension FO[$\Game$] of first-order logic by regular game quantifiers, and investigated decidability, determinacy of the associated games, and the expressive power of this logic on automatic structures. Using alternating automata he proved that FO[$\Game$] on ω-automatic structures is decidable, and that all regular relations are definable by means of the game quantifer $\Game$ on the basis of just the successor relations. Game quantifiers thus provide a highly interesting and extremely powerful operator for automatic structures. The domain of regular relations is preserved, but all regular relations are definable with minimal prerequisites.

This can be exploited for the construction of model checking games for FO[$\eth$], and hence also for FO, which is based on the general notion of *hierarchical infinite games*. These are zero-sum games between two coalitions, organised in a strictly hierarchical way, so that each player can observe only the moves of those players (in both coalitions) that are below her in the hierarchy. Besides the applications to logic there are also purely game-theoretic reasons for organising the imperfect information in a game in this way; indeed other kinds of imperfect information lead to undecidable problems even for very simple winning conditions (such as reachability conditions in games with three players). The order in which players have to move turns out to be a fundamental criterion for such games. Without restrictions on the interaction patterns between players, already Büchi winning conditions (a 'good' state should be seen infinitely often) can lead to nondetermined hierarchical games. On the other side, alternating hierarchical games (where player take turns in a fixed order) are determined for all Muller winning conditions. The proof shows that winning regions for hierarchical alternating games are definable in FO[$\eth$]. Conversely the evaluation of an arbitrary FO[$\eth$] formula on an automatically presented infinite structure can be presented as an alternating hierarchical Muller game. This proves that such games are indeed model checking games for FO[$\eth$] on automatic structures.

9 Automatic structures with advice and uniformly automatic classes

Solving a long-standing open problem, Tsankov [59] has proved that the additive group of the rational numbers $(\mathbb{Q}, +)$ is not automatic. However, it is "almost" automatic, in the sense that there is a presentation in which addition is automatic but the domain is not a regular set [50]. Further, as shown in [41], also the domain can be recognized by an automaton, provided that it has access to a specific infinite advice string. This advice string itself has a decidable monadic second-order theory, which is sufficient to give an automata-based decision procedure for the first-order theory of $(\mathbb{Q}, +)$.

This motivated the study of *advice automatic structures*, which admit automatic presentations of the same kind as $(\mathbb{Q}, +)$ does: they can be presented by automata that have access to some fixed advice. This setting had appeared occasionally in the literature [18, 34] but a systematic investigation has only been done in [1, 3]. Advice automatic structures generalise the domain of automatic structures while preserving their good algorithmic and model-theoretic properties, in particular the decidability of their first-order theories. But there is a further very interesting twist: Automata with advice permit us to lift the notion of an automatic presentation from single structures to classes of structures that can be represented by a single presentation, but with a set of different advices, which leads to the concept of *uniformly automatic classes of structures*.

There are in fact several variants of this concept. Not all advice sets give classes of structures with a decidable theory since one can easily encode undecidable problems inside the advice set, or even in a single advice. But any class of structures that admits an automatic presentation with an advice set that has a decidable monadic second-order theory does indeed have an automata-based decision procedure for its first-order theory, and even for the extension of first-order logic by different variants of cardinality quantifiers and by Ramsey quantifiers. This result shows that automatic presentations with advice provide relevant generalisations of the concept of automata-based representations of infinite structures, and that the algorithmic properties, which make automatic structures suitable for applications, survive under these generalisations.

We can identify classes of structures, such as trees and Abelian groups, where automatic presentations with advice are capable of presenting significantly more complex structures than ordinary automatic presentations. For instance there exists uniformly ω-automatic presentation of the torsion-free Abelian groups of rank one and a uniformly ω-tree automatic presentation for the class of all countable divisible Abelian groups and the class of all Abelian groups up to elementary equivalence. On the other side, there are also limitations of this concept. For Boolean algebras, for instance, we do not gain anything essential from the access to an advice, and every uniformly ω-automatic class of countable Abelian groups must have bounded rank. Further, it turns out that an advice does not help for representing some particularly relevant examples of structures with decidable theories, most notably for the field of reals.

We extend the semantics of classical models of finite automata to define languages that are *regular with advice*.

Definition 9.1. A *parameterised (Büchi-)automaton* over Σ is an automaton $\mathcal{A}$ over an extended alphabet $\Sigma \times \Gamma$. In the standard sense, $\mathcal{A}$ recognizes a language $L(\mathcal{A}) \subseteq (\Sigma \times \Gamma)^\omega$. We say that $\mathcal{A}$ recognises $L \subseteq \Sigma^\omega$ with advice $\alpha \in \Gamma^\omega$ if $L = \{\beta \in \Sigma^\omega : \beta \otimes \alpha \in L(\mathcal{A})\}$. A language L is called *ω-regular with advice α* if there is a parameterised automaton $\mathcal{A}$ recognizing L with advice α. Parameterised automata on finite words and finite or infinite trees are defined analogously.

This leads, in the obvious way, to the notion of an *ω-automatic presentation with advice α* of a structure $\mathfrak{A}$, and we say that a structure $\mathfrak{A}$ is *ω-automatic with advice* if there is a parameterised ω-automatic presentation $\mathfrak{d}$, that presents $\mathfrak{A}$ with advice α, in short $\mathfrak{A} = \mathfrak{d}[\alpha]$, for some parameter α. By changing the automata model we obtain analogous notions of structures that are, for instance, word-automatic with advice or tree-automatic with advice.

Examples of structures that are automatic with advice, but not automatic in the classical sense, include certain generalisations of the tree structures $\mathrm{Tree}(p)$, the structure $(\mathbb{Q},+)$ and more generally, all torsion-free Abelian groups of rank one, i.e., all subgroups of $(\mathbb{Q},+)$ [3]. This relies on the encoding of rationals as digit sequences $(d_i)_i$ of their *factorial base representation.* Every rational number $r \in [0,1)$ can be written as $r = \sum_{i=2}^{n} \frac{d_i}{i!}$ with $d_i < i$.

Automata with advice further admit to generalise automatic presentations from single structures to notions of *uniformly automatic classes of structures.*

Definition 9.2. A class of C of τ-structures is *uniformly ω-automatic* if there is a parameterised ω-automatic presentation $\mathfrak{d}$ of vocabulary τ, and a set $P \subseteq \Gamma^{\omega}$ of parameters, so that $C = \{\mathfrak{d}[\alpha]) : \alpha \in P\}$, up to isomorphism. If P has a decidable MSO-theory we say that C is *strongly ω-automatic.* If P is even regular, then we say that C is *regularly ω-automatic.* In this case we call a tuple $(\mathfrak{d}, \mathcal{A}_p)$ of automata with $L(\mathcal{A}_p) = P$ a *regularly ω-automatic presentation* of C.

We outline how to obtain an effective decision procedure for the first-order theory of a strongly automatic class. The decision procedure works by recursively building the union, complement or projection automaton from automata that recognise the relations defined by subformulae. Since advice automata are basically ordinary synchronous multi-tape automata with a designated advice tape, advice regular relations are also effectively closed under union, complement and projection.

Proposition 9.3. *There is an algorithm which, given a parameterised ω-automatic presentation $\mathfrak{d}$ and a first-order formula $\varphi(\overline{x})$ constructs an automaton $\mathcal{A}_\varphi$ with $L(\mathcal{A}_\varphi) = \{\overline{a}\otimes\alpha : \mathfrak{d}[\alpha]) \models \varphi(\overline{a})\}$.*

Given a *sentence* φ the algorithm constructs an automaton with $L(\mathcal{A}_\varphi) = \{\alpha : \mathfrak{d}[\alpha]) \models \varphi\}$. Deciding whether φ is in the FO-theory of the class presented by $\mathfrak{d}$ and P thus reduces to deciding the inclusion problem $P \subseteq L(\mathcal{A}_\varphi)$. The well-known correspondence theorems between MSO and regular languages imply that there is a MSO-sentence ψ with $\alpha \models \psi$ if, and only if, $\alpha \in L(\mathcal{A}_\varphi)$. Thus the inclusion problem reduces to checking whether ψ holds in every $\alpha \in P$, which proves claims 2 and 3 of the following corollary. Claims 1 and 4 follow from Proposition 9.3 analogously to the case of automatic structures.

Corollary 9.4.
1. *The ω-automatic structures with advice α are effectively closed under* FO*-interpretations.*
2. *The* FO*-theory of a structure that is ω-automatic with advice α is decidable if the MSO-theory of α is decidable.*
3. *The* FO*-theory of a strongly ω-automatic class is decidable.*
4. *If C is* FO*-interpretable in a uniformly ω-automatic class $\mathcal{D}$ then C is also uniformly ω-automatic.*

The analogous automatic, tree- and ω-tree-automatic versions of these statements are true as well.

Analogous versions of Proposition 9.3 further hold for extensions of FO by regularity preserving quantifiers, such as counting or Ramsey quantifiers, i.e., the evaluation of a formula with regularity preserving quantifiers in an automatic structure yields effectively a regular relation again.

Also the result that every countable ω-automatic structure is automatic [32] generalises to uniformly ω-automatic classes of countable structures [3]. We say that an ω-automatic presentation is a presentation over finite words if the elements of the domain(s) of the structure(s) are encoded in a subset of $\Sigma^*\{\square\}^{\omega}$. When a finite words presentation is given we will for brevity often write w for $w\square^{\omega}$.

Theorem 9.5. *A class C of countable structures has a parameterised ω-automatic presentation with parameter set P, if, and only if, it has an injective parameterised ω-automatic presentation over finite words with the same parameter set P. Moreover the transformation to an injective presentation is effective.*

We mentioned above that each individual torsion-free Abelian group of rank 1 is automatic with advice. This can be extended to an automaticity result for the entire class of such groups [3].

The torsion-free Abelian groups of rank $\leq n$ coincide, up to isomorphism, with the subgroups of $(\mathbb{Q}^n,+)$. For rank one, a classification of the subgroups of $(\mathbb{Q},+)$ is given by sequences $c := (c_p)_{p\in\mathbb{P}}$ of numbers $c_p \in \mathbb{N}\cup\{\infty\}$ indexed by primes $p \in \mathbb{P}$. Each such sequence defines a subgroup $(\mathbb{Q}_c,+) \subseteq (\mathbb{Q},+)$ with

$$\mathbb{Q}_c := \{z/(p_1^{d_1}\cdots p_k^{d_k}) \mid z \in \mathbb{Z}, p_i \in \mathbb{P}, d_i \in \mathbb{N}, d_i \leq c_{p_i}\}$$

and every subgroup of $(\mathbb{Q},+)$ is isomorphic to a group $(\mathbb{Q}_c,+)$ for some c. One can generalise the factorial base representation of rational numbers so that precisely the elements of a subgroup $(\mathbb{Q}_c,+)$ have a representation in a generalised factorial base which depends on c. This is used to construct a parametrised automatic presentation, which given as advice a sequence of (binary representations of) natural numbers, presents a structure $(\mathbb{Q}_c,+,<,\mathbb{Z})$, and each such structure can be obtained by some advice. For details, see [3].

Theorem 9.6. *The class of torsion-free Abelian groups of rank 1 is regularly ω-automatic.*

On the other side, it is also the case that every uniformly ω-automatic class of Abelian groups has bounded rank, and hence every Abelian group that is ω-automatic with advice automatic must have finite rank. This relies on combinatorial limitations of the properties of definable functions in structures presentable by parametrised automata, similar to the ones proved for automatic and ω-automatic structures without advice. It follows further that none of the following classes is uniformly automatic:

- The class $\{(\mathbb{Z}^n, +) \mid n \in N\}$ for any infinite set $N \subseteq \mathbb{N} \setminus \{0\}$.
- The class of all free abelian groups.
- The class $\{(\mathbb{N}^n, +) \mid n \in N\}$ for any infinite set $N \subseteq \mathbb{N} \setminus \{0\}$.

Moving to automata over infinite trees and uniformly ω-tree-automatic classes one can establish closure properties of such classes under products and disjoint unions, and get a number of interesting consequences. From the regularly ω-automatic presentation of $(\mathbb{Q}_c, +, <, \mathbb{Z})$ one can obtain a regularly automatic presentation of the subgroups of $\mathbb{Q}/\mathbb{Z}$ via a first-order interpretation. The class of countable divisible Abelian groups can be written as the product closure of $(\mathbb{Q}, +)$ and the subgroups of $\mathbb{Q}/\mathbb{Z}$ generated by $\{n^{-k} : k \in \mathbb{N}\}$. The closure under products of the strongly ω-tree-automatic classes then gives the following.

Proposition 9.7. *The class of countable divisible Abelian groups is strongly ω-tree-automatic.*

Even in cases when a class of structures does not admit a regular presentation up to isomorphism (possibly already due to cardinality reasons), a regular presentation up to elementary equivalence may still be possible, and we can thereby still get a decision procedure for the first-order theory of the class. This is also possible for the class of all Abelian groups itself, since every Abelian group is elementary equivalent to a Szmielew group and every Szmielew group is isomorphic to a countably infinite direct sum of subgroups of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ [26].

Corollary 9.8. *There is a regularly ω-tree-automatic presentation of the class of all Abelian groups up to elementary equivalence.*

In particular, this implies Szmielew's decidability result for the first-order theory of Abelian groups [55].

These results show that in certain domains, in particular the domain of Abelian groups, the concepts of automatic presentations with advice and unformly automatic classes extend the power of presentations significantly. On the other side, the combinatorial limitations for definable functions can be used to show that, on some other domains, parametrizations and advice do not provide significant additional power. We summarize a few results to that effect from [1, 3]:

- The free semigroup with two generators is not a substructure of any countable structure that is ω-automatic with advice.
- $(\mathbb{N}, \cdot)$ is not a substructure of any countable structure that is ω-automatic with advice.
- Infinite integral domains do not admit any injective ω-automatic presentations with advice.
- The field of reals is not ω-automatic with advice.

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References

[1] Faried Abu Zaid. 2016. *Algorithmic Solutions via Model Theoretic Interpretations.* Ph.D. Dissertation. RWTH Aachen University.

[2] Faried Abu Zaid, Erich Grädel, Łukasz Kaiser, and Wied Pakusa. 2013. Model-Theoretic Properties of omega-Automatic Structures. *Theory of Computing Systems, Special Issue dedicated to STACS 2012* (2013).

[3] Faried Abu Zaid, Erich Grädel, and Fréderic Reinhardt. 2017. Advice Automatic Structures and Uniformly Automatic Classes. In *26th EACSL Annual Conference on Computer Science Logic (CSL 2017).*

[4] Faried Abu Zaid, Dietrich Kuske, and Peter Lindner. 2018. Climbing up the Elementary Complexity Classes with Theories of Automatic Structures. In *27th EACSL Annual Conference on Computer Science Logic, CSL 2018 (LIPIcs)*, Vol. 119. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 3:1–3:16.

[5] Vince Bárány. 2006. Invariants of Automatic Presentations and Semi-Synchronous Transductions. In *Proceedings of the International Symposium on Theoretical Aspects of Computer Science, STACS 2006 (LNCS)*, B. Durand and W. Thomas (Eds.), Vol. 3884. Springer, 289–300.

[6] Vince Bárány. 2007. *Automatic Presentations of Infinite Structures.* Dissertation. RWTH Aachen University.

[7] Vince Bárány, Erich Grädel, and Sasha Rubin. 2011. Automata-Based Presentations of Infinite Structures. In *Finite and Algorithmic Model Theory*, Javier Esparza, Christian Michaux, and Charles Steinhorn (Eds.). 1–76.

[8] Nikolay Bazhenov, Matthew Harrison-Trainor, Iskander Kalimullin, Alexander Melnikov, and Keng Meng Ng. 2019. Automatic and Polynomial-Time Algebraic Structures. *Journal of Symbolic Logic* 84, 4 (2019), 1630–1669.

[9] Achim Blumensath. 1999. Automatic Structures. Diploma thesis, RWTH Aachen University.

[10] Achim Blumensath and Erich Grädel. 2000. Automatic Structures. In *Proceedings of IEEE Symposium on Logic in Computer Science LICS 2000.* 51–62.

[11] Achim Blumensath and Erich Grädel. 2004. Finite Presentations of Infinite Structures: Automata and Interpretations. *Theory of Computing Systems* 37 (2004), 641 – 674.

[12] Véronique Bruyère, Georges Hansel, Christian Michaux, and Roger Villemaire. 1994. Logic and p-recognizable sets of integers. *Bull. Belg. Math. Soc* 1 (1994), 191–238.

[13] J. Richard Büchi. 1960. Weak second-order arithmetic and finite automata. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 6 (1960), 66–92.

[14] J. Richard Büchi. 1962. On a Decision Method in Restricted Second Order Arithmetic. In *Proceedings of the International Congress on Logic, Methodology and Philosophy of Science.* 1–11.

[15] Wesley Calvert and Julia F. Knight. 2006. Classification from a Computable Viewpoint. *Bulletin of Symbolic Logic* 12, 2 (2006), 191–218.

[16] Douglas Cenzer and Jeffrey Remmel. 1991. Polynomial-time versus recursive models. *Annals of Pure and Applied Logic* 54 (1991), 17–58.

[17] Thomas Colcombet. 2004. *Propriétés et représentation de structures infinies.* Thèse de Doctorat. Universit'e de Rennes I.

[18] Thomas Colcombet and Christof Löding. 2007. Transforming structures by set interpretations. *Logical Methods in Computer Science* 3, 2 (2007).

[19] Christian Delhommé. 2004. Automaticité des ordinaux et des graphes homogènes. *Comptes Rendus de L'Académie des Sciences, Mathématiques* 339 (2004), 5–10.

[20] Calvin Elgot and Michael Rabin. 1966. Decidability and Undecidability of Extensions of Second (First) Order Theory of (Generalized) Successor. *Journal of Symbolic Logic* 31, 2 (1966), 169–181.

[21] David Epstein, James Cannon, Derek Holt, Silvio Levy, Mike Paterson, and William Thurston. 1992. *Word Processing in Groups*. Jones and Bartlett Publishers, Boston.

[22] Benson Farb. 1992. Automatic Groups: A Guided Tour. *L' Enseignement Mathématique* 38 (1992), 291–313.

[23] Olivier Finkel and Stevo Todorcevic. 2010. The Isomorphism Relation Between Tree-Automatic Structures. *Central European Journal of Mathematics* 8 (2010), 299–313.

[24] Erich Grädel. 1990. Simple Interpretations among Complicated Theories. *Inform. Process. Lett.* 35 (1990), 235–238.

[25] Greg Hjorth, Bakhadyr Khoussainov, Antonio Montalbán, and André Nies. 2008. From Automatic Structures to Borel Structures. In *Proceedings of the IEEE Symposium on Logic in Computer Science, LICS 2008*. 431–441.

[26] Wilfrid Hodges. 1993. *Model Theory*. Cambridge University Press.

[27] Bernard R. Hodgson. 1976. *Théories décidables par automate fini*. Ph.D. Thesis. Université de Montréal.

[28] Bernard R. Hodgson. 1982. On Direct Products of Automaton Decidable Theories. *Theor. Comput. Sci.* 19 (1982), 331–335.

[29] Bernard R. Hodgson. 1983. Décidabilité par automate fini. *Ann. sc. math. Québec* 7, 1 (1983), 39–57.

[30] Martin Huschenbett. 2016. *The model-theoretic complexity of automatic linear orders*. Ph.D. Dissertation. Technische Universität Ilmenau, Germany.

[31] Łukasz Kaiser. 2011. *Logic and Games on Automatic Structures - Playing with Quantifiers and Decompositions*. Lecture Notes in Computer Science, Vol. 6810. Springer.

[32] Łukasz Kaiser, Sasha Rubin, and Vince Bárány. 2008. Cardinality and counting quantifiers on ω-automatic structures. In *Proceedings of the International Symposium on Theoretical Aspects of Computer Science, STACS 2008*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Germany, 385–396.

[33] Alexander Kartzow. 2011. *First-Order Model Checking on Generalisations of Pushdown Graphs*. Ph.D. Thesis. TU Darmstadt.

[34] Alexander Kartzow and Philipp Schlicht. 2013. Structures without Scattered-Automatic Presentation. In *The Nature of Computation. Logic, Algorithms, Applications*. Lecture Notes in Computer Science, Vol. 7921. Springer Berlin Heidelberg, 273–283.

[35] Olga Kharlampovich, Bakhadyr Khoussainov, and Alexei Miasnikov. 2011. From automatic structures to automatic groups. *CoRR* abs/1107.3645 (2011).

[36] Bakhadyr Khoussainov and Mia Minnes. 2007. Three lectures on automatic structures. In *Proceedings of Logic Colloquium*. 132—–176.

[37] Bakhadyr Khoussainov and Anil Nerode. 1995. Automatic presentations of structures. In *Logic and Computational Complexity*. Number 960 in Lecture Notes in Computer Science. Springer, 367–392.

[38] Bakhadyr Khoussainov, Andre Nies, Sasha Rubin, and Frank Stephan. 2007. Automatic Structures: Richness and Limitations. *Logical Methods in Computer Science* 3, 2 (2007).

[39] Bakhadyr Khoussainov, Sasha Rubin, and Frank Stephan. 2004. Definability and Regularity in Automatic Structures. In *STACS 2004, 21st Annual Symposium on Theoretical Aspects of Computer Science (Lecture Notes in Computer Science)*, Vol. 2996. Springer, 440–451.

[40] Bakhadyr Khoussainov, Sasha Rubin, and Frank Stephan. 2005. Automatic linear orders and trees. *ACM Trans. Comput. Log.* 6, 4 (2005), 675–700.

[41] Alex Kruckman, Sasha Rubin, John Sheridan, and Ben Zax. 2012. A Myhill-Nerode theorem for automata with advice. In *Symposium on Games, Automata, Logics and Formal Verification, GandALF 2012 (EPTCS)*, Vol. 96. 238–246.

[42] Dietrich Kuske. 2009. Theories of automatic structures and their complexity. In *Proceedings of CAI 2009*. 81–98.

[43] Dietrich Kuske. 2010. Is Ramsey's Theorem ω-automatic?. In *Proceedings of the International Symposium on Theoretical Aspects of Computer Science, STACS 2010*. 537–548.

[44] Dietrich Kuske, Jiamou Liu, and Markus Lohrey. 2010. The Isomorphism Problem on Classes of Automatic Structures. In *Proceedings of IEEE Symposium on Logic in Computer Science, LICS 2010*. 160–169.

[45] Dietrich Kuske, Jiamou Liu, and Markus Lohrey. 2013. The isomorphism problem for ω-automatic trees. *Ann. Pure Appl. Log.* 164, 1 (2013), 30–48.

[46] Dietrich Kuske and Markus Lohrey. 2008. First-order and counting theories of ω-automatic structures. *Journal of Symbolic Logic* 73 (2008), 129–150.

[47] Dietrich Kuske and Markus Lohrey. 2008. Hamiltonicity of automatic graphs. In *Fifth IFIP International Conference On Theoretical Computer Science - TCS 2008 (IFIP)*, Vol. 273. Springer, 445–459.

[48] Dietrich Kuske and Markus Lohrey. 2011. Automatic structures of bounded degree revisited. *Journal of Symbolic Logic* 76, 4 (2011), 1352–1380.

[49] Markus Lohrey. 2003. Automatic Structures of Bounded Degree. In *Proceedings of LPAR 2003 (Lecture Notes in Computer Science)*, Vol. 2850. Springer, 346–360.

[50] André Nies. 2007. Describing Groups. *Bulletin of Symbolic Logic* 13, 3 (2007), 305–339.

[51] Dominique Perrin and Jean-Eric Pin. 2004. *Infinite words - automata, semigroups, logic and games*. Pure and applied mathematics series, Vol. 141. Elsevier Morgan Kaufmann.

[52] Michael O. Rabin. 1969. Decidability of second-order theories and automata on infinite trees. *Transactions of the AMS* 41 (1969), 1–35.

[53] Sascha Rubin. 2004. *Automatic Structures*. PhD Thesis. University of Auckland.

[54] Sasha Rubin. 2008. Automata Presenting Structures: A Survey of the Finite String Case. *Bulletin of Symbolic Logic* 14, 2 (2008), 169–209.

[55] Wanda Szmielew. 1955. Elementary properties of Abelian groups. *Fundamenta Mathematicae* 41, 2 (1955), 203–271.

[56] Alfred Tarski. 1951. *A Decision Method for Elementary Algebra and Geometry*. University of California Press.

[57] W. Thomas. 1990. Automata on infinite objects. In *Handbook of Theoretical Computer Science*, J. van Leeuwen (Ed.). Vol. B. Elsevier, Amsterdam, 133–191.

[58] Wolfgang Thomas. 1997. Languages, Automata, and Logic. In *Handbook of formal languages, vol. 3*, Grzegorz Rozenberg and Arto Salomaa (Eds.). Springer-Verlag New York, Inc., New York, NY, USA, 389–455.

[59] Todor Tsankov. 2011. The additive group of the rationals does not have an automatic presentation. *Journal for Symbolic Logic* 76, 4 (2011), 1341–1351.

[60] Thomas Wilke. 1993. An algebraic theory for regular languages of finite and infinite words. *International Journal of Algebra and Computation* 3, 4 (1993), 447–489.