# Generalized Quantifiers and 0-1 Laws

Anuj Dawar*
Department of Computer Science
University of Wales, Swansea
Swansea SA2 8PP, U.K.
a.dawar@swansea.ac.uk

Erich Grädel
Mathematische Grundlagen der Informatik
RWTH Aachen
D-52056 Aachen, Germany
graedel@informatik.rwth-aachen.de

## Abstract

*We study 0-1 laws for extensions of first-order logic by Lindström quantifiers. We state sufficient conditions on a quantifier $Q$ expressing a graph property, for the logic $FO[Q]$ – the extension of first-order logic by means of the quantifier $Q$ – to have a 0-1 law. We use these conditions to show, in particular, that $FO[Rig]$, where $Rig$ is the quantifier expressing rigidity, has a 0-1 law. We also show that $FO[Ham]$, where $Ham$ is the quantifier expressing Hamiltonicity, does not have a 0-1 law. Blass and Harary pose the question whether there is a logic which is powerful enough to express Hamiltonicity or rigidity and which has a 0-1 law. It is a consequence of our results that there is no such regular logic (in the sense of abstract model theory) in the case of Hamiltonicity, but there is one in the case of rigidity. We also consider sequences of vectorized quantifiers, and show that the extensions of first-order logic obtained by adding such sequences generated by quantifiers that are closed under substructures have 0-1 laws.*

## 1 Introduction

The study of random graphs in combinatorics has focused attention on the asymptotic probabilities of graph properties. Informally, the asymptotic probability $\mu(P)$ of a graph property $P$ is the limit, as $n$ goes to infinity, of the proportion of graphs of cardinality $n$ that satisfy $P$, if this limit exists. It turns out that many interesting properties of graphs have asymptotic probability 0 or 1. Intuitively, these properties are either false or true in almost all graphs. Thus, for instance, $\mu(\text{connectivity}) = 1$, $\mu(\text{3-colourability}) = 0$, $\mu(\text{rigidity}) = 1$, $\mu(\text{planarity}) = 0$ and $\mu(\text{Hamiltonicity}) = 1$ (see [3]). In contrast, it is clear that evenness – the property of the cardinality of a graph being even – does not have an asymptotic probability.

The study of the *logical* properties of random structures has focused on the existence of 0-1 laws, and other limit laws, for a variety of logics. We say that a logic $L$ has a 0-1 law if, for every property that is expressible by a sentence of $L$, the asymptotic probability is defined and is either 0 or 1. Glebskiĭ *et al.* [9] and Fagin [7] independently showed that first-order logic has a 0-1 law. Such laws have also been established for fragments of second-order logic [12], extensions of first-order logic by inductive operators [1, 15, 16] and the infinitary logic with finitely many variables [13] (see [4] for a survey of results on 0-1 and limit laws).

Most of the known 0-1 laws in logic are proved by means of *extension axioms*. For atomic types $s, t$ where $s \subseteq t$, the $s$-$t$-extension axiom is a first-order sentence stating that every tuple realizing the type $s$ can be extended to a tuple realizing $t$. It can be proved that every extension axiom has asymptotic probability 1 [7]. For graphs this amounts to saying that for all $k \leq m$ and all collections $v_1, \ldots, v_m$ of $m$ nodes there almost surely exists a node $w$ with an edge to each of $v_1, \ldots, v_k$ but to none of $v_{k+1}, \ldots, v_m$. Since every extension axiom holds in almost all graphs, the same is true for any property which is a consequence of a finite collection of extension axioms. Some of the results on asymptotic probabilities of graph properties mentioned above can be derived in this way. For instance, the property of having diameter two is expressed by the conjunction of two extension axioms. As a consequence, we obtain that $\mu(\text{connectivity}) = 1$, even though connectivity is not a first-order property. Similarly, given any graph $H$, the extension axioms imply that almost all graphs contain $H$ as an induced subgraph. Thus, every non-trivial property which is closed under taking subgraphs has asymptotic probability 0; in particular this proves that $\mu(\text{planarity}) = 0$ and $\mu(\text{3-colourability}) = 0$.

However, there are important graph properties which have asymptotic probability 0 or 1 and for which this does not follow from the extension axioms, the

*Supported by EPSRC grant GR/H 81108.

most notable being Hamiltonicity and rigidity. Blass and Harary [2] prove that there is no first-order sentence with asymptotic probability 1 which implies either Hamiltonicity or rigidity. They pose the question of whether there is any natural logic which can express Hamiltonicity or rigidity and which has a 0-1 law. This problem is also commented on in an informal way and reported as "still wide open" in [11].

In this paper we investigate 0-1 laws for extensions of first-order logic by Lindström quantifiers. Such extensions were also considered, from the point of view of 0-1 laws by Fayolle *et al.* [8], where a sufficient condition was established on a quantifier $Q$, for a restricted fragment of the logic FO[$Q$] to have a 0-1 law. We extend such results and formulate other sufficient conditions on quantifiers $Q$ associated with graph properties which guarantee that the logic FO[$Q$] has a 0-1 law in the language of graphs.

We use our conditions to establish, in particular, that FO[Rig] has a 0-1 law, where Rig is the quantifier associated with the class of rigid graphs. By contrast, we show that FO[Ham] does not have a 0-1 law, where Ham is the quantifier associated with the class of Hamiltonian graphs. We also extend the result for FO[Rig] to its closure under relativizations. This enables us to establish that there is no regular logic (in the sense ain which this term is used in abstract model theory, see [6]) which can express Hamiltonicity and which has a 0-1 law, but there is one in the case of rigidity.

Finally, we also consider extensions of first-order logic by means of vectorized quantifiers. In particular, we show that for any quantifier that is closed under substructures, the corresponding extension of first-order logic by means of a vectorized sequence of quantifiers has a 0-1 law, greatly generalizing a result of [8]. This establishes 0-1 laws for the extensions of first-order logic by the sequences of quantifiers obtained by vectorizing the graph quantifiers for 3-colourability and planarity.

# 2 Preliminaries

Let $\sigma, \tau$ be finite relational signatures. We denote structures by $\mathfrak{A}, \mathfrak{B}, \ldots$ and their universes by $A, B, \ldots$. Let $\mathrm{Str}(\sigma)$ and $\mathrm{Str}_n(\sigma)$ denote, respectively, the set of all finite $\sigma$-structures and the set of all $\sigma$-structures with universe $[n] = \{0, \ldots, n-1\}$. For a $\sigma$-structure $\mathfrak{A}$ and a formula $\psi(x_1, \ldots, x_k)$, we write $\psi^{\mathfrak{A}}$ to denote $\{\bar{a} \in A^k : \mathfrak{A} \models \psi(\bar{a})\}$, i.e. the relation that $\psi$ defines on $\mathfrak{A}$. Similarly, if $\psi$ has additional free variables $\bar{y}$, then for any valuation $\bar{b}$ of those variables, we define $\psi^{\mathfrak{A},\bar{b}}$ as $\{\bar{a} \in A^k : \mathfrak{A} \models \psi(\bar{a}, \bar{b})\}$, i.e. the relation defined by $\psi$ on $\mathfrak{A}$ by fixing the interpretation of the parameters $\bar{y}$ to be $\bar{b}$. For a sentence $\varphi$, we write $\mathrm{Mod}(\varphi)$ to denote the set of all (finite) models of $\varphi$. A structure $\mathfrak{B}$ is a substructure of $\mathfrak{A}$, if $B \subseteq A$, and the relations on $\mathfrak{B}$ are the restrictions of the corresponding relations on $\mathfrak{A}$ to the universe $B$.

**Definition 2.1** *An* atomic type in $x_1, \ldots, x_k$ over $\sigma$ *is a maximal consistent set of $\sigma$-atoms and negated $\sigma$-atoms in the variables $x_1, \ldots, x_k$. Often, we call an atomic type in $k$ variables a $k$-type. We denote atomic types by $t, t', s, \ldots$ or by $t(x_1, \ldots, x_k), \ldots$ to display the variables. By abuse of notation, we do not distinguish between an atomic type and the conjunction over all formulae in it.*

The following lemma is immediate.

**Lemma 2.2** *Every quantifier-free formula is equivalent to a disjunction of atomic types.*

Proof. Let $\varphi(x_1, \ldots, x_k)$ be a quantifier-free formula over $\sigma$. Then

$$\varphi(x_1, \ldots, x_k) \equiv \bigvee_{t \models \varphi} t(x_1, \ldots, x_k),$$

where $t$ ranges over the atomic types in $x_1, \ldots, x_k$ over $\sigma$. ■

## 2.1 Asymptotic Probabilities

Let $0 \leq p \leq 1$. A *Bernoulli trial with mean* $p$ is a random variable $X$ that takes only the values 0 and 1 and such that $P[X = 1] = p$.

Let $\Gamma(\sigma) = (\Gamma_n(\sigma))_{n \in \mathbb{N}}$ be a sequence of probability spaces over $\sigma$-structures, where $\Gamma_n(\sigma)$ is obtained by assigning a probability distribution $\mu_n$ to $\mathrm{Str}_n(\sigma)$. Some important examples are:

- $\Omega_n(\sigma, 1/2)$ denotes the probability space with the uniform probability distribution, i.e. every structure $\mathfrak{A} \in \mathrm{Str}_n(\sigma)$ has the same probability $\mu(\mathfrak{A}) = 1/|\mathrm{Str}_n(\sigma)|$.

- For arbitrary functions $p : \mathbb{N} \to [0,1]$ we define the probability spaces $\Omega_n(\sigma, p)$ as follows: the truth of all instances $R(i_1, \ldots, i_r)$ of $\sigma$-atoms over universe $[n]$ are determined by independent Bernoulli trials with mean $p(n)$.

  It is clear that when $p$ is the constant function $1/2$, this indeed gives the uniform probability distribution.

- $\mathcal{G}(n,p)$ is the probability space of random graphs with edge probability $p$ (again $p$ may depend on $n$). We write $\mathcal{G}(p)$ for the sequence $(\mathcal{G}(n,p))_{n\in\mathbb{N}}$. Note that $\mathcal{G}(n,p)$ is not the same space as $\Omega_n(\{E\},p)$, since a graph is assumed to be undirected and loop–free.

For a fixed sequence $\Gamma(\sigma) = (\Gamma_n(\sigma))_{n\in\mathbb{N}}$ of probability spaces, define the probability $\mu_n(P)$ of a class $P$ of $\sigma$-structures as the probability that a structure $\mathfrak{A}$ with universe $\{0,\ldots,n-1\}$ is in the class $P$. Define the asymptotic probability of $P$ as $\mu(P) = \lim_{n\to\infty}\mu_n(P)$, if this limit exists. If the limit does not exist, we say that $P$ has no asymptotic probability for $\Gamma(\sigma)$.

For any logic $L$, we define the asymptotic probability $\mu(\varphi)$ of a sentence of $L$ to be $\mu(\mathrm{Mod}(\varphi))$. If every sentence of $L$ in the vocabulary $\sigma$ has an asymptotic probability for $\Gamma(\sigma)$, we say that $L$ has a limit law for $\Gamma(\sigma)$. Furthermore, if $\mu(\varphi)$ is 0 or 1 for every $\sigma$-sentence $\varphi$ of $L$, we say that $L$ has a 0-1 law for $\Gamma(\sigma)$.

The following theorem is at the core of the proof due to Glebskiĭ *et al.* [9] that first order logic has a 0-1 law (see also [10]). It can be seen as establishing an "almost sure" quantifier elimination property for the theory of finite structures.

**Theorem 2.3 ([9])** *For every formula $\psi(\bar{x})$ of first-order logic, there is a quantifier free formula $\theta(\bar{x})$ such that the sentence $\forall\bar{x}(\theta\leftrightarrow\psi)$ has asymptotic probability 1.*

For formulae $\psi$ and $\theta$ as in Theorem 2.3 above, we will say that $\psi$ and $\theta$ are equivalent *almost everywhere*.

### 2.2 Interpretations and Quantifiers

Let the signature $\tau$ be $\{R_1,\ldots,R_m\}$ where $R_i$ is a relation symbol of arity $r_i$. A sequence $\Psi = \psi_1(\bar{x}_1),\ldots,\psi_m(\bar{x}_m)$ of formulae of signature $\sigma$, where $\psi_i(\bar{x}_i)$ has the free variables $x_1,\ldots,x_{r_i}$ defines an interpretation

$$\begin{array}{rcl}\Psi:\mathrm{Str}(\sigma) & \to & \mathrm{Str}(\tau)\\ \mathfrak{A} & \mapsto & \Psi\mathfrak{A} = (A,\psi_1^{\mathfrak{A}},\ldots,\psi_m^{\mathfrak{A}}).\end{array}$$

An *interpretation with parameters* is given by a sequence $\Psi(\bar{y}) = \psi_1(\bar{x}_1,\bar{y}),\ldots,\psi_m(\bar{x}_m,\bar{y})$ of $\sigma$-formulae $\psi_i$ which may contain besides $\bar{x}_i$ additional free variables $\bar{y}$. For any $\sigma$-structure $\mathfrak{A}$ and any valuation $\bar{a}$ for $\bar{y}$ we obtain an interpreted structure

$$\Psi(\mathfrak{A},\bar{a}) = (A,\psi_1^{\mathfrak{A},\bar{a}},\ldots,\psi_m^{\mathfrak{A},\bar{a}}).$$

The following definition of a generalized quantifier is essentially due to Lindström [14].

**Definition 2.4** *Let $K$ be a collection of structures of some fixed signature $\tau$, which is closed under isomorphisms, i.e. if $\mathfrak{A}\in K$ and $\mathfrak{A}\cong\mathfrak{B}$ then $\mathfrak{B}\in K$. With $K$ we associate the* generalized quantifier *$Q_K$, which can be adjoined to first-order logic to form an extension $FO[Q_K]$, which is defined by closing FO under the following rule for building formulae:*

*If $\Psi(\bar{y}) = (\psi_1,\ldots,\psi_k)$ is an interpretation with parameters $\bar{y}$ from $\sigma$ to $\tau$ then $Q_K\bar{x}(\psi_1,\ldots,\psi_k)$ is a formula of $FO[Q_K]$ of signature $\sigma$ with free variables $\bar{y}$.*

*The semantics of $Q_K$ is given by the following rule: for a $\sigma$-structure $\mathfrak{A}$ and a valuation $\bar{a}$ for $\bar{y}$,*

$$(\mathfrak{A},\bar{a})\models Q_K\bar{x}(\psi_1,\ldots,\psi_k)\iff\Psi(\mathfrak{A},\bar{a})\in K.$$

An interpretation $\Psi$ from $\sigma$-structures to $\tau$-structures also maps a probability space $\Gamma_n(\sigma)$ to a new probability space $\Psi\Gamma_n(\sigma)$ of $\tau$-structures, defined by assigning to $\mathfrak{B}\in\mathrm{Str}_n(\tau)$ the probability

$$\nu(\mathfrak{B}) = \sum_{\Psi\mathfrak{A}=\mathfrak{B}}\mu(\mathfrak{A}),$$

where $\mu(\mathfrak{A})$ is the probability of $\mathfrak{A}$ in $\Gamma_n(\sigma)$.

On the other hand, if we are given $\Psi(\bar{y})$, an interpretation with parameters, it does not define a map from $\sigma$-structures to $\tau$-structures. Rather, it defines a map from pairs $(\mathfrak{A},\bar{a})$, where $\mathfrak{A}$ is a $\sigma$-structure and $\bar{a}$ is a valuation of the parameters $\bar{y}$ in $\mathfrak{A}$, to $\tau$-structures. Thus, we will assume we are given a probability space $\Gamma_n(\sigma,\bar{y})$ that assigns a probability to $(\mathfrak{A},\bar{a})$ for each $\mathfrak{A}\in\mathrm{Str}_n(\sigma)$ and each valuation $\bar{a}$ of the parameters $\bar{y}$ in $\mathfrak{A}$. We then define the probability space $\Psi\Gamma_n(\sigma,\bar{y})$ by assigning to $\mathfrak{B}\in\mathrm{Str}_n(\tau)$ the probability

$$\nu(\mathfrak{B}) = \sum_{\Psi(\mathfrak{A},\bar{a})=\mathfrak{B}}\mu(\mathfrak{A},\bar{a}).$$

One of the goals of this paper is to elucidate the structure of $\Psi\Gamma_n(\sigma,\bar{y})$.

### 2.3 Graph quantifiers

In this paper, a graph always means a loop-free undirected graph $G = (V,E)$. A graph quantifier is a generalized quantifier given by an isomorphism closed class $\mathcal{H}$ of graphs. It is applied to interpretations that map graphs to graphs. Thus, a graph quantifier binds two variables, say $x$ and $y$, and is applied to a single formula $\varphi(x,y,\bar{z})$ of signature $\{E\}$. A little complication arises because we have to make sure that the interpreted structure $\Phi(G,\bar{a})$ is indeed a graph. To avoid the necessity of verifying the semantic condition that a formula does indeed define an irreflexive and symmetric relation (a condition that has to be met for all valuations of the parameters), we impose no restriction on the formulae, but modify the interpretation of formulae.

**Definition 2.5** *For any class $L$ of formulae over $\{E\}$ and any isomorphism closed class $\mathcal{H}$ of graphs, we define the logic $L[Q^G_{\mathcal{H}}]$ by closing $L$ under the following rule: given any formula $\varphi(x, y, \bar{z})$, we can build also the formula*

$$Q^G_{\mathcal{H}} x, y\ \varphi$$

*with free variables $\bar{z}$.*

*The semantics is given by the equivalence*

$$Q^G_{\mathcal{H}} x, y\ \varphi \equiv Q_{\mathcal{H}} x, y\ (x \neq y \wedge (\varphi(x, y) \vee \varphi(y, x))).$$

*(where $\varphi(y, x)$ is $\varphi(x, y)$ with variables $x$ and $y$ interchanged.)*

For any formula $\varphi(x, y, \bar{z})$ we will refer to the interpretation with parameters $\Phi(\bar{z})$ defined by the formula $x \neq y \wedge (\varphi(x, y, \bar{z}) \vee \varphi(y, x, \bar{z}))$ as the *graph interpretation* associated with $\varphi$. We also call $Q^G_{\mathcal{H}}$ the graph quantifier associated with $\mathcal{H}$.

## 3 Eulerian and Hamiltonian Graphs

32 We now proceed to investigate conditions that can be imposed on a class of graphs $\mathcal{H}$ in order for the logic $\mathrm{FO}[Q^G_{\mathcal{H}}]$ to have a 0-1 law. We begin, in this section, by formulating some necessary conditions and showing that they are not sufficient. We also obtain a necessary and sufficient condition for a certain fragment of $\mathrm{FO}[Q^G_{\mathcal{H}}]$ to have a 0-1 law.

For any property $K$, let $\mathrm{FO}^r[Q_K]$ denote those sentences of $\mathrm{FO}[Q_K]$ of the form $Q_K \bar{x} \varphi$, where $\varphi$ is first-order, i.e. $\mathrm{FO}^r[Q_K]$ can express exactly those properties that are reducible to $K$ by means of a first-order interpretation *without* parameters. This fragment was considered by Fayolle *et al.* [8], who showed for any generalized quantifier $Q_K$, and any signature $\sigma$, a sufficient condition for the logic $\mathrm{FO}^r[Q_K]$ to have a 0-1 law on the class of $\sigma$-structures is that $K$ is monotone and closed under extensions. Clearly a necessary condition is that $K$ itself has asymptotic probability 0 or 1, because the property $K$ can be trivially expressed by a sentence of $\mathrm{FO}^r[Q_K]$. We first show that this latter condition is not sufficient, by means of an example.

**Example 3.1** *Let $\mathcal{E}$ be the class of Eulerian graphs. It is well known that a connected graph $G$ is Eulerian if, and only if, every vertex in $G$ has even degree. Furthermore it follows from known results about degrees in random graphs (see [3], chapter 3) that $\mathcal{E}$ has asymptotic probability 0 for $\mathcal{G}(p)$ for any constant $0 \leq p < 1$. However, consider the following sentence of $FO^r[Eul]$:*

$$\varphi \equiv (Eul\, x, y\ (x \neq y))$$

*It is clear that a graph $G$ satisfies $\varphi$ if and only if it has an odd number of vertices. It follows that $\varphi$ does not have an asymptotic probability for $\mathcal{G}(p)$ for any $p$.*

The above example shows that a graph property might have asymptotic probability 0 or 1 for $\mathcal{G}(p)$ for a fixed $p$, without the logic $\mathrm{FO}^r[Q^G_{\mathcal{H}}]$, let alone $\mathrm{FO}[Q^G_{\mathcal{H}}]$, having a 0-1 law. The next result establishes a necessary *and* sufficient condition for $\mathrm{FO}^r[Q^G_{\mathcal{H}}]$ to have a 0-1 law for $\mathcal{G}(p)$.

**Theorem 3.2** *For any graph property $\mathcal{H}$, $FO^r[Q^G_{\mathcal{H}}]$ has a 0-1 law for $\mathcal{G}(p)$ if and only if $\mathcal{H}$ has asymptotic probability 0 or 1 for each of $\mathcal{G}(0)$, $\mathcal{G}(1)$, $\mathcal{G}(p)$ and $\mathcal{G}(1-p)$.*

Proof. Let $\varphi \equiv Q^G_{\mathcal{H}} x, y\ \psi$ be a sentence of $\mathrm{FO}'[Q^G_{\mathcal{H}}]$. By Theorem 2.3, there is a quantifier free formula $\theta$ that is equivalent to $\psi$ almost everywhere. The asymptotic probability of $\varphi$ is given by the asymptotic probability of $\mathcal{H}$ on $\Theta\mathcal{G}(p)$, where $\Theta$ is the graph interpretation defined by the formula $\theta$.

Up to equivalence, there are only four quantifier-free formulae in two variables that define an irreflexive and symmetric relation:

True, False, $Exy$ and $\neg Exy$.

Thus, $\Theta\mathcal{G}(p)$ is one of $\mathcal{G}(1)$, $\mathcal{G}(0)$, $\mathcal{G}(p)$ or $\mathcal{G}(1-p)$. ■

In Example 3.1 above, the class of Eulerian graphs does not have an asymptotic probability defined for $\mathcal{G}(1)$.

To take another example, recall that the class of Hamiltonian graphs has asymptotic probability 1 for $\mathcal{G}(p)$ for any constant $p > 0$ (see [3]). Since it is clear that this class has asymptotic probability 0 for $\mathcal{G}(0)$, it follows from Theorem 3.2 that $\mathrm{FO}^r[\mathrm{Ham}]$ has a 0-1 law. We will show next that there is no such law for FO[Ham], which implies that the condition in Theorem 3.2 is, in general, not sufficient to establish a 0-1 law for the unrestricted logic $\mathrm{FO}[Q^G_{\mathcal{H}}]$.

**Example 3.3** *Consider the sentence:*

$$\varphi \equiv \exists z(Ham\, x, y\, \psi), \text{ where } \psi \equiv Exz \wedge \neg Eyz$$

*The interpretation $\Psi(z)$ defined by $\psi$ maps a pair $(G, v)$ (where $G = (V, E)$ is a graph and $v$ a distinguished vertex of $G$) to the complete bipartite graph $H = (V, E')$, where $E' = \{(a, b) \mid (v, a) \in E$ and $(v, b) \notin E\}$. Letting $D(v)$ denote the set $\{a \in V \mid (v, a) \in E\}$, it can be verified that the graph $H$ has a Hamiltonian cycle if and only if $|D(v)| = |V - D(v)|$. In particular, if $G$ is a graph of odd size, then it cannot satisfy the sentence $\varphi$. On the other hand, $\varphi$ is true in a graph of cardinality $2n$ just in case the graph contains a vertex of degree $n$. But, as $n$ goes to infinity this happens almost surely (see [3], p. 57). We conclude that $\varphi$ does not have an asymptotic probability for $\mathcal{G}(1/2)$.*

The contrast between Theorem 3.2 and Example 3.3 shows that quantifier-free interpretations with parameters can be much more complex than those without parameters. We take up the analysis of the case with parameters in the next section.

## 4 Graph Interpretations with Parameters

In order to formulate a condition on the graph quantifiers $Q$ which guarantees that the logic FO$[Q]$ has a 0-1 law, we will construct an argument by quantifier elimination. That is, we will state sufficient conditions on $Q$ so that, for every quantifier free formula $\psi$, $Qx, y\,\psi$ is itself equivalent, almost everywhere, to a quantifier free formula. This, along with Theorem 2.3 will then enable us to derive the required result.

To establish this quantifier elimination, we consider the action of a quantifier-free interpretaion with parameters $\Psi(\bar{y})$ on a probability distribution that assigns probabilities to structures $(G, \bar{a})$. For this, we consider each atomic type of the tuple $\bar{a}$ separately. That is, for each atomic type $t$, we define a distribution $\mathcal{G}_{t,n}(p)$ which assigns a probability to each pair $(G, \bar{a})$, where $\bar{a}$ is a tuple of elements of $G$. This probability is 0 if $\bar{a}$ is not of type $t$ in $G$, and otherwise it is the same for all tuples of type $t$ in $G$.

More formally, let $t$ be an atomic type in the variables $\bar{z} = z_1, \ldots, z_m$ such that for each $1 \le i < j \le m$, $t \models z_i \neq z_j$. We denote by $\mathcal{G}_{t,n}(p)$ the probability space obtained from $\mathcal{G}_n(p)$ as follows: for each graph $G \in \mathcal{G}_n$, and each $m$-tuple $\bar{a}$ of elements in $G$, let the probability $\mu_{t,n}(G, \bar{a}) = 0$ if $G \not\models t[\bar{a}]$ and $\mu_{t,n}(G, \bar{a}) = \mu_n(G)/k$ otherwise, where $k$ is the number of distinct tuples in $G$ of type $t$. We write $\mathcal{G}_t(p)$ for any sequence of probability spaces $(\Gamma_n)_{n \in \mathbb{N}}$ such that $\Gamma_n$ is $\mathcal{G}_{t,n}(p)$ for $n \ge m$.

**Definition 4.1** *Let $\Psi$ be an interpretation in $m$ parameters and $t$ a type in $m$ variables, as above. A class of graphs $\mathcal{H}$* converges quickly *to 1 (resp. 0) for $\Psi\mathcal{G}_t(p)$ if $\mu_{t,n}(\mathcal{H}) = 1 - o(n^{-m})$ (resp. $o(n^{-m})$).*

This definition enables us to formulate the following lemma.

**Lemma 4.2** *If $\psi(x, y, \bar{z})$ is a first-order formula, $\Psi(\bar{z})$ is the associated graph interpretation, and $\mathcal{H}$ is a class of graphs which converges quickly to 1 for $\Psi\mathcal{G}_t(p)$, then the sentence $\forall\bar{z}(t(\bar{z}) \rightarrow (Q^G_{\mathcal{H}} x, y\, \psi))$ has asymptotic probability 1 for $\mathcal{G}(p)$. Similarly, if $\mathcal{H}$ converges quickly to 0 for $\Psi\mathcal{G}_t(p)$, then the sentence $\exists\bar{z}(t(\bar{z}) \wedge (Q^G_{\mathcal{H}} x, y\, \psi))$ has asymptotic probability 0 for $\mathcal{G}(p)$.*

Proof. *(Sketch)* We sketch the proof for the universal case, the existential case being dual.

The number of tuples of type $t$ in a graph $G$ of cardinality $n$ tends to $n^m/c$ for some constant $c$ as $n$ goes to infinity. Thus, if $\mathcal{H}$ converges quickly to 1 on $\Psi\mathcal{G}_t(p)$, then in almost all graphs, for all tuples $\bar{a}$ of type $t$, $\Psi(G, \bar{a}) \in \mathcal{H}$. Hence $\forall\bar{z}(t(\bar{z}) \rightarrow (Q^G_{\mathcal{H}} x, y\, \psi))$ has asymptotic probability 1 for $\mathcal{G}(p)$. ∎

Lemma 4.2 is used in proving the next result, which defines the conditions for one step of our quantifier elimination.

**Lemma 4.3** *If $\psi$ is a formula defining a graph interpretation $\Psi(\bar{z})$ in $m$ parameters, and $\mathcal{H}$ is a class of graphs which converges quickly to 0 or 1 for $\Psi\mathcal{G}_t(p)$, for every $m$-type $t$, then there is a quantifier free formula $\theta(\bar{z})$ such that the sentence $\forall\bar{z}(\theta \leftrightarrow Q^G_{\mathcal{H}} x, y\, \psi)$ has asymptotic probability 1 for $\mathcal{G}(p)$.*

Proof. *(Sketch)* Let

$$\theta \equiv \bigvee\{t(\bar{z}) : \mathcal{H} \text{ converges quickly to 1 for } \Psi\mathcal{G}_t(p)\}.$$

∎

We now proceed to study the structure of the spaces $\Psi\mathcal{G}_t(p)$. By Theorem 2.3, it suffices to consider the case where the interpretation $\Psi(\bar{z})$ is given by a quantifier free formula. For the remainder of this section, we will also confine ourselves to the case where $p$ is the constant function $1/2$.

Let $\psi(x, y, \bar{z})$ be a quantifier free formula defining a graph interpretation $\Psi(\bar{z})$ with $m$ parameters. By

Lemma 2.2, $\psi$ is equivalent to a disjunction of $(m+2)$-types. Let $S$ be the collection of the types in this disjunction that extend the $m$-type $t$. Clearly, $\Psi\mathcal{G}_t(1/2)$ is completely determined by which types are in $S$.

Furthermore, if $s$ is a type in the variables $x, y, \bar{z}$ extending $t(\bar{z})$, then $s$ is determined by its subtypes $s_1(x, \bar{z})$, $s_2(y, \bar{z})$ and whether or not $s \models Exy$. Moreover, in the case where either $s \models x = z_i$ or $s \models y = z_i$ for some $i$, the last of these is already determined by the two $(m+1)$-types $s_1$ and $s_2$. Thus, given two $(m+1)$-types $s_1$ and $s_2$ extending $t$, there may be one or two $(m+2)$-types consistent with $s_1$ and $s_2$. For our purposes, we can identify a set $S$ of $(m+2)$-types extending $t$ with a function $f$ that maps pairs of $(m+1)$-types extending $t$ into the set $\{0, 1, 1/2\}$. Thus, $f(s_1, s_2) = 0$ if there is no type in $S$ that extends $s_1$ and $s_2$; $f(s_1, s_2) = 1/2$ if there are two $(m+2)$-types that extend $s_1$ and $s_2$ and exactly one of them is in $S$; and $f(s_1, s_2) = 1$ if all the $(m+2)$-types that extend $s_1$ and $s_2$ (whether there are one or two of them) are in $S$.

Now, there are $m+2^m$ distinct types in the variables $x, \bar{z}$, extending $t$. These are obtained by taking $x = z_i$ for some $i$, yielding $m$ distinct types, and for the case when $x \neq z_i$ for all $i$, by taking the $2^m$ ways in which $x$ can be connected by edges to $z_1, \ldots, z_m$.

Thus, given a random graph $G$ and a tuple $\bar{a}$ such that $G \models t[\bar{a}]$, we can divide the vertices $b \in G$ into $m + 2^m$ sets according to the $(m+1)$-type of $(b, \bar{a})$. Of these sets, $m$ are singletons (containing the vertices that are in the tuple $\bar{a}$) and the rest of the vertices are distributed randomly among the other $2^m$ sets. The probability that a pair $(b_1, b_2)$ satisifies $G \models \psi[b_1, b_2, \bar{a}]$, and therefore that there is an edge $(b_1, b_2)$ in $\Psi(G, \bar{a})$, is then given by $f(s_1, s_2)$ where $s_i$ is the $(m+1)$-type of $(b_i, \bar{a})$. This discussion motivates the following definitions.

**Definition 4.4** *A pair $(m, f)$ is an* interpretive measure *for $\mathcal{G}(1/2)$ if and only if $m \in \mathbb{N}$ and there are disjoint sets $P = \{p_1, \ldots, p_m\}$ and $Q = \{q_1, \ldots, q_{2^m}\}$ such that $f$ is a function from $(P \cup Q)^2$ to $\{0, 1, 1/2\}$ subject to the following conditions: $f(x, y) = f(y, x)$ and if either $x \in P$ or $y \in P$, then $f(x, y) \in \{0, 1\}$.*

**Definition 4.5** *For any interpretive measure $(m, f)$, and any $n \geq m$, let $\mathcal{T}_n$ be the collection of all functions $T : \{0, \ldots, n-1\} \to (P \cup Q)$, for which there are $m$ distinguished points $0 \leq a_1, \ldots, a_m < n$ such that $T(x) = p_i$ if and only if $x = a_i$.*

*For each $T \in \mathcal{T}_n$, the probability space $\Gamma_T$ is obtained by determining for each pair of points $a, b \in \{0, \ldots, n-1\}$ whether there is an edge between them by means of independent Bernoulli trials with mean $f(T(a), T(b))$.*

*Finally, the probability space $\Gamma_n(m, f)$ is defined by assigning to each graph $G$ with $n$ vertices the probability $(\sum_{T \in \mathcal{T}_n} \mu_T(G))/card(\mathcal{T}_n)$, where $\mu_T(G)$ is the probability assigned to $G$ in the probability space $\Gamma_T$.*

*We write $\Gamma(m, f)$ for the sequence $(\Gamma_n(m, f))_{n \in \mathbb{N}}$, where for $n < m$, $\Gamma_n$ is chosen arbitrarily.*

The relevance of the above definition to $\Psi\mathcal{G}_t(1/2)$ emerges in the following lemma.

**Lemma 4.6** *If $\psi(x, y, \bar{z})$ is a quantifier-free first-order formula, $\Psi(\bar{z})$ is the associated graph interpretation, and $t$ is a type in the variables $\bar{z}$, then $\Psi\mathcal{G}_t(1/2)$ is $\Gamma(m, f)$ for some interpretive measure $(m, f)$.*

Proof. Let $\psi^*$ be the formula $x \neq y \wedge (\psi(x, y) \vee \psi(y, x))$, i.e. the formula that defines the interpretation $\Psi$. By Lemma 2.2, we can assume that $\psi^*$ is presented as a disjunction over a set $R$ of atomic types in the variables $x, y, \bar{z}$. Let $S$ be the set of all types $s$ in the variables $x, y, \bar{z}$ such that:

$s \models x \neq y$; and

$s \models t$, i.e. $s$ extends $t$.

Clearly, $\Psi\mathcal{G}_t(1/2)$ is completely determined by the set $R \cap S$.

We proceed to define the measure $(m, f)$. Let $m$ be the number of *distinct* parameters in $\bar{z}$, i.e. it is the cardinality of a maximal set $P = \{z_{i_1}, \ldots, z_{i_m}\}$ of variables from $\bar{z}$ such that $t \models z_{i_j} \neq z_{i_k}$. We will assume without loss of generality, by renaming variables if necessary, that $P$ consists of the variables $\{z_1, \ldots, z_m\}$. Let $Q = \{q_1, \ldots, q_{2^m}\}$ be the power set of $P$.

Intuitively, $P \cup Q$ represents the $m + 2^m$ sets of vertices as mentioned in the discussion preceding Definition 4.4. Therefore, each pair $(x, y) \in (P \cup Q)^2$ either uniquely determines a type $s \in S$ (if either $x$ or $y$ is in $P$), or it determines two types $s_0, s_1 \in S$. Thus, for each pair $(x, y) \in (P \cup Q)^2$, we will determine the value of the function $f$ based on whether or not the corresponding types are in $R$.

We formally define $f$ as follows:

1. $f(z, z) = 0$ for all $z \in P$;
2. For $z_i, z_j \in P, i < j$, let $s$ be the unique type in $S$ such that $s \models x = z_i \wedge y = z_j$. We let $f(z_i, z_j) = f(z_j, z_i) = 1$ if $s \models \psi$, and $f(z_i, z_j) = f(z_j, z_i) = 0$ otherwise.
3. For $z_i \in P$ and $q \in Q$, let $s$ be the unique type in $S$ satisfying:

$s \models x = z_i$;
$s \models y \neq z_j$, for $1 \leq j \leq m$;
$s \models Eyz_j$, for $z_j \in q$; and
$s \models \neg Eyz_j$, for $z_j \notin q$.

We let $f(z_i, q) = f(q, z_i) = 1$ if $s \models \psi$ and $f(z_i, q) = f(q, z_i) = 0$ otherwise.

4. for $q_i, q_j \in Q, i \leq j$, let $s_0$ and $s_1$ be the two types in $S$ satisfying:

$s_c \models x \neq z_k$, for $z_k \in P$ and $c = 0, 1$;
$s_c \models y \neq z_k$, for $z_k \in P$ and $c = 0, 1$;
$s_c \models Exz_k$, for $z_k \in q_i, c = 0, 1$;
$s_c \models \neg Exz_k$, for $z_k \notin q_i$ and $c = 0, 1$;
$s_c \models Eyz_k$, for $z_k \in q_j$ and $c = 0, 1$;
$s_c \models \neg Eyz_k$, for $z_k \notin q_j$ and $c = 0, 1$;
$s_0 \models Exy$; and
$s_1 \models \neg Exy$.

We let:
$$f(q_i, q_j) = f(q_j, q_i) = p,$$
where,
$$p = \begin{cases} 0 & \text{if } s_0 \not\models \psi \text{ and } s_1 \not\models \psi \\ 1 & \text{if } s_0 \models \psi \text{ and } s_1 \models \psi \\ 1/2 & \text{if } s_0 \models \psi \text{ and } s_1 \not\models \psi \\ 1/2 & \text{if } s_0 \not\models \psi \text{ and } s_1 \models \psi \end{cases}$$

It then follows from the discussion preceeding Definition 4.4 that $\Psi\mathcal{G}_t(1/2)$ is $\Gamma(m, f)$. ∎

Lemma 4.6 tells us the structure of the probability spaces on which $\mathcal{H}$ must converge quickly in order for us to be able to apply Lemma 4.3 to eliminate an occurrence of a quantifier. If this can be done for every $\Gamma(m, f)$, then starting with an arbitrary sentence $\varphi$ of FO$[Q^G_{\mathcal{H}}]$, by repeated application of this procedure, we can obtain a quantifier free sentence that is equivalent to $\varphi$ almost everywhere. This then yields the main theorem of this section:

**Theorem 4.7** *For any graph property $\mathcal{H}$, if $\mathcal{H}$ converges quickly to 0 or 1 for $\Gamma(m, f)$ for every interpretive measure $(m, f)$, then FO$[Q^G_{\mathcal{H}}]$ has a 0-1 law for $\mathcal{G}(1/2)$*

Proof. Let $\varphi$ be a sentence of FO$[Q^G_{\mathcal{H}}]$. We prove, by induction on the total number of quantifiers in $\varphi$, that $\varphi$ is equivalent almost everywhere to a quantifier free sentence, i.e. to True or False. This is trivially true when this number is 0. Let $\varphi$ contain $q+1$ quantifiers. There is a subformula $\chi$ of $\varphi$ which is either of the form $\exists x\psi$, or of the form $Q^G_{\mathcal{H}}x, y\psi$, where $\psi$ is quantifier free. In either case, $\chi$ is equivalent almost everywhere to a quantifier free formula $\theta$. In the first case this is true by Theorem 2.3 while in the second it follows from Lemma 4.3. Thus, by replacing $\chi$ by $\theta$ in $\varphi$, we obtain a sentence $\varphi'$ that is equivalent to $\varphi$ over a class with asymptotic probability 1, and that has only $q$ quantifiers. But then, by the induction hypothesis, $\varphi'$ is equivalent to a quantifier free sentence, on a class of asymptotic probability 1. Since the intersection of two classes that have asymptotic probability 1 must itself have asymptotic probability 1, we conclude that $\varphi$ is equivalent almost everywhere to a quantifier free sentence. ∎

We have assumed throughout this paper that we are working with purely relational signatures. It is well known that when we have constants in our signature, than even the 0-1 law for first order logic fails. However, one can still show that every sentence is equivalent almost everywhere to a quantifier free sentence (*cf.* Theorem 2.3). This extends also to the above Theorem 4.7. Thus, for any $\mathcal{H}$ that satisfies the hypotheses of the theorem, any sentence of FO$[Q^G_{\mathcal{H}}]$, perhaps including constants, is equivalent almost everywhere to a quantifier free sentence.

## 5 Rigidity

We now use the characterization provided by Theorem 4.7 to show that FO[Rig] has a 0-1 law, where Rig is the graph quantifier formed from the class of rigid graphs.

**Theorem 5.1** *For every interpretive measure $(m, f)$, the probability that a graph is rigid converges exponentially fast to either 0 or 1 for $\Gamma(m, f)$.*

Proof. We distinguish three cases for interpretive measures $(m, f)$. Recall that $f : P \cup Q \to \{0, 1, 1/2\}$.

*(i)* There exists a non-trivial permutation $\pi$ on $P$ such that $f(p, p') = f(\pi p, \pi p')$ for all $p, p' \in P$, and $f(p, q) = f(\pi p, q)$ for all $p \in P$, $q \in Q$.

*(ii)* There exists a $q \in Q$ such that $f(q, q') \in \{0, 1\}$ for all $q' \in Q$.

*(iii)* All other cases.

In case *(i)* the permutation $\pi$ defines a non-trivial automorphism on all $G \in \Gamma_n(m, f)$. In case *(ii)* we have a non-trivial automorphism for $G \in \Gamma_n(m, f)$ provided $G$ contains at least two nodes in the class defined by $q$; this holds with probability tending to 1 exponentially fast. We prove that in all other cases, the graphs $G \in \Gamma_n(m, f)$ are almost surely rigid.

The random process of constructing $G \in \Gamma_n(m, f)$ can be split into two subprocesses. In the first stage, the nodes from $[n] = \{0, \ldots, n-1\}$ are distributed over the $m + 2^m$ classes $P \cup Q$. In the second stage, edges are determined according to the probabilities given by $f$.

Recall that the first subprocess randomly selects $m$ points to form the singleton sets $p \in P$, and then distributes the remaining $n - m$ nodes over the sets $q \in Q$. For every $q \in Q$, the probability that $q$ gets precisely $k$ points is described by a binomial distribution $b(k; n-m, 2^{-m})$, where $b(k; n, p)$ is the usual abbreviation for

$$\binom{n}{k} p^k (1-p)^{n-k}.$$

Obviously, the expected number of elements in every class $q$ is $2^{-m}(n-m)$. More precisely, basic facts on binomial distributions (see e.g. [3, pp. 10-14]) imply that for every $\delta > 0$, the probability that some class $q$ contains less than $(1-\delta)2^{-m}(n-m)$ or more than $(1+\delta)2^{-m}(n-m)$ elements, is bounded by $2^{-\varepsilon n}$ for some $\varepsilon > 0$.

It is convenient to exclude from further consideration those rare events, where the nodes are not 'evenly' distributed over $Q$. Fix a constant $d > 0$, and let $\Gamma_n^d(m, f)$ be a new probability space, obtained from $\Gamma_n(m, f)$ by throwing away all those graphs where the first stage of the construction produces any class $q \in Q$ with less than $(1-d)2^{-m}n$ elements. Since these graphs form a set whose measure goes to 0 exponentially fast, it suffices to prove our result for the sequence of probability spaces $\Gamma_n^d(m, f)$.

Let $X(G)$ be the number of non-trivial automorphisms of $G$. We will prove that the expectation $E(X)$ on $\Gamma_n^d(m, f)$ tends exponentially fast to 0 as $n$ goes to infinity. Since, by Markov's inequality, $P[X \geq 1] \leq E(X)$, this immediately implies the desired result. For $\pi \in S_n$, let $X_\pi$ be the indicator random variable, defined by

$$X_\pi(G) = \begin{cases} 1 & \text{if } \pi \in \mathrm{Aut}(G) \\ 0 & \text{otherwise.} \end{cases}$$

By linearity of expectation we have that

$$E(X) = \sum_{\pi \in S_n - \{id\}} E(X_\pi).$$

The *support* of a permutation $\pi$, denoted $supp(\pi)$ is the set of points moved by $\pi$. Let $h = |supp(\pi)|$ and $T_{n,h} = \{\pi \in S_n : |supp(\pi)| = h\}$.

It is sufficient to prove the following claim.

**Claim.** *There exists a $\delta > 0$ such that*

$$E(X_\pi) \leq 2^{-\delta h n}$$

*for all $h$ and all $\pi \in T_{n,h}$.*

Indeed, the claim implies that

$$\begin{aligned} E(X) &\leq \sum_{h=1}^{n} |T_{n,h}| 2^{-\delta h n} \\ &\leq \sum_{h=1}^{n} 2^{h \log n} 2^{-\delta h n} \leq 2^{-\varepsilon n} \end{aligned}$$

for some $\varepsilon > 0$.

We first prove a bound on $E(X_\pi)$ that holds for arbitrary size of the support.

**Lemma 5.2** *If $\pi \neq id$, then $E(X_\pi) \leq 2^{-\varepsilon n}$ for some $\varepsilon > 0$ and sufficiently large $n$.*

PROOF. $\pi$ moves at least one point, say $\pi(i) = j$. Assume that the first subprocess produces $p = \{i\}$ and $p' = \{j\}$ for $p, p' \in P$. Then, since condition *(i)* does not hold, there exists a class $q \in Q$ such that $f(p, q) \neq f(p', q)$. Thus, to be an automorphism of $G$, $\pi$ has have to move the whole class $q$. But this means that $\pi$ must preserve $\Omega(n^2)$ non-trivial edge events.

Otherwise, at least one of the nodes $i$ and $j$ is put into a class $q \in Q$. But then, there exists an entire class $q' \in Q$ such that the edge-probabilities from this node to $q'$ are $1/2$. Since $q'$ has $\Omega(n)$ elements, the result follows. ∎

Note that Lemma 5.2 proves the claim for $h < k$ where $k$ is fixed (independent of $n$).

Before we prove the claim for permutations that move more points, we make some general observations that hold for arbitrary probability spaces of graphs.

Let $\pi \in S_n$ and $K$ be the set of potential edges, i.e. the set of unordered pairs of elements of $[n]$. We call $R \subseteq K$ a *witness set* for $\pi$ if $K - R$ intersects every orbit of the operation of $\pi$ on $K$; in other words, for every pair $(i, j) \in R$ there exists $k \in \mathbb{N}$ such that $(\pi^k(i), \pi^k(j)) \notin R$. If $R$ is a witness set for $\pi \in \mathrm{Aut}(G)$ and we fix the edges and non-edges of $G$ outside of $R$, then those inside $R$ are determined as well.

The following is a possibile way to construct witness sets: Let $B, C \subseteq supp(\pi)$ such that $B \cap C = \emptyset$ and $C$ contains, for every $b \in B$, precisely one element of the orbit of $b$ under $\pi$. Further, let $D = [n] - (B \cup C)$. Then $B \times D$ is a witness set.

Thus, given a permutation $\pi \in S_n$ we can establish an upper bound for $E(X_\pi)$ as follows: We choose suitable sets $B, C$ and prove that the first stage of the construction of a random graph must assign edge probability $1/2$ to at least $r$ pairs in the associated witness set $B \times D$. Then $E(X_\pi) \leq 2^{-r}$.

**Lemma 5.3** *Let $c < (1-d)2^{-m}$ and let $2m < h \leq cn$. Then there exists an $\varepsilon > 0$ such that $E(X_\pi) \leq 2^{-\varepsilon(h/2-m)n}$ for $\pi \in T_{n,h}$.*

PROOF. Let $C \subset supp(\pi)$ be any set obtained by picking precisely one element out of every nontrivial cycle of $\pi$, and let $B = supp(\pi) - C$. Thus, $D = [n] - (B \cup C)$ coincides with the set of fixed points of $\pi$ and therefore contains at least $(1-c)n$ elements. $B$ contains at least $h/2$ nodes, since the support of $\pi$ is decomposed into cycles of length $\geq 2$ and $C$ contains only one element of each cycle. Thus, at least $h/2 - m$ of the nodes of $B$ are put into some $q \in Q$ so that each of these has nontrivial edge-probabilities to at least one entire class $q' \in Q$. Since $|q'| \geq (1-d)2^{-m}n$, it follows that $|D \cap q'| \geq \varepsilon n$ where $\varepsilon = (1-d)2^{-m} - c$. Thus $B \times D$ contains at least $\varepsilon(h/2 - m)n$ pairs with edge probability $1/2$. Thus the probability that a random graph $G \in \Gamma_n^d(m, f)$ is fixed by $\pi$ is bounded by $2^{-\varepsilon(h/2-m)n}$. ∎

**Lemma 5.4** *For the same constant $c$ as in the previous lemma and $h > cn$, there exists a $\delta > 0$ such that $E(X_\pi) \leq 2^{-\delta n^2}$.*

PROOF. Let $B, C$ be disjoint subsets of $supp(\pi)$ such that $|B| = cn/2$ and $C$ contains precisely one element of each cycle of $\pi$ that intersects with $B$. Again $D = [n] - (B \cup C)$ has at least $(1-c)n$ elements. With precisely the same reasoning as in the previous lemma, we infer that $B \times D$ contains at least $(cn/2 - m)\varepsilon n$ pairs with edge probability $1/2$. By choosing $\delta > 0$ such that $\delta n^2 \leq (cn/2 - m)\varepsilon n$, the result follows. ∎

Together, the three lemmata prove the claim, and therefore the theorem. ∎

Theorem 5.1, together with Theorem 4.7 yields:

**Corollary 5.5** *FO[Rig] has a 0-1 law on $\mathcal{G}(1/2)$.*

## 6 Regular Logics

In model theory, the notion of a regular logic has been introduced, in order to make precise ideas of what constitutes a natural extension of first-order logic. A regular logic can be described as a logic that can express all atoms, and is closed under negation, conjunction, particularization (or existential quantification), relativization and substitution. We refer to [6] for precise definitions.

**Proposition 6.1** *For any class $K$ of $\sigma$-structures, FO$[Q_K]$ is the minimal logic closed under negation, conjunction, particularization and substitution that can express $K$.*

Note that FO$[Q_K]$ is not necessarily regular since it need not be closed under relativization. Nevertheless, on the negative side, we get the following consequence of this proposition and Example 3.3.

**Theorem 6.2** *There is no regular logic that can express Hamiltonicity and has a 0-1 law for $\mathcal{G}(1/2)$.*

In order to show that there is a regular logic that can express rigidity on graphs and has a 0-1 law, we need to consider the closure of FO[Rig] under relativization. This can be obtained by considering a relativized version of the rigidity quantifier, denoted Rig$'$, which binds two formulae $\delta(x, \bar{z})$ and $\varphi(x, y, \bar{z})$. Let $\psi = (x \neq y) \wedge (\varphi(x, y, \bar{z}) \vee \varphi(y, x, \bar{z}))$ be the irreflexive and symmetric formula associated with $\varphi$. Then the meaning of a formula Rig$'$ $x, y(\delta, \varphi) \in$ FO[Rig$'$] in a structure $\mathfrak{A}$ with valuation $\bar{b}$ for $\bar{z}$, is that the graph $(\delta^{\mathfrak{A},\bar{b}}, \psi^{\mathfrak{A},\bar{b}})$ is rigid.

A simple modification of the proof of Theorem 5.1 gives the following result.

**Theorem 6.3** *FO[Rig$'$] is a regular logic that has a 0-1 law for $\mathcal{G}(1/2)$.*

## 7 Vectorized Quantifiers

In this section, we consider extensions of first-order logic formed by adding vectorized quantifiers. A single Lindström quantifier can be seen as giving rise to an infinite sequence of quantifiers formed by vectorization. This allows us to consider interpretations that are not bound by the universe of a given structure and can map it to potentially larger structures. Vectorized interpretations and quantifiers capture a natural notion of logical reduction. For a discussion of this and its significance for descriptive complexity, see [5].

We begin with some definitions. Let $\tau = \{R_1, \ldots, R_m\}$ be a signature where $R_i$ has arity $r_i$. A vectorized interpretation of $\tau$ in $\sigma$ of width $k$ is given by a sequence of $\sigma$-formulas, $\psi_1(\bar{x}_1, \bar{y}), \ldots, \psi_m(\bar{x}_m, \bar{y})$, where the length of $\bar{x}_i$ is $k \cdot r_i$. The variables in $\bar{y}$ are parameters. The interpretation maps a $\sigma$-structure $\mathfrak{A}$ along with an interpretation $\bar{a}$ of the parameters in $\mathfrak{A}$ to a $\tau$-structure $\mathfrak{B}$, whose universe is $A^k$, with the relation $R_i^{\mathfrak{B}}$ given by $\psi_i^{\mathfrak{A},\bar{a}}$.

For any graph quantifier $Q_{\mathcal{H}}$, we define its $k$th vectorization $Q^k_{\mathcal{H}}$ as a quantifier that binds $2k$ variables and whose semantics is given by the following rule: if $\psi(\bar{x}, \bar{y})$ defines a vectorized interpretation $\Psi(\bar{y})$ of width $k$, then $(G, \bar{a}) \models Q^k_{\mathcal{H}} \bar{x} \psi$ if and only if $\Psi(G, \bar{a}) \in \mathcal{H}$. We define FO$[Q^{\star}_{\mathcal{H}}]$ to be the extension of first-order logic by the infinite sequence of quantifiers $\{Q^k_{\mathcal{H}} | k \in \mathbb{N}\}$.

Let $\Phi$ be a vectorized interpretation of width $k$ given by a quantifier free formula, with $m$ parameters. Let $G$ and $H$ be graphs and $\bar{a}$, $\bar{b}$ be $m$ tuples of vertices from $G$ and $H$ respectively, such that there is an isomorphic embedding $f : H \rightarrow G$ with $f(\bar{b}) = \bar{a}$. Let $\Phi f$ denote the map from $\Phi(H, \bar{b})$ to $\Phi(G, \bar{a})$ given by the natural extension of $f$ to $k$ tuples. The following lemma is based on the observation that quantifier free formulas are preserved under isomorphic embeddings.

**Lemma 7.1** *$\Phi f$ is an isomorphic embedding of $\Phi(H, \bar{b})$ in $\Phi(G, \bar{a})$.*

PROOF. If $\bar{h}_1$ and $\bar{h}_2$ are two $k$-tuples in $H$, then whether or not there is an edge between them in $(\Phi H)(\bar{b})$ is determined by the quantifier free formulas in $\Phi$. However, quantifier free formulas are clearly preserved under the isomorphic embedding $f$, and therefore $\Phi f(\bar{h}_1)$ and $\Phi f(\bar{h}_2)$ have an edge if and only if $\bar{h}_1$ and $\bar{h}_2$ do. ∎

Let $H$ be any fixed graph, $\bar{b}$ an $m$-tuple of vertices in $H$ and $t$ the atomic type of $\bar{b}$ in $H$. Recall that $\mathcal{G}_{t,n}(p)$ is a probability space on structures $(G, \bar{a})$, for graphs $G$ of cardinality $n$ and $m$-tuples $\bar{a}$ of vertices of $G$, such that the probability $\mu_{t,n}(G, \bar{a})$ is non zero only if $\bar{a}$ has type $t$ in $G$. Let $F_{(H,\bar{b})}$ denote those structures $(G, \bar{a})$ for which there is an isomorphic embedding $f : (H, \bar{b}) \rightarrow (G, \bar{a})$.

**Lemma 7.2** *For any graph $H$, and any $m$-tuple $\bar{b}$ of vertices of $H$,*

$$\mu_{t,n}(F_{(H,\bar{b})}) = 1 - o(n^{-m}).$$

PROOF. The proof is immediate from the fact that the probability of each of the extension axioms converges exponentially quickly to 1 [7]. ∎

Let $\mathcal{H}$ be a collection of graphs that is closed under taking substructures. The following lemma, which is analogous to to Lemma 4.3, is derived from Lemmas 7.1 and 7.2.

**Lemma 7.3** *For any quantifier free formula $\psi$ defining a vectorized interpretation $\Psi(\bar{y})$ of width $k$, with parameters $\bar{y}$, there is a quantifier free formula $\theta$ such that the sentence $\forall \bar{y}(\theta \leftrightarrow Q^k_{\mathcal{H}} \bar{x}\, \psi)$ has asymptotic probability 1 for $\mathcal{G}(p)$, for any constant $p$.*

PROOF. We show that for any $m$-type $t$, either there is no pair $(H, \bar{b})$, such that $\bar{b}$ has type $t$ in $H$, and $(H, \bar{b}) \models Q^k_{\mathcal{H}} \bar{x}\, \psi$, and therefore $\mathcal{H}$ converges quickly to 0 for $\mathcal{G}_t(p)$; or $\mathcal{H}$ converges quickly to 1 for $\mathcal{G}_t(p)$. It then follows that we can take $\theta$ to be the disjunction of types $t$ such that there is such a pair $(H, \bar{b})$.

Suppose now, that for a given $t$, there is a graph $H$ and a tuple $\bar{b}$ of type $t$ in $H$ such that $(H, \bar{b}) \models Q^k_{\mathcal{H}} \bar{x}\, \psi$. It then follows by Lemma 7.1 that for any $(G, \bar{a}) \in F_{(H,\bar{b})}$, $(G, \bar{a}) \models Q^k_{\mathcal{H}} \bar{x}\, \psi$. In other words, for every $(G, \bar{a}) \in F_{(H,\bar{b})}$, $\Psi(G, \bar{a}) \in \mathcal{H}$. Therefore, by Lemma 7.2 $\mathcal{H}$ converges quickly to 1 for $\mathcal{G}_t(p)$. ∎

This enables us to prove the following theorem, by an elimination of quantifiers along the lines of Theorem 4.7.

**Theorem 7.4** *For any class of graphs $\mathcal{H}$ closed under taking substructures, the logic FO$[Q^{\star}_{\mathcal{H}}]$ has a 0-1 law for $\mathcal{G}(p)$, for any constant $p$.*

Observe that, by duality, the argument outlined above also works for classes of graphs that are closed under extensions rather than substructures. This should be compared with a result in [8] which shows that the logic FO$^r[Q_{\mathcal{H}}]$ has a 0-1 law if $\mathcal{H}$ is monotone and closed under extensions. We have weakened the hypothesis by dropping the requirement of monotonicity and greatly strengthened the theorem by allowing both vectorization and nesting of quantifiers.

Writing 3-col for the graph quantifier defined by the class of 3-colourable graphs, and Plan for the graph quantifier corresponding to the class of planar graphs, the following two corollaries of Theorem 7.4 are immediate.

**Corollary 7.5** *FO[3-col$^{\star}$] has a 0-1 law.*

**Corollary 7.6** *FO[Plan$^{\star}$] has a 0-1 law.*

Moreover, these results are easily extended to the closure of these logics under relativizations. Neither 3-colourability nor planarity has previously been shown to be expressible in a regular logic that is closed under vectorization and has a 0-1 law. Corollary 7.5 also answers a question posed by Iain Stewart.

## References


[1] A. Blass, Y. Gurevich, and D. Kozen, *A zero-one law for logic with a fixed point operator,* Information and Control 67 (1985), 70–90.

[2] A. Blass and F. Harary, *Properties of almost all graphs and complexes,* Journal of Graph Theory 3 (1979), 225–240.

[3] B. Bollobás, *Random Graphs*, Academic Press, 1985.

[4] K.J. Compton, *0-1 laws in logic and combinatorics*, in: I. Rival (Ed.), NATO Advanced Study Institute on Algorithms and Order, pages 353–383, Kluwer, 1989.

[5] A. Dawar, *Generalized quantifiers and logical reducibilities,* to appear in Journal of Logic and Computation.

[6] H.-D. Ebbinghaus, *Extended logics: The general framework*, in: J. Barwise and S. Feferman (Eds.), Model-Theoretic Logics, pages 25–76, Springer-Verlag, New York, 1985.

[7] R. Fagin, *Probabilities on finite models,* Journal of Symbolic Logic 41 (1976), 50–58.

[8] G. Fayolle, S. Grumbach, and C. Tollu, *Asymptotic probabilities of languages with generalized quantifiers,* Proc. 8th IEEE Symp. on Logic in Computer Science (1993).

[9] Y. V. Gelbskiĭ, D. I. Kogan, M.I. Ligon'kiĭ, and V. A. Talanov, *Range and degree of realizability of formulas in the restricted predicate calculus,* Kibernetika, 2 (1969), 17–28.

[10] E. Grandjean, *Complexity of the first-order theory of almost all structures,* Information and Control, 52 (1983), 180–204.

[11] Y. Gurevich, *Zero-One Laws*, in: The Logic in Computer Science column, EATCS Bulletin 46 (1992), 90–106.

[12] Ph. G. Kolaitis and M. Y. Vardi, *0-1 laws and decision problems for fragments of second-order logig,* Information and Computation, 87 (1990), 302–338.

[13] Ph. G. Kolaitis and M. Y. Vardi, *Infinitary logics and 0-1 laws,* Information and Computation, 98 (1992), 258–294.

[14] P. Lindström, *First order predicate logic with generalized quantifiers,* Theoria, 32 (1966), 186–195.

[15] V. Talanov, *Asymptotic Solvability of Logical Formulae*, in: Combinatorial-Algebraic Methods in Applied Mathematics, Gorky University (1981), 118–126 (in Russian). Math. Reviews 85i:03081.

[16] V. Talanov and V. Knyazev, *The Asymptotic Truth Value of Infinite Formulae*, Proc. of the All-Union Seminar of Discrete Math. and its Applications, Moscow (1986), 55–61 (in Russian). Math. Reviews 89g:03054.