On Asymptotic Probabilities in Logics That Capture DSPACE($\log n$) in Presence of Ordering*

Jerzy Tyszkiewicz

Institute of Informatics, University of Warsaw,
ul. Banacha 2, 02-097 Warszawa, Poland.
jurekty@mimuw.edu.pl

Abstract. We show that for logics that capture DSPACE($\log n$) over ordered structures, and for recursive probability distributions on the class of finite models of the signature, the 0–1 law and the convergence law hold if and only if certain boundedness conditions are satisfied. As one of the applications, we consider the conjecture of Kolaitis and Vardi, stating that for arbitrary probability distributions the 0–1 law holds for the logic $L^{\omega}_{\omega_1\omega}$ iff the same law holds for fixpoint logic.

1 Introduction

1.1 About the theory of asymptotic probabilities

The problems considered in this paper belong to the research area called *random structure theory,* and, more specifically, to its logical aspect.

To explain (very imprecisely and incompletely) what it means, let us consider a class of some structures (say: finite ones over some fixed signature), equipped with a probability space structure (this probability is usually assumed to be only *finitely* additive). Then we draw one structure at random and ask:

- how does the drawn structure look like?
- does the drawn structure have some particular property?

Those questions are typical in random structure theory. To turn to the logical part of it, look at the drawn structure through the logical glasses: we can only notice properties definable in some particular logic. Then new questions become natural:

- does every property we can observe have a probability (is it measurable)?
- if so, what is this probability equal to?
- can we compute this probability, and, eventually, how difficult is it?

It becomes clear from the above that the random structure theory is closely connected to combinatorics, finite model theory, mathematical logic, and, last but not least, computer science. An exposition of the logical part of the random structure theory may be found in a nice survey of Compton [1].

Generally, it seems that the emergence of the theories of asymptotic probabilities and of descriptive complexity reflect the same trend that was observed in continuous

* Research partially supported by KBN grant GR–71.

mathematics. Namely, in order to better understand the nature of the set of reals, mathematicians equipped it with various structures (such as the structure of probability space, topological space, field, Banach algebra, etc.), and studied the resulting object.

To better understand the expressive power of a logic over finite structures, logicians equip the family of sets of finite models defined by sentences of the logic under consideration with various structures.

One of possible choices is to add the successor relation to the signature, and then every structure can be treated as a word, so that the mentioned above family becomes a family of languages. The great discovery of the theory of descriptive complexity is that for most of natural logics the resulting families of languages naturally correspond to well known complexity classes.

The other choice is to add the structure of a (asymptotic) probability space. The resulting theory of asymptotic probabilities allows to make some other nontrivial observations. One of them is, e.g., connection between decidability of finite satisfiability problem for a prefix class $\mathcal{F}$ of first order formulas and uniform, labelled 0–1 law for the collection of existential second order formulas with first order part in $\mathcal{F}$: the class $\Sigma^1_1(\mathcal{F})$ (details can be found in the paper [6] of Kolaitis and Vardi). Some other examples are presented in Gurevich's article [3].

1.2 About the paper

In the current paper we are interested in the problem settled by Kolaitis and Vardi, in the form of the following conjecture:

Conjecture 1 (Kolaitis and Vardi [8]). *Let $\boldsymbol{\mathcal{A}}$ be any class of finite structures and let μ_n, $n \geq 1$ be any sequence of probability measures on the structures of $\boldsymbol{\mathcal{A}}$ with n elements. Then the 0–1 law holds for the infinitary logic $L^{\omega}_{\omega_1\omega}$ on $\boldsymbol{\mathcal{A}}$ relative to the measures μ_n if and only if the 0–1 law holds for fixpoint logic on $\boldsymbol{\mathcal{A}}$ relative to the measures μ_n.*

We show that the above conjecture is true when restricted to measures whose values can be, roughly speaking, approximated in recursive way with arbitrary precision, while in general it fails.

However, our method, used to deal with the conjecture, is much more general and applies to many logics other than fixpoint logic. Some of its consequences are discussed in the paper.

We keep the paper on rather abstract level. In particular, we do not give complete definitions of the logics we consider, but use results about them. The reader, who is not familiar with them, will probably have to consult some textbooks. There are also no examples – we decided to use the limited space for more theorems and more details in the proofs, instead.

2 Formal definitions

Throughout the paper we assume that we are dealing with logics over some fixed, finite signature σ (with equality). It is assumed to contain only relation and constant symbols, and therefore eventual functions are represented as restricted relations. Let $\boldsymbol{\mathcal{A}}$ be the set of *all* finite structures over signature σ, whose carrier set is some initial segment of natural numbers. Let $\boldsymbol{\mathcal{A}}(n)$ be the set of all structures $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ with carrier set (of cardinality) $|\mathbf{A}| = n = \{0, \ldots, n-1\}$.

Let there be a probability distribution μ_n on each $\boldsymbol{\mathcal{A}}(n)$. Writing $\boldsymbol{\mu}$ for $\{\mu_n\}_{n \in \mathbb{N}}$ we may consider $\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mu} \rangle$ as *randomized set of finite structures,* and make it an object of our study. Since $\boldsymbol{\mathcal{A}}$ is fixed, $\boldsymbol{\mu}$ itself determines this randomized set, and therefore in the sequel we deal with distributions only. (In the literature it is usually adopted that $\boldsymbol{\mathcal{A}}$ is the set of those structures $\mathbf{A}$ for which $\mu_{|\mathbf{A}|}(\{\mathbf{A}\}) > 0$. Of course our approach does not cause any loss of generality.)

Then for any subset $D \subseteq \boldsymbol{\mathcal{A}}$ we define

$$\mu_n(D) = \mu_n(D \cap \boldsymbol{\mathcal{A}}(n)).$$

We often consider D being the set of those structures in $\boldsymbol{\mathcal{A}}$ that satisfy some sentence φ in some logic over σ, and then we write $\mu_n(\varphi)$, instead of $\mu_n(D)$. We are interested in asymptotic properties of $\mu_n(\varphi)$, and especially whether the limit $\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$ exists, for φ being a sentence of the logic under consideration. If it exists, we call it an *asymptotic probability of* φ. If this is the case for every sentence of the logic L, we say that the *convergence law* holds (for L and $\boldsymbol{\mu}$). If, in addition, every sentence has probability either 0 or 1, we say that the *0–1 law* holds.

We fix some method of coding structures in $\boldsymbol{\mathcal{A}}$ as natural numbers. The coding bijection $\boldsymbol{\mathcal{A}} \to \mathbb{N}$ we denote by code, while its converse $\mathbb{N} \to \boldsymbol{\mathcal{A}}$ by struct. They will be used to speak about recursive sets of elements of $\boldsymbol{\mathcal{A}}$, first structure in a $D \subseteq \boldsymbol{\mathcal{A}}$, etc. The particular choice of coding function is immaterial, except that we require it to satisfy: the function $a \mapsto |\mathrm{struct}(a)|$ is recursive, and for a relation symbol R of arity k from σ and $k-$tuple $\mathbf{b} \in |\mathrm{struct}(a)|^k$, it is decidable whether $\mathrm{struct}(a) \models R(\mathbf{b})$.

The *representing relation* of a distribution $\boldsymbol{\mu}$ is then a ternary relation $m_{\boldsymbol{\mu}}$ on natural numbers, defined as follows:

$$m_{\boldsymbol{\mu}}(a, b, c) \qquad \text{iff} \qquad \mu_{|\mathrm{struct}(a)|}(\{\mathrm{struct}(a)\}) \leq b/c.$$

A more intuitive, but less formal, is to think about $m_{\boldsymbol{\mu}}$ to be the relation

$$\mu_{|\mathbf{A}|}(\{\mathbf{A}\}) \leq q$$

included in $\boldsymbol{\mathcal{A}} \times \mathbb{Q}$. Now we have an uniform method of representing distributions, which is independent of representation of the values of $\boldsymbol{\mu}$ (which may be e.g. irrational), so we can speak about *complexity* of the distribution.

There is, however, one thing to consider before. Namely, there may be some very simple (of very low complexity) distribution, which can be converted into one of very high complexity by a perturbation that vanishes as $n \to \infty$. But, as we are interested in asymptotical properties of the resulting distribution only, it should be still regarded as simple.

The formalisation of this idea is as follows:

Definition 2. We call two distributions $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ *asymptotically equivalent* if and only if for every $D \subseteq \boldsymbol{\mathcal{A}}$

$$\lim_{n\to\infty} \mu_n(D) - \mu'_n(D) = 0.$$

Now we define the *arithmetical complexity* of a distribution $\boldsymbol{\mu}$: we say that the distribution $\boldsymbol{\mu}$ is Δ_k (Σ_k, Π_k, resp.), $k > 0$, iff there exists distribution $\boldsymbol{\mu}'$, asymptotically equivalent to $\boldsymbol{\mu}$, and such that its representing relation $m_{\boldsymbol{\mu}'}$ is Δ_k (Σ_k, Π_k, resp.). □

We are particularly interested in Δ_1 distributions, whose values can be approximated with arbitrary precision in recursive way.

First we prove a simple fact, stating that essentially the arithmetical hierarchy collapses to the Δ hierarchy for distributions:

Proposition 3. *If a distribution $\boldsymbol{\mu}$ is Σ_k (Π_k, resp.), then it is Δ_k, as well. Moreover, there exists a distribution $\boldsymbol{\mu}'$, asymptotically equivalent to $\boldsymbol{\mu}$, such that the function $\mathbf{A} \mapsto \mu'_{|\mathbf{A}|}(\{\mathbf{A}\})$ is rational valued and Δ_k.*

Proof. We consider only the case of $k = 1$ and Σ_1. The converse implication can be obtained by symmetric argumentation, while for $k > 1$ it suffices to replace Turing machines, appearing in the proof below, by Turing machines with suitable oracle.

Suppose that the distribution $\boldsymbol{\mu}$ is Σ_1. We construct a distribution $\boldsymbol{\mu}'$, which is asymptotically equivalent to $\boldsymbol{\mu}$, rational valued and, moreover, the function $\mathbf{A} \mapsto \mu'_{|\mathbf{A}|}(\{\mathbf{A}\})$ is recursive.

Let M be the Turing machine that enumerates all true inequalities of the form $\mu_{|\mathbf{A}|}(\{\mathbf{A}\}) \le q$, $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, $q \in \mathbb{Q}$.

We let the machine computing function $\mathbf{A} \mapsto \mu'_{|\mathbf{A}|}(\{\mathbf{A}\})$, on the input $\mathbf{A} \in \boldsymbol{\mathcal{A}}(n)$, to simulate the behaviour of M and store all the inequalities it generates until the first moment there is a subset of them

$$\{\mu_n(\{\mathbf{B}\}) \le q_{\mathbf{B}} \mid \mathbf{B} \in \boldsymbol{\mathcal{A}}(n)\}$$

with the property

$$\sum_{\mathbf{B}\in\mathcal{A}(n)} q_{\mathbf{B}} \ge 1 - 1/n.$$

At that moment it outputs the value

$$\mu'_n(\{\mathbf{A}\}) = \begin{cases} 1 - \sum_{\mathbf{B}\in\mathcal{A}(n)\setminus\{\mathbf{A}\}} q_{\mathbf{B}} & \text{if } \mathbf{A} \text{ is the first element of } \boldsymbol{\mathcal{A}}(n), \\ q_{\mathbf{A}} & \text{otherwise.} \end{cases}$$

The machine always halts since the equality $\sum_{\mathbf{B}\in\mathcal{A}(n)} \mu_n(\{\mathbf{B}\}) = 1$ holds. The distribution $\boldsymbol{\mu}'$ is clearly asymptotically equivalent to $\boldsymbol{\mu}$, as for every $D \subseteq \boldsymbol{\mathcal{A}}$ we have $|\mu_n(D) - \mu'_n(D)| \le 1/n$. □

It is quite natural to call Δ_1 distributions *recursive,* and we do so in the sequel.

3 The main theorem

The distribution $\boldsymbol{\mu}$ is assumed to be fixed through the exposition in this paragraph, and therefore all the notions we introduce should be understood to refer to the fixed pair $\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mu} \rangle$.

Let us adopt that a recursive function $h : \mathbb{N} \longrightarrow \mathbb{N}$ is called *space constructible* iff there exists an on-line Turing Machine M which computes h, and for some constant c_h and all $n \in \mathbb{N}$, machine M uses at most $c_h \cdot \mathrm{length}(h(n))$ tape cells during computation with input n (input and output are represented as binary strings). Note that our definition is slightly more liberal than the usual one. The name is chosen to indicate that all space constructible (in standard sense) functions satisfy our definition.

Lemma 4. *If $h : \mathbb{N} \longrightarrow \mathbb{N}$ is strictly growing and space constructible, then the binary relation $(k \leq h(m))$, of arguments k, m being natural numbers written in unary expansion, is in* DSPACE$(\log k)$.

Proof. Let the machine recognizing our relation first check whether $k \leq m$, given input (k, m). (It can be done without reading entire input, when $k < m$!) If this is the case, it decides that $k \leq h(m)$ (from strict monotonicity of h it follows that $k \leq m \leq h(m)$). Otherwise it first converts m into binary, and then tries to compute the value of $h(m)$ in $c_h \lceil \log k \rceil$ space cells. If it succeeds and the result is less than k, then $k \not\leq h(m)$; if either it does not succeed (due to lack of space), or it succeeds and the result is $\geq k$, then it decides that $k \leq h(m)$. The computation requires $c_h \lceil \log k \rceil + \lceil \log k \rceil + O(1)$ space cells, and so the relation is in DSPACE$(\log k)$. □

Now we introduce one more definition, being the crucial definition in our paper:

Definition 5. Let L be a logic, and let $Int(\mathbf{x})$, $Eq(\mathbf{x}, \mathbf{y})$, $Succ(\mathbf{x}, \mathbf{y})$ and $Edge(\mathbf{x}, \mathbf{y})$ be L–formulas with ℓ, 2ℓ, 2ℓ and 2ℓ free variables, respectively. Let $\mathcal{J}$ be the quadruple $\langle Int, Eq, Succ, Edge \rangle$, $D \subseteq \boldsymbol{\mathcal{A}}$ be a class of finite structures, and let C be a complexity class. We say that L is *C–expressive in D for $\mathcal{J}$* if and only if:

1. In every structure $\mathbf{A} \in D$ the interpretation $Eq^{\mathbf{A}}$ of formula Eq is an equivalence relation, and the structure $\mathcal{J}(\mathbf{A}) = \langle Int^{\mathbf{A}}, Succ^{\mathbf{A}}, Edge^{\mathbf{A}} \rangle \big/ Eq^{\mathbf{A}}$ is isomorphic to a graph with successor. The cardinality of this graph is denoted by $\|\mathbf{A}\|$.
2. For every set S of graphs with successor in C there exists a sentence φ in L such that for every $\mathbf{A} \in D$:

$$\mathbf{A} \models \varphi \iff \mathcal{J}(\mathbf{A}) \in S.$$

□

The above definition generalizes in some aspect the notion of a logic that captures complexity class (see e.g. [5]), where the formulas Int, Eq, $Succ$ and $Edge$ were *a priori* chosen (to be atomic formulas) and fixed.

Namely, the standard notion is: one can express by sentences of L all those properties of a structure $\mathbf{A} \in D$ that are computable in C, and only those.

Our notion is: one has $\mathcal{J}$, which interprets in every $A \in D$ another structure $\mathcal{J}(\mathbf{A})$ (being graph with successor). Then one is able to express all those properties of $\mathcal{J}(\mathbf{A})$ that are computable in C (with respect to size of $\mathcal{J}(\mathbf{A})$!).

We say that L *is almost surely C–expressive for $\mathcal{J}$* if and only if there is a subset $D \subseteq \boldsymbol{\mathcal{A}}$ such that $\mu(D) = 1$ and L is C–expressive for $\mathcal{J}$ in D.

In this paper we focus our attention on $C = \mathrm{DSPACE}(\log n)$. We will also disregard the edge relation, and this is why we take $\mathcal{J}$ to be a 3–tuple, and $\mathcal{J}(\mathbf{A})$ to be, in general, isomorphic to an interval of natural numbers with successor.

We present now a theorem that will allow us to prove the Kolaitis and Vardi conjecture for all recursive distributions.

Theorem 6. *Let $\boldsymbol{\mu}$ be a recursive distribution. Suppose that a logic L is almost surely* DSPACE$(\log n)$*–expressive for $\mathcal{J}$. Then*

1. *If there is a constant $\alpha < 1$ such that for every natural d*

$$\liminf_{n\to\infty} \mu_n(\{\mathbf{A} \in \boldsymbol{\mathcal{A}} \mid \|\mathbf{A}\|\} \leq d) < \alpha,$$

   then the convergence law does not hold for L and $\boldsymbol{\mu}$.

2. *If there is no natural d such that*

$$\mu(\{\mathbf{A} \in \boldsymbol{\mathcal{A}} \mid \|\mathbf{A}\|\} \leq d) = 1,$$

   then the 0–1 law does not hold for L and $\boldsymbol{\mu}$.

Proof. Throughout the proof we will simply write DSPACE$(\log n)$ properties of $\|\mathbf{A}\|$ as the formulas of L.
We start proving 1.

Without any loss of generality we may assume the distribution $\boldsymbol{\mu}$ to be so that the function $\mathbf{A} \mapsto \mu_{|\mathbf{A}|}(\{\mathbf{A}\})$ is rational valued and recursive. Moreover, we may also assume that whenever $\mu_{|\mathbf{A}|}(\{\mathbf{A}\}) > 0$, the structure $\mathcal{J}(\mathbf{A})$ is an interval with successor.

If for some $d \in \mathbb{N}$ the value

$$a(d) = \mu(\|\mathbf{A}\| \leq d)$$

does not exist, we are done, since then the sentence in L expressing property $\|\mathbf{A}\| \leq d$ has not asymptotic probability. (Recall that all finite sets are in DSPACE$(\log n)$.)

So for the rest of the proof we assume that the values $a(d)$ do exist. Then observe that $a(d)$ is nondecreasing function of d, bounded by 1, and therefore the limit

$$\lim_{d\to\infty} \mu(\|\mathbf{A}\| \leq d)$$

exists (we denote it by a). Our assumption becomes now equivalent to $a < 1$.

So let $\delta > 0$ be a rational number, satisfying $1.5\delta > 1 - a > \delta$. Choose M large enough to have $1.5\delta > 1 - a(M) > \delta$, and N large enough to have $1.5\delta > 1 - \mu_n(\|\mathbf{A}\| \leq M) > \delta$ for all $n > N$.

Now let $g : \mathbb{N} \to \mathbb{N}$ be defined as follows: $g(0) = 0$ and for $n > 0$

$$g(n) = 1 + \big(\text{the least number } m > g(n-1) \text{ such that } \mu_m(\|\mathbf{A}\| > n) > \delta\big).$$

The function g is strictly growing and recursive since $\boldsymbol{\mu}$ is rational valued and recursive.

Now let $\overline{g} : \mathbb{N} \to \mathbb{N}$ be strictly growing, space constructible function with $\overline{g}(n) > (g(n))^{\ell}$. (Observe that always $\|\mathbf{A}\| \leq |\mathbf{A}|^{\ell}$.) Then we define space constructible and strictly growing $f : \mathbb{N} \to \mathbb{N}$ to be

$$f(m) = \begin{cases} 1 & \text{if } m = 0, \\ \overline{g}\,\overline{g} f(m-1) & \text{if } m > 0. \end{cases}$$

Then by assumption about L, and by lemma 4 (recall that functions f, $\overline{g}$ are both space constructible and strictly growing), the following property can be decided in DSPACE($\log n$), and therefore is expressed by some L sentence (say ϑ) :

$$\|\mathbf{A}\| \geq M \wedge \bigwedge_{k=0}^{\infty} \left[(\|\mathbf{A}\| \geq f(k)) \ \rightarrow \ (\|\mathbf{A}\| \geq \overline{g}(f(k))) \right].$$

Let us make sure that ϑ is indeed a DSPACE($\log n$) property (recall that $n = \|\mathbf{A}\|$ is written in unary expansion). Indeed, it follows from strict monotonicity that $f(k) \geq k$, so ϑ is equivalent to

$$\|\mathbf{A}\| \geq M \wedge \bigwedge_{k=0}^{s-1} \left[(\|\mathbf{A}\| \geq f(k)) \ \rightarrow \ (\|\mathbf{A}\| \geq \overline{g}(f(k))) \right],$$

where s is a minimal number such that $f(s) > \|\mathbf{A}\|$. Now the quantified part of the above property can be verified in DSPACE($\log n$) by systematic checking of all $k < s$, according to lemma 1.

We claim that ϑ has no asymptotic probability.

* For all sufficiently large k and $n = g(f(k))$ such that $n > N$ and $f(k) > M$ (there are infinitely many such n), we have that

  $$\mu_n(\|\mathbf{A}\| \geq f(k)) \geq \delta,$$

  and

  $$\begin{aligned} &\mu_n(\|\mathbf{A}\| \geq \overline{g}(f(k))) \leq \\ &\mu_n(\|\mathbf{A}\| > (g(f(k)))^{\ell}) = \\ &\mu_n(\|\mathbf{A}\| > n^{\ell}) = 0. \end{aligned}$$

  Hence

  $$\begin{aligned} &\mu_n(\vartheta) \leq \\ &1 - \mu_n(\|\mathbf{A}\| < M) - \mu_n(\|\mathbf{A}\| \geq f(k)) \leq \\ &(1 - \mu_n(\|\mathbf{A}\| < M)) - \delta < 1.5\delta - \delta = 0.5\delta. \end{aligned}$$

* For all sufficiently large k and $n = g(\overline{g}(f(k)))$ such that $n > N$ and $\overline{g}(f(k)) > M$ (there are infinitely many such n), we have that

$$\mu_n(\|\mathbf{A}\| \geq \overline{g}(f(k)) > M) > \delta,$$

and

$$\mu_n(\|\mathbf{A}\| \geq f(k+1)) =$$

$$\mu_n(\|\mathbf{A}\| \geq \overline{g}(\overline{g}(k))) \leq$$

$$\mu_n(\|\mathbf{A}\| > (g(\overline{g}(k)))^{\ell}) =$$

$$\mu_n(\|\mathbf{A}\| > n^{\ell}) = 0.$$

Combining we get

$$\mu_n(\vartheta) > \delta.$$

We conclude that $\mu_n(\vartheta)$ cannot be convergent since both inequalities $\mu_n(\vartheta) > \delta$ and $\mu_n(\vartheta) < 0.5\delta$ hold infinitely often, and $\delta > 0$.

To prove (ii) suppose contrary that the 0–1 law holds, but there is no natural d such that $\mu(\|\mathbf{A}\| \leq d) = 1$. Then, as the 0–1 law holds and all properties $\|\mathbf{A}\| \leq d$ are L–definable, we conclude that for every d we have $\mu(\|\mathbf{A}\| \leq d) = 0$. But then 1. applies, and even the convergence law fails to hold, which yields a contradiction, and thus finishes the proof. □

This theorem subsumes the results in [10] and [11], where it was assumed that the length of the defined interval should be estimated from the below by a nondecreasing, recursive function, with no too small probability. In particular, most of examples of applications presented there can be now formulated in stronger versions, and the proofs become easier.

4 The consequences: recursive distributions

4.1 Kolaitis and Vardi Conjecture

Before we turn to the Conjecture itself, let us recall basic definitions concerning fixpoint logic and infinitary logic $L^{\omega}_{\omega_1\omega}$.

The *infinitary logic* $L^{\omega}_{\omega_1\omega}$ is an extension of first order logic, where, except all first order formula formation rules, there is an infinitary rule: if Φ is arbitrary countable set of $L^{\omega}_{\omega_1\omega}$ formulas such that only finitely many distinct variables occur in formulas in Φ, then the *infinitary disjunction* $\bigvee\Phi$ is a formula of $L^{\omega}_{\omega_1\omega}$. We can also define certain fragments of $L^{\omega}_{\omega_1\omega}$: e.g, the recursive (recursively enumerable, ...) fragment of $L^{\omega}_{\omega_1\omega}$ is obtained by restricting infinitary disjunctions to recursive (recursively enumerable, ...) sets of formulas.

The *fixpoint logic* is a logic obtained from first order logic by adding *fixpoint formulas* of the form φ^{∞}, where φ is a first order formula, has ℓ free variables (say $x_1, \ldots, x_\ell$), and a new ℓ-ary relation symbol $X^{\ell} \notin \sigma$ occurs *positively* (i.e. under even number of negations) in it. The set of formulas fixpoint logic is the least set of expressions, containing all first order formulas, all fixpoint formulas and closed under standard first order formula formation rules: propositional connectives and quantification.

The semantics of such fixpoint formula φ^∞ is as follows: Let $\mathbf{A}$ be a finite σ-structure. Then there is an operator from ℓ-ary relations on $|\mathbf{A}|$ to ℓ-ary relations on $|\mathbf{A}|$, defined by

$$\Phi(R) = \{\mathbf{a} \in |\mathbf{A}|^\ell \mid \mathbf{A}, \mathbf{x}\!:\!\mathbf{a}, X^\ell\!:\!R \models \varphi.\}$$

Since X^ℓ occurs only positively in φ, the operator Φ is monotone: if $R \subseteq R'$, then $\Phi(R) \subseteq \Phi(R')$. Therefore it gives rise to an increasing sequence of *stages*, $\Phi(\emptyset) \subseteq \Phi(\Phi(\emptyset)) \subseteq \ldots$ Elements of this sequence we denote $\Phi^1, \Phi^2, \ldots$ Since $\mathbf{A}$ is finite, it follows that there exists a minimal m such that $\Phi^m = \Phi^{m+1}$. Moreover, the value of m cannot exceed $|\mathbf{A}|^\ell$. Φ^m is the *least fixed point of* Φ, and we denote it Φ^∞. Now we define:

$$\mathbf{A}, \mathbf{x}\!:\!\mathbf{a} \models \varphi^\infty \qquad \Longleftrightarrow \qquad \mathbf{A}, \mathbf{x}\!:\!\mathbf{a}, X^\ell\!:\!\Phi^\infty \models \varphi.$$

The value of m defined above we call the *closure ordinal of* φ *in* $\mathbf{A}$, and denote $\|\mathbf{A}\|_\varphi$.

We say that formula φ^∞ is *bounded on a class* $D \subseteq \boldsymbol{\mathcal{A}}$ iff there is a constant d such that $\|\mathbf{A}\|_\varphi \leq d$ for every $\mathbf{A} \in D$. If this is the case for every fixpoint formula, we say that fixpoint logic is *bounded on* D. Either of these properties is said to hold almost surely iff, in addition, $\mu(D) = 1$.

It should be noted that if fixpoint logic is bounded on D, then, in particular, it collapses to first order logic on D. It is also known that fixpoint logic is essentially a sublogic of $L^\omega_{\omega_1\omega}$, as proved by Kolaitis and Vardi in [7].

The following theorem is the key one:

Theorem 7. *For an arbitrary fixpoint formula* φ^∞ *there exist fixpoint formulas* Int_φ, $Succ_\varphi$ *and* Eq_φ, *such that LFP is* PTIME*–expressive (and, in consequence, also* DSPACE$(\log n)$*–expressive) for* $\mathcal{J}_\varphi \equiv \langle Int_\varphi, Succ_\varphi, Eq_\varphi\rangle$ *in* $\boldsymbol{\mathcal{A}}$.

Observe that the above theorem is essentially an amalgamate of: Moschovakis Stage Comparison Theorem from [9], which provides point 1 of the definition, and, due to Immerman [4] and Vardi [12]: classical result about capturing PTIME by fixpoint logic over structures with standard successor together with normal form theorem for fixpoint logic, the latter assuring that one can, on the semantical level, substitute formulas resulting from Moschovakis Theorem into the place of standard successor.

Theorem 8. *Let* $\boldsymbol{\mu}$ *be arbitrary recursive probability distribution on the class* $\boldsymbol{\mathcal{A}}$ *of all finite structures over the signature* σ. *Then:*

1. *The convergence law holds for the fixpoint logic relative to* $\boldsymbol{\mu}$ *if and only if the same law holds for first order logic and for every* $\varepsilon > 0$ *there is a subset* $D \subseteq \boldsymbol{\mathcal{A}}$ *with* $\mu(D) > 1 - \varepsilon$, *such that fixpoint logic is bounded on* D.
2. *The 0–1 law holds for the fixpoint logic relative to* $\boldsymbol{\mu}$ *if and only if the same law holds for first order logic, and fixpoint logic is almost surely bounded on* $\boldsymbol{\mathcal{A}}$.

Proof. We prove only the nontrivial direction: from left to right. According to theorem 7, the premises of theorem 1 are satisfied. It then follows that:

1. if the fixpoint convergence law holds, then for every $\varepsilon > 0$, and for every formula φ^∞, there exists a subset $D \subseteq \boldsymbol{\mathcal{A}}$ with $\mu(D) > 1-\varepsilon$ on which φ^∞ is bounded.
2. if the fixpoint 0–1 law holds, then for every $\varepsilon > 0$, and for every formula φ^∞, there exists a subset $D \subseteq \boldsymbol{\mathcal{A}}$ with $\mu(D) = 1$ on which φ^∞ is bounded.

It remains to be shown that we can exchange the quantifiers "for every formula" and "there exists a subset". This is a standard construction: E.g., for the convergence law we let φ_1^∞, φ_2^∞, ... to be any enumeration of all fixpoint formulas. Let M_j be a sequence of natural numbers such that

$$\lim_{n\to\infty} \mu_n(\|\mathbf{A}\|_{\varphi_j} \leq M_j) \geq 1 - \varepsilon/2^j .$$

Moreover, let n_i for $i = 1, 2, \ldots$ be the number such that

$$\mu_n(\|\mathbf{A}\|_{\varphi_j} \leq M_j) \geq 1 - (\varepsilon/2^j) - (1/i^2)$$

whenever $j \leq i$ and $n \geq n_i$.

Then we set

$$D = \{\mathbf{A} \in \boldsymbol{\mathcal{A}} \mid \bigwedge_{j=1}^{\infty}(|\mathbf{A}| \geq n_j \to \|\mathbf{A}\|_{\varphi_j} \leq M_j)\}.$$

First observe that fixpoint logic is bounded on D. Indeed, for every fixpoint formula φ^∞, say $\varphi \equiv \varphi_i$, we have that φ^∞ is bounded on $\{\mathbf{A} \in D \mid |\mathbf{A}| > n_i\} \subseteq D$ by construction, and there are only finitely many other structures in D.

Now let $n_i \leq n < n_{i+1}$. We have

$$\begin{aligned}
&\mu_n(D) = \\
&\mu_n(\{\mathbf{A} \in \boldsymbol{\mathcal{A}} \mid n_i \leq |\mathbf{A}| < n_{i+1} \wedge \textstyle\bigwedge_{j=1}^{\infty}(|\mathbf{A}| \geq n_j \to \|\mathbf{A}\|_{\varphi_j} \leq M_j)\}) = \\
&\mu_n(\{\mathbf{A} \in \boldsymbol{\mathcal{A}} \mid \textstyle\bigwedge_{j=1}^{i} \|\mathbf{A}\|_{\varphi_j} \leq M_j\}) = \\
&1 - \mu_n(\{\mathbf{A} \in \boldsymbol{\mathcal{A}} \mid \textstyle\bigvee_{j=1}^{i} \|\mathbf{A}\|_{\varphi_j} > M_j\}) \geq \\
&1 - \textstyle\sum_{j=1}^{i} \mu_n(\{\mathbf{A} \in \boldsymbol{\mathcal{A}} \mid \|\mathbf{A}\|_{\varphi_j} > M_j\}) = \\
&1 - \textstyle\sum_{j=1}^{i}(1 - \mu_n(\{\mathbf{A} \in \boldsymbol{\mathcal{A}} \mid \|\mathbf{A}\|_{\varphi_j} \leq M_j\})) \geq \\
&1 - \textstyle\sum_{j=1}^{i}(\varepsilon/2^j + 1/i^2) \\
&\geq 1 - \varepsilon - 1/i.
\end{aligned}$$

It follows that D, with some structures eventually removed to make it measurable, satisfies $\mu(D) \geq 1-\varepsilon$, as desired. □

We can derive now several consequences, the first of them being solution of the Kolaitis and Vardi conjecture for recursive distributions. But before we cite here one deep result from their paper:

Theorem 9 (Kolaitis and Vardi [8], theorem 4.1). *$L^\omega_{\omega_1\omega}$ collapses to first order logic on a class $D \subseteq \boldsymbol{\mathcal{A}}$ if and only if fixpoint logic is bounded on D.* □

Theorem 10. *Let $\boldsymbol{\mu}$ be arbitrary recursive probability distribution on the class $\boldsymbol{\mathcal{A}}$ of all finite structures over the signature σ. Then:*

1. *The convergence law holds for the infinitary logic $L^\omega_{\omega_1\omega}$ relative to $\boldsymbol{\mu}$ if and only if the convergence law holds for fixpoint logic relative to $\boldsymbol{\mu}$.*
2. *The same is true when "convergence law" is replaced by "0–1 law".*

Proof. Part (ii) is immediate from theorems 8 and 9. Part (i) is almost immediate (once more we prove only the nontrivial direction): take $D \subseteq \boldsymbol{\mathcal{A}}$ with $\mu(D) \geq 1 - \varepsilon$, on which fixpoint logic is bounded. Then $L^\omega_{\omega_1\omega}$ collapses to first order logic on D, by theorem 9. Since the convergence law for first order logic holds, it follows that for no φ in $L^\omega_{\omega_1\omega}$ the difference between $\limsup_{n\to\infty} \mu_n(\varphi)$ and $\liminf_{n\to\infty} \mu_n(\varphi)$ exceeds ε, which implies the thesis. □

As a trivial consequence we get the following

Corollary 11. *The same thesis as in theorem 10 holds for every logic whose expressive power over finite structures is between fixpoint logic and $L^\omega_{\omega_1\omega}$. These include, among others, partial fixed point logic, recursive fragment of $L^\omega_{\omega_1\omega}$, recursively enumerable fragment of $L^\omega_{\omega_1\omega}$.* □

Now we turn to relationships between fixpoint logic and first order logic. The first consequence we derive is the following, having "excluded middle principle" flavour:

Corollary 12. *Let $\boldsymbol{\mu}$ be arbitrary recursive probability distribution on the class $\boldsymbol{\mathcal{A}}$ of all finite structures over the signature σ. Then if the fixpoint convergence law holds, then asymptotic probability of every sentence of fixpoint logic is a limit of a sequence of asymptotic probabilities of first order sentences.*

In particular, if first order 0–1 law holds, then either fixpoint 0–1 law holds, or even fixpoint convergence law fails to hold.

Proof. Let fixpoint 0–1 law hold for $\boldsymbol{\mu}$. Suppose that φ is any sentence of fixpoint logic.

Let D_m be any subset of $\boldsymbol{\mathcal{A}}$ with $\mu(D_m) \geq 1 - 1/m$, and such that fixpoint logic is bounded on D_m. It exists by theorem 8. Let φ_m be any first order sentence equivalent to φ on D_m. Then we immediately get $|\mu(\varphi) - \mu(\varphi_m)| \leq 1/m$, and consequently, $\mu(\varphi) = \lim_{m\to\infty} \mu(\varphi_m)$. □

The other consequence touches the problem of computing asymptotic probabilities of fixpoint properties:

Let us call the L *almost sure theory* the set of those formulas φ of the logic L, for which $\mu(\varphi) = 1$. It is clear what decidability of this theory means. Additionally, we say that this theory is *recursively approximable* iff for every $\varepsilon > 0$ there exists a recursive set S of sentences of fixpoint logic such that if $\mu(\varphi) = 1$ then $\varphi \in S$, and if $\liminf_{n\to\infty} \mu_n(\varphi) < 1 - \varepsilon$, then $\varphi \notin S$.

We cite now a result from [11] (after conversion to our terminology):

Theorem 13. *Let $\boldsymbol{\mu}$ be arbitrary (not necessarily recursive) probability distribution on the class $\boldsymbol{\mathcal{A}}$ of all finite structures over the signature σ. Suppose that the convergence law for first order logic holds. Then:*

1. *The fixpoint almost sure theory is recursive iff the first order almost sure theory is recursive and fixpoint logic is almost surely bounded on $\boldsymbol{\mathcal{A}}$.*
2. *The fixpoint almost sure theory is recursively approximable iff the first order almost sure theory is recursively approximable and for every $\varepsilon > 0$ there is a subset $D \subseteq \boldsymbol{\mathcal{A}}$ with $\mu(D) > 1 - \varepsilon$, such that fixpoint logic is bounded on D.* □

Now we can easily combine the above theorem with theorem 8, getting

Corollary 14. 1. *Let $\boldsymbol{\mu}$ be a Δ_1 distribution such that the first order 0–1 law holds, and the first order almost sure theory is recursive. Then the fixpoint 0–1 law holds if and only if the fixpoint almost sure theory is recursive.*
2. *The same as above is true, when "the 0–1 law" is replaced by "the convergence law", and "recursive" by "recursively approximable".* □

4.2 Deterministic transitive closure logic

In this section we want to show (but in less details), that most of results from the previous section is true also for *deterministic transitive closure logic* (DTC, in short).

This logic allows formulas expressing transitive closure of relations (of even arity) definable in this logic, provided that they are single valued. Namely, we augment first order logic with the following formula formation rule: if $\varphi(\mathbf{x}, \mathbf{x}')$ is a formula (the length ℓ of $\mathbf{x}$ is equal to the length of $\mathbf{x}'$), then $\varphi^{DTC}(\mathbf{x}, \mathbf{x}')$ is also a formula. The least set of expressions containing all first order formulas and closed under such enriched family of formula formation rules is the DTC.

The semantics of DTC formulas is as follows: let $\mathbf{A}$ be any finite σ-structure. Then $\mathbf{A}, \mathbf{x} : \mathbf{a}, \mathbf{x}' : \mathbf{a}' \models \varphi^{DTC}$ iff the pair $(\mathbf{a}, \mathbf{a}')$ belongs to the *transitive closure* of the relation $R = \{(\mathbf{a}, \mathbf{a}') \in (|\mathbf{A}|^\ell)^2 \mid (\exists! \mathbf{b}\ \mathbf{A}, \mathbf{x} : \mathbf{a}, \mathbf{x}' : \mathbf{b} \models \varphi) \ \wedge\ \mathbf{A}, \mathbf{x} : \mathbf{a}, \mathbf{x}' : \mathbf{a}' \models \varphi\}$.

To apply the proofs we presented in previous section, we need suitable substitutes of the closure ordinal related definitions, and of theorem 7. The closure ordinal $\|\mathbf{A}\|_\varphi$ is a straightforward construction: it is a diameter of the set $(|\mathbf{A}|^\ell)^2$ under the distance measured in number of arcs of the relation R, necessary to get from one tuple to the other. The substitute of theorem 7 is a simple exercise in the part of constructing a successor structure, while the part of expressing DSPACE($\log n$) properties (rather than PTIME ones) can be found in [5].

These constructions allow us to formulate and prove counterparts of theorems 8, and corollaries 12 and 14. The last of them, however, requires additional effort: it is also necessary to look at the paper [11] to see that the necessary and sufficient conditions for recursiveness and recursive approximability of the fixpoint almost sure theory can be easily transformed into ones for almost sure theory of DTC sentences.

However, counterparts of theorem 10 and corollary 11 do not hold. In fact, one of the examples presented by Grädel and McColm in [2] (namely, the one with hypercubes, in section 4 of that paper), shows that there are classes of finite structures in which deterministic transitive closure logic collapses to first order logic, while unrestricted transitive closure logic does not. The example can be suitably modified so that it gives a recursive distribution with the 0–1 law for deterministic transitive closure logic, but without convergence law for fixpoint logic. In particular, convergence law for $L^{\omega}_{\omega_1\omega}$ and this distribution also fails.

5 Nonrecursive distributions

It is quite natural to ask now: "What about distributions which are not recursive?"

The answer is essentially (but implicitly) given in [11]): there exists a distribution for which neither of theses of theorems 6, 8, 9 holds. Of course this distribution is not recursive. We sketch briefly (after cf. [11]) the construction now to analyze its complexity.

Let for natural numbers $p < n$ the structure $\mathbf{A}(p, n)$ be a directed graph with carrier set $\{0, \ldots, n-1\}$ and with edges $(i-1, i)$ for $1 \leq i \leq p$. Thus $\mathbf{A}(p, n)$ contains one chain of length p and $n-p-1$ isolated points.

Let $F \subseteq \mathbb{N}$ satisfy the following condition: for every recursively enumerable subset $R \subseteq \mathbb{N}$ either $F \cap R$ is finite or $F \setminus R$ is finite. This set can be constructed by standard diagonalization technique, and then it is Σ_2.

Once we have defined F we define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \max\{m \in F \mid n \geq 2m\}.$$

Now consider the distribution $\boldsymbol{\mu}^f$, defined by:

$$\mu^f_n(\{\mathbf{A}\}) = \begin{cases} 1 & \text{if } \mathbf{A} = \mathbf{A}(f(n), n) \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that $\boldsymbol{\mu}^f$ is Σ_2, like F, and hence by proposition 3 it is also Δ_2. Therefore the following theorem both falsifies the Kolaitis and Vardi Conjecture in general, and gives a tight (with respect to the arithmetical hierarchy) bound on the distributions for which it remains true.

Theorem 15 ([11]). *The fixpoint 0–1 law for the distribution $\boldsymbol{\mu}^f$ holds, but even the $L^{\omega}_{\omega_1\omega}$ convergence law for $\boldsymbol{\mu}^f$ does not hold.*

Proof. It is an easy observation that if φ is in $L^{\omega}_{\omega_1\omega}$ with k variables (free and bound) and $n-p-1,\ m-p-1 \geq k$, then

$$\mathbf{A}(p, n) \models \varphi \qquad \text{iff} \qquad \mathbf{A}(p, m) \models \varphi,$$

which may be immediately proved by application of the infinitary Ehrenfeucht-Fraïssé game with k pebbles.

Moreover, it is easily observed that f is nondecreasing, unbounded, $f(n) \leq n/2$, so $n - f(n) \geq n/2$. Therefore for every sentence φ in $L^{\omega}_{\omega_1\omega}$ with at most k variables, and for all sufficiently large n the fact whether $\mathbf{A}(f(n), n) \models \varphi$ or not depends on the value $f(n)$ only. As we are interested in asymptotical properties of $\mu^f_n(\varphi)$, we may think that φ simply recognizes some set R of natural numbers – the lengths of chains in models of positive probability. (The set of all chain lengths in models of positive probability is F.)

Let φ be any sentence of fixpoint logic and let R be the set of chain lengths it recognizes. Clearly R is recursive. Then either $F \setminus R$ is finite, so $\mu_n(\varphi) = 1$ for all large n, or $F \cap R$ is finite, so $\mu_n(\varphi) = 0$ for all large n. Therefore the fixpoint 0–1 law holds for $\boldsymbol{\mu}^f$. This completes the first part of the thesis.

We construct now a sentence in $L^{\omega}_{\omega_1\omega}$ without asymptotic probability. Let first order sentence φ_m be:

$$\exists x(\exists y E(x,y) \wedge (\exists x E(y,x) \wedge (\exists y E(x,y) \wedge (\ldots \wedge (\exists \{x|y\} E(\{x|y\},\{y|x\}))\ldots)))),$$

with $m+1$ occurrences of quantifier "$\exists$". We take $\{u|v\}$ to be u if m is even and v if m is odd. Then, essentially, φ_m expresses the property "there exists a chain of length m".

Let R be an arbitrary infinite and coinfinite subset of F. Then the $L^2_{\omega_1\omega}$ sentence $\bigvee_{m\in R} \varphi_m \wedge \neg\varphi_{m+1}$ has not asymptotic probability. □

On the other hand, the reader will easily guess how to prove the following theorem:

Theorem 16. *Let S be arbitrary subset of $\mathcal{P}(\mathbb{N}^3) \cup \mathcal{P}(\mathbb{N})$ closed under Turing reductions and complements. Then for arbitrary probability distribution $\boldsymbol{\mu}$ on $\boldsymbol{\mathcal{A}}$ such that $m_{\boldsymbol{\mu}}$ is in S, one has*

1. *The convergence law holds for the S–fragment of the infinitary logic $L^{\omega}_{\omega_1\omega}$ relative to $\boldsymbol{\mu}$ if and only if the convergence law holds for whole $L^{\omega}_{\omega_1\omega}$ relative to $\boldsymbol{\mu}$.*
2. *The same is true when "convergence law" is replaced by "0–1 law".* □

Acknowledgment I am very indebted to Erich Grädel, who suggested many improvements, clarifying the exposition and leading to more "reader–friendly" paper.

References

1. Compton, K.J. 0–1 laws in logic and combinatorics, *Proc. NATO Advanced Study Institute on Algorithms and Order* (I. Rival, ed.), Reidel, Dordrecht (1988).
2. Grädel,. E., and McColm, G.L., Deterministic versus nondeterministic transitive closure logic, in: *Proc. 7th IEEE Symp. on Logic in Computer Science, 1992*, pp. 58–63.
3. Gurevich, Yu., Zero–one laws, *Bull. EATCS,* Spring 1992, pp. 90–106.
4. Immerman, N.: Upper and lower bounds for first–order expressibility, *Journal of Computer and System Sciences* 25(1982), pp. 76–98
5. Immerman, N.: Languages that capture complexity classes, *SIAM Journal of Computing* 16(1987).
6. Kolaitis, Ph.G., and Vardi, M.Y., 0–1 laws for fragments of second order logic: an overview, *MSRI Workshop on Logic from Computer Science (ed. Y.N. Moschovakis), Berkeley 1989, to appear.*
7. Kolaitis, Ph., and Vardi, M.Y., On the expressive power of Datalog: tools and a case study, in: *Proc. 9th IEEE Symp. on Principles of Database Systems, 1990,* pp. 61–71.
8. Kolaitis, Ph., and Vardi, M.Y., Fixpoint logic vs. infinitary logic in finite–model theory, in: *Proc. 7th IEEE Symp. on Logic in Computer Science, 1992*, pp. 46–57.
9. Moschovakis, Y.N. *Elementary induction on abstract structures,* North Holland, 1974.
10. Tyszkiewicz, J., Infinitary queries and their asymptotic probabilities I: Properties definable in Transitive Closure Logic *in: E. Börger et al. (eds.), Proc. Computer Science Logic '91, LNCS 626, Springer Verlag, pp. 396–410.*

11. Tyszkiewicz, J., Infinitary queries and their asymptotic probabilities II: Properties definable in Least Fixed Point Logic *to appear in: A. Frieze et al. eds., Proc. Random Graphs '91, Wiley (?).*
12. Vardi, M.Y.: Complexity of relational query languages, *14th Symposium on Theory of Computation* 1982, pp. 137–146.