Generalized Quantifiers for Simple Properties

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Abstract

We consider extensions of fixed-point logic by means of generalized quantifiers in the context of descriptive complexity. By the well-known theorem of Immerman and Vardi, fixed-point logic captures PTime over linearly ordered structures. It fails, however, to express even most fundamental structural properties, like simple cardinality properties, in the absence of order.

In the present investigation we concentrate on extensions by generalized quantifiers which serve to adjoin simple *or* basic *structural properties. An abstract notion of simplicity is put forward which isolates those structural properties, that can be characterized in terms of a concise* structural invariant*. The key examples are provided by all monadic and cardinality properties in a very general sense.*

The main theorem establishes that no extension by any family of such simple quantifiers can cover all of PTime. These limitations are proved on the basis of the semantically motivated notion of simplicity; in particular there is no implicit bound on the arities of the generalized quantifiers involved. Quite to the contrary, the natural applications concern infinite families of quantifiers adjoining certain structural properties across all arities in a uniform way.

1 Introduction

An important branch of finite model theory deals with the problem to find exact matches between computational complexity and the expressiveness of logical systems: Given a complexity class, e.g. Ptime, find a logic or language L such that any PTime query can be defined in L, and vice versa. Since a logic is a specification of descriptive resources in much the same way a complexity class specifies computational resources, this issue is referred to as *descriptive complexity*, cf. [13, 15]. Logical equivalents of all major complexity classes have been found in the case of linearly ordered structures. In particular, fixed-point logic allows to define exactly those classes of linearly ordered structures that can be recognized in PTime, [12, 21]. In the general case of not necessarily ordered structures, however, the problem remains open for all complexity classes below NPTime. See [8, 9, 13] for overviews.

In database theory in particular it would be desirable to have a query language, i.e. a logic, which exactly expresses those properties that can be checked in PTime. In the many cases that the database does not carry a natural ordering inherent to the problem, the use of the inevitable representational ordering in the query language is not a priori safe, since it allows to express properties which are dependent on the choice of this purely representational ordering.

From the theoretical point of view it is not even known whether the class of all PTime queries pertaining to not necessarily ordered structures is recursively enumerable. In a precise formulation of the problem, as given in [9], this turns out to be equivalent with the existence of a logic for PTime.

Certainly fixed-point logic fails to express all PTime properties in the absence of order, e.g. it completely lacks the expressive means for even the most simple cardinality properties. Hence the question becomes:

- Is there any extension of fixed-point logic that captures PTime in the general case?

In abstract model theory the concept of *generalized quantifiers* or *Lindström extensions*, [18], provides a uniform and universal formalism for the treatment of the extension of a logic, [6]. A generalized quantifier adjoins the expressive means for one particular structural property in a systematic way. The observation that fixed-point logic lacks even most fundamental expressive features justifies the following ramification of the above question:

- Is there any extension by means of simple or fundamental structural properties which reaches up to PTime?

To make this question precise, we have to specify a notion of *simplicity of structural properties*: We shall regard a structural property as simple if it can be determined in terms of a concise invariant, as is provided by simple counting in the case of all cardinality properties. We require that invariant to take only sub-exponentially many values in terms of the size of the structure, so that in particular it does in general really reduce the *information content*.

Our main theorem shows that even the entire class of such structural properties, all added to fixed-point logic in the shape of infinitely many quantifiers, cannot compensate for the lack of recursive power suffered by fixed-point logic over structures which provide too little order.

Main Theorem *The extension of fixed-point logic by the family of all quantifiers that are based on simple invariants does not contain all PTime.*

The main theorem in particular applies to all quantifiers based on cardinality invariants. From the precise statement of the theorem it is seen that the extension of fixed-point logic by all generalized quantifiers based on cardinality information falls short even of the expressive power of proper fixed-point with counting: The latter was introduced by Immerman in [13] as the natural (and non-Lindström type) extension of fixed-point logic which allows to assess and process cardinalities in a recursive way. We hereby reproduce in the present much more general setting a previous result from [19]. The proposed criterion of simplicity, however, extends beyond the analysis of cardinality invariants. As a natural class of examples of a completely different flavour we exhibit the PTime properties of rigid graphs.

There are a number of known negative results concerning Lindström extensions of fixed-point logic in the finite. Hella [11] shows that no finite Lindström extension of fixed-point logic can capture PTime, a result that is extended by Dawar and Hella in [4]. The essential point of the finiteness requirement there is that it implicitly restricts the arity of structural properties which are adjoined. Apart from this no further semantical restriction is put on the extensions considered.

In a different and combinatorially sophisticated approach Kolaitis and Väänänen [17] show that extensions by properties of one monadic predicate fail to capture even very simple cardinality properties of two unary predicates like 'equal size' (the Härtig quantifier). Since our results are applicable to the class of all properties based on cardinality invariants, and hence also to all properties of monadic structures, some results in [17] are found to be generalized in the present treatment, albeit with completely different methods.

Unlike these results on the limitations of extensions of fixed-point logic by means of generalized quantifiers, our approach does not require the extensions to be finite or of bounded arity. The limitations found here are direct consequences of the semantically motivated simplicity criterion, and the central result is applicable to natural infinite families which adjoin certain structural properties across all arities.

The present results indicate that in finite model theory other types of extensions than quantifier extensions in the Lindström sense must play an essential rôle. This is all the more so if in descriptive complexity we are not merely interested in purely extensional semantic equivalence between logical systems and complexity classes, but also in closer low-level logical modelling of elementary algorithmical processes in the analysis of structures.

2 Basic definitions and preliminaries

2.1 Fixed-point logic

Fixed-point logic, FP, is obtained as an extension of first-order logic through closure with respect to the generation of *inductive fixed points*. Let $\varphi = \varphi(X, \overline{x})$ be a formula with a free predicate variable X of some arity r and designated free individual variables $\overline{x} = (x_1, \ldots, x_r)$ of matching arity. With φ associate a global operator F_φ on r-ary predicates: Over a structure $\mathfrak{A}$, which interprets φ up to the designated free variables, F_φ is defined through

$$F_\varphi^{\mathfrak{A}} : R \longmapsto R \cup \{\overline{a} \mid \mathfrak{A} \models \varphi[R, \overline{a}]\},$$

for all r-ary predicates $R \subset A^r$. Since this operator is cumulative, the sequence generated according to $X_0 = \emptyset$, $X_{i+1} = F_\varphi(X_i)$ saturates over $\mathfrak{A}$ within at most $|A|^r$ many steps. Its limit is called the *inductive fixed point* of φ. For fixed-point logic we admit, in addition to the usual first-order rules of formation, the formation of a formula $\psi(\overline{z}) = [\mathrm{IFP}_{X,\overline{x}}\varphi(X, \overline{x})]\, \overline{z}$, from any formula φ and matching tuple of variables $\overline{z}$. The semantics of this formula is defined as follows: $\mathfrak{A} \models \psi[\overline{a}]$ if $\overline{a}$ is an element of the inductive fixed point of φ, i.e. if $\overline{a} \in \bigcup_i \big(F_\varphi^{\mathfrak{A}}\big)^i(\emptyset)$. The interest in FP in finite model theory is due to

Theorem 1 (Immerman [12],Vardi [21]) *A class of linearly ordered finite structures is recognized by a PTime algorithm if and only if it is the class of finite models of a sentence in* FP*.*

Over not necessarily linearly ordered structures, however, FP is strictly contained in PTime: E.g. it is not difficult to see that over the empty vocabulary, i.e. for pure sets, FP collapses to first-order logic. It follows in particular that even very low complexity cardinality properties are not FP-definable in the absence of order. E.g., the class EVEN $= \{A \mid |A| \text{ is even}\}$ is not FP-definable.

2.2 Interpretations and generalized quantifiers

Generalized quantifiers provide the means to assert structural properties of definably interpreted structures. We therefore first fix some nomenclature concerning interpretations.

Direct interpretations: Consider a tuple of formulae $\overline{\varphi} = \big(\varphi_i(\overline{x}^{(i)})\big)_{i=1\ldots l}$, where each φ_i has a tuple of free variables $\overline{x}^{(i)}$ of arity r_i. Over structures $\mathfrak{A}$ of appropriate type each φ_i defines an r_i-ary predicate $\varphi_i[\mathfrak{A}] := \{\overline{a} \in A^{r_i} \mid \mathfrak{A} \models \varphi_i[\overline{a}]\}$. We take $\overline{\varphi}$ to *interpret* the following structure over $\mathfrak{A}$:

$$\big(A, \overline{\varphi}[\mathfrak{A}]\big) := \big(A, (\varphi_i[\mathfrak{A}])_{i=1\ldots l}\big).$$

As usual the formulae φ may have other free variables than the ones displayed for the purpose of the interpretation, these are then regarded as parameters in the interpretation.

For a more systematic treatment of *uniform families* of quantifiers we shall have occasion to consider two generalizations of this concept:

Interpretations over powers of the universe: The structure interpreted over the s-th power of $\mathfrak{A}$ has universe A^s. The predicates over A^s are defined by formulae whose free variables come in groups of s.

Interpretations as quotients (over powers): Suppose that some structure $\mathfrak{C}$ is interpreted over the s-th power of $\mathfrak{A}$, and that an additional formula $\psi(x_1, \ldots, x_s, x'_1, \ldots, x'_s)$ interprets an equivalence relation $\sim$ over the s-th power of $\mathfrak{A}$. If $\sim$ is a congruence relation for $\mathfrak{C}$, then we regard the quotient structure $\mathfrak{C}/\!\sim$ as interpreted as a quotient.

Examples: Consider as base structures graphs $\mathfrak{G} = (V, E)$, E a binary relation symbol for the edge relation. Let $\varphi(x, z)$ be a fixed-point formula which asserts that x is pathwise connected to z. Then for every value v of the parameter z, the formula $\varphi(x, z)$ interprets the structure with universe V and a monadic predicate for the connected component of v over $(\mathfrak{G}, v)$. For a directed graph $\mathfrak{G} = (V, E)$, its dual can be interpreted over the second power of its universe by the formula $\varphi_1(x_1, x_2, x'_1, x'_2) := (x_1 = x'_1 \wedge \neg\, x_2 = x'_2) \vee (\neg\, x_1 = x'_1 \wedge x_2 = x'_2)$. An additional monadic predicate corresponding to the set of original edges can be interpreted simply through $\varphi_2(x_1, x_2) := Ex_1x_2$. For the dual of an undirected graph $\mathfrak{G} = (V, E)$ it is natural not to distinguish the edge (v_1, v_2) from the edge (v_2, v_1). The obvious idea is to factorise the result of the previous interpretation with respect to the congruence relation interpreted by $\psi(x_1, x_2, x'_1, x'_2) := (x_1 = x'_1 \wedge x_2 = x'_2) \vee (x_1 = x'_2 \wedge x_2 = x'_1)$. In this case the dual gets interpreted as a quotient.

Generalized quantifiers: A generalized quantifier is associated with an isomorphism-closed class of structures – its defining class, i.e. the class that represents the structural property at issue. Let K stand for such a class. The corresponding quantifier Q_K binds formulae which are apt to interpret structures of the type appropriate for K. Semantically such quantification allows to assert membership in K of that structure which is directly interpreted by the quantified formulae.

More formally, let K be of type $\overline{R} := (R_1, \ldots, R_l)$, with R_i of arity r_i. The syntax of any logic L can be extended to allow the construction of formulae

$$\psi := Q_K\big(\overline{x}^{(i)}; \varphi_i(\overline{x}^{(i)})\big)_{i=1\ldots l}$$

with semantics:

$$\mathfrak{A} \models \psi \quad \text{iff} \quad (A, \overline{\varphi}[\mathfrak{A}]) \in K.$$

We write FP(Q_K) for the logic obtained as the simultaneous closure with respect to the formation rules of FP and the above Q_K-quantification (in particular FP(Q_K) allows for the interleaving of Q_K-quantifications and FP-constructions). Similarly, we write FP$(\mathcal{Q})$ for the simultaneous adjunction of any family $\mathcal{Q}$ of quantifiers.

For a simple example consider the class K_{even} consisting of those structures of one monadic predicate U for which the cardinality of U is even. With the induced quantifier $Q_{\text{even}} := Q_{K_{\text{even}}}$ the class EVEN $= \{\mathfrak{A} \mid |A| \text{ is even}\}$ becomes trivially definable as $\{\mathfrak{A} \mid \mathfrak{A} \models Q_{K_{\text{even}}}(x; x = x)\}$. Two more cardinality properties which are naturally rendered as generalized quantifiers are: The Härtig quantifier,

[10], which expresses equicardinality, as induced by the class $\{(B, U_1, U_2) \mid |U_1| = |U_2|\}$ of two monadic predicates U_1, U_2, and the Rescher quantifier, [20], which similarly expresses cardinality comparison, as induced by $\{(B, U_1, U_2) \mid |U_1| < |U_2|\}$.

Uniform families: If we think of the class K as specifying a structural property that is to be made expressible, it is natural to consider more uniform ways of adjoining this property. For a typical example consider transitive closure logic; here the underlying structural property is graph reachability. It is natural, however, to assert reachability not only of graphs that are directly interpreted, i.e. with elements of the underlying structure as nodes, but also of graphs whose nodes are tuples of elements – as is in fact done in the standard formalization of transitive closure logic. Thus one is lead to interpretations over powers of the universe. Similarly, interpretations as quotients arise naturally in other context.

In general, a highly uniform way to adjoin a particular structural property across all arities is induced by these more general forms of interpretations. The concept of quantifiers need not be modified, if we work with suitable modifications in the defining classes K:

1) Let K_s contain the representations of members of K as realized in interpretations over the s-th power, for each $s \geq 1$.

$$K_s := \{(B, \overline{R}') \mid \mathfrak{C} = (B^s, \overline{R}') \in K\},$$

is of type $\overline{R}' = (R'_1, \ldots, R'_l)$, with R'_i of arity sr_i, where R_i was of arity r_i. Q_{K_s} asserts of bound formulae $\overline{\varphi}$ of the appropriate kind (with variables coming in blocks of s), that they interpret a structure in K over the s-th power of the universe.

2) Let $K^\sim$ consist of the representations of members of K as realized in interpretations as quotients with respect to a congruence relation,

$$K^\sim := \{(B, \overline{R}, \sim) \mid \sim \text{ a congruence}, (B, \overline{R})/{\sim} \in K\}.$$

Thus $K_s^\sim$ contains the representations of members of K as quotients over the s-th power of the universe, so that $Q_{K_s^\sim}$ asserts of bound formulae that they interpret a structure in K as a quotient in power s.

Applying these generalizations to the class $K = K_{\text{even}}$ we get

1) Quantifiers which allow to assert the evenness of a relation of arity s, and

2) Quantifiers which assert that some s-ary relation consists of an even number of equivalence classes with respect to a congruence.

Definition 2 *For a class $\mathcal{Q}$ of quantifiers, let $\mathcal{Q}^\omega$ (resp. $\mathcal{Q}^\sim$) contain all the variants that express the properties represented in $\mathcal{Q}$ of interpretations in power s (resp. of quotients in power s). More formally:*

$$\begin{aligned} \mathcal{Q}^\omega &:= \{Q_{K_s} \mid Q_K \in \mathcal{Q}, s \geq 1\}, \\ \mathcal{Q}^\sim &:= \{Q_{K_s^\sim} \mid Q_K \in \mathcal{Q}, s \geq 1\}. \end{aligned}$$

The obvious inclusions $\mathrm{FP}(\mathcal{Q}) \subset \mathrm{FP}(\mathcal{Q}^\omega) \subset \mathrm{FP}(\mathcal{Q}^\sim)$ are strict in general.

Two quantifiers can be regarded as semantically equivalent if they express the same structural property up to quantifier-free translations. We use the following notion of reduction:

Definition 3 *Call K'* reducible *to K and write $K' \lhd K$, (or $Q_{K'}$* reducible *to Q_K, $Q_{K'} \lhd Q_K$), if there is a tuple of quantifier-free formulae $\overline{\varphi}$ over the vocabulary of K' and of appropriate arities, such that*

$$K' = \{\mathfrak{B} \mid (B, \overline{\varphi}[\mathfrak{B}]) \in K\}.$$

We extend this notion to classes, $\mathcal{Q}' \lhd \mathcal{Q}$, to say that for all $Q_{K'} \in \mathcal{Q}'$ there is a $Q_K \in \mathcal{Q}$ such that $Q_{K'} \lhd Q_K$.

Obviously, if K' is reducible to K, then any application of quantifier $Q_{K'}$ can be replaced by an application of Q_K to formulae obtained as boolean combinations of the original ones.

Let $\mathcal{Q}_{\text{mon}}$ be the class of all quantifiers of monadic type. Consider the defining class K of such quantifier, K of type $\overline{R} = (R_1, \ldots, R_t)$ for unary relation symbols R_i. Let $\mathrm{BK}(\overline{R})$ stand for the tuple of all boolean combinations over the R_i. Then membership in K is completely determined in terms of the numerical predicate S consisting of all those tuples $(n, (m_R)_{R \in \mathrm{BK}(\overline{R})})$ of natural numbers, for which there is a structure $\mathfrak{B} = (B, R_1, \ldots, R_l) \in K$ with cardinalities $|B| = n$, $|R| = m_R$ for all R in $\mathrm{BK}(\overline{R})$. In the sense of the following definition K reduces to a monadic *cardinality quantifier*:

Definition 4 *With any numerical predicate $S \subset \mathbf{N}^{l+1}$ and a tuple of arities $r_1, \ldots, r_l$ we associate a* cardinality Lindström quantifier *Q_K with defining class*

$$K := \{(B, R_1, \ldots, R_l) \mid (|B|, |R_1|, \ldots, |R_l|) \in S\},$$

where R_i is of arity r_i. Let $\mathcal{Q}_{\text{card}}$ stand for the family of all cardinality Lindström quantifiers.

The Härtig and Rescher quantifiers, as well as the quantifier which expresses evenness, are all cardinality quantifiers, of course. Note that in the general definition there is no restriction with respect to the number of formulae bound, arities, or even recursiveness of the underlying numerical relation. Obviously $\mathcal{Q}^{\omega}_{\text{card}} = \mathcal{Q}_{\text{card}}$. The following is also straightforward.

Lemma 5 *With respect to quantifier-free reduction the relations between monadic and cardinality quantifiers are as follows:*

$$\mathcal{Q}_{\text{mon}} \lhd \mathcal{Q}_{\text{card}} \lhd \mathcal{Q}^{\omega}_{\text{mon}} \quad \textit{and} \quad \mathcal{Q}^{\sim}_{\text{mon}} \lhd \mathcal{Q}^{\sim}_{\text{card}} \lhd \mathcal{Q}^{\sim}_{\text{mon}}.$$

3 Limiting the abstracted information: Simple invariants

We next formulate certain restrictions in terms of the defining classes K in order to obtain a notion of simple or basic structural properties.

For us, an *invariant* is a functor which extracts the *relevant information* about structures, relevant in the sense that the structural property at issue can be determined in terms of the invariant.

Definition 6 *A map I defined on the class of all finite σ-structures provides an* invariant *for a class K of finite σ-structures, if membership in K only depends on the image under I: There is a subset $Im(I)^+ \subset Im(I)$ of the image of I such that $\mathfrak{A} \in K$ if and only if $I(\mathfrak{A}) \in Im(I)^+$.*

Let I be an invariant for K. Then the evaluation of a single Q_K-quantification $Q_K\overline{\varphi}$ over $\mathfrak{A}$ can be separated into three consecutive steps:

i) evaluation of $\overline{\varphi}$ in $\mathfrak{A}$ to obtain the structure of which property K is asserted:
$\mathfrak{B} := \big(A, \overline{\varphi}[\mathfrak{A}]\big)$,

ii) abstraction of the relevant information:
$\mathfrak{B} \longmapsto I(\mathfrak{B})$,

iii) decision whether $\mathfrak{B}$ has property K, as though provided by an oracle upon $I(\mathfrak{B})$.

Our criterion for simple structural properties is formulated in terms of the invariant I alone; there is in particular no restriction whatsoever on the nature of the decision procedure for K in terms of I, this is not even required to be recursive.

In our key example, namely properties based on cardinality information, the relevant invariant will just be a counting function. These invariants will be regarded as simple because any function that counts tuples is of only polynomial variation.

For the following condition certain trivial extensions, which increase the size of the structure without really adding to the structural complexity, are important. Any structure $\mathfrak{A}\dot{\cup}B$ for a pure set B is a *trivial extension* of $\mathfrak{A}$.

Definition 7 *An invariant I is* simple *if the following conditions are satisfied:*

i) $I(\mathfrak{A})$ can be computed from $\mathfrak{A}$ in polynomial time.

ii) I takes sub-exponentially many values: For all polynomials p, $\big|\,\{I(\mathfrak{A}) \mid |A| = n\}\,\big| \cdot p(n)$ is asymptotically less than 2^n.

iii) I is conservative with respect to trivial extensions: $I(\mathfrak{A}\dot{\cup}B)$, for a pure set B, only depends on $I(\mathfrak{A})$ and $|B|$, i.e. there is a function h such that for all $\mathfrak{A}$ and all B, $I(\mathfrak{A}\dot{\cup}B) = h(I(\mathfrak{A}), |B|)$.

Some comments: The essential point is ii). In general it enforces some real restriction in the information content in the passage from $\mathfrak{A}$ to $I(\mathfrak{A})$. There are, however, classes of 'simple' structures that admit a complete invariant, i.e. one which characterizes structures up to isomorphism, and still satisfying this condition. E.g. classes of monadic structures, or classes of equivalence relations, both have simple cardinality invariants which are complete. In its specific form ii) is due to technical requirements of the proof to come. Condition iii) may be regarded as a weak locality or preservation condition. Some such condition is obviously necessary to keep with the intention to model with I the abstraction of the essential information: otherwise, for PTime K, the characteristic function of K itself would of course always constitute the ideal invariant (Compare, however, the examples based on properties of rigid structures in the last section, where this condition poses no real restriction.) As it stands, however, this condition would always remain ineffective as a condition on the underlying structural property, since any Q_K is reducible to a quantifier $Q_{K'}$ whose defining class K' does not contain any trivial extensions. To remedy this, we finally make the simplicity condition robust with respect to reduction:

Definition 8 *Q_K is* based on a simple invariant *if any class K' which is reducible to K admits a simple invariant. A family $\mathcal{Q}$ of quantifiers is based on simple invariants if all its members are.*

The family $\mathcal{Q}_{\text{card}}$ of all cardinality Lindström quantifiers is based on simple invariants, namely the obvious counting functions. In fact we shall show that even $\mathcal{Q}^{\sim}_{\text{card}}$ (hence also $\mathcal{Q}^{\sim}_{\text{mon}}$) is based on simple invariants.

We think that the proposed criterion of simplicity grasps some essential aspect of the intuitive notion of a simple structural property. At any rate it provides a framework for the treatment of a very general class of properties related to cardinality invariants and counting. And of course many invariants arising naturally are of numerical type and furthermore directly related to cardinalities. But we shall also give other examples below to indicate that it covers cases essentially different from ordinary counting.

4 The main theorem

Consider a fixed-point generation in arity r over some structure which realizes few isomorphism-types of r-tuples. Then the fixed-point recursion terminates in a number of steps polynomial in the number of such types rather than in the size of the structure. In fact it suffices to consider the number of types in some L^k, first-order logic with only k variables, to obtain an even better bound. Over such structures even the following logic FP^* gains much expressive power beyond that of fixed-point logic.

With any relational structure $\mathfrak{A}$ associate the two-sorted structure $\mathfrak{A}^* := (\mathfrak{A}; (\{1, \ldots, |A|\}, <))$ consisting of a copy of $\mathfrak{A}$ itself for the first sort and the canonical ordered domain of the same size for the second sort.

Definition 9 FP^* *is just full fixed-point logic for the two-sorted variants $\mathfrak{A}^*$ of structures $\mathfrak{A}$. We allow first-order quantification in both sorts, and fixed-point constructions over mixed-sorted predicate variables.*

Obviously, FP^* is still strictly contained in PTime. In fact it corresponds to a weak fragment of Immerman's fixed-point logic with counting. This becomes evident from the natural two-sorted formalization of fixed-point with counting, see [7, 19]. We point out that fixed-point with counting can be obtained from FP^* through the extension by the Härtig or equicardinality quantifier mentioned above; this single quantifier serves in particular to equate cardinalities of definable subsets of the first sort with initial segments of the second sort and thus is appropriate for modelling *counting terms*. Fixed-point with counting in its turn does fall short of capturing PTime by the result of Cai, Fürer and Immerman [3]. The rôle of FP^* in the present context lies in the fact that it allows While-recursion over $\mathfrak{A}$ of polynomial time complexity. As observed above, FP only captures While-recursion of time complexity polynomial in the index with respect to L^k-equivalence for some k rather than in the size of the structure; see [1, 2, 5] for further analysis.

The main theorem states that the gap in recursive power between FP and FP^* cannot be bridged by the adjunction of simple structural properties:

Theorem 10 (Main Theorem)
Let $\mathcal{Q}$ be a family based on simple invariants.
Then $\mathrm{FP}(\mathcal{Q})$ does not contain PTime, in fact even $\mathrm{FP}(\mathcal{Q}) \not\supseteq \mathrm{FP}^*$.

Rather than give the details of the proof in this condensed version we just indicate the main steps. We choose trivial extensions of linearly ordered structures as one type of structures that allow to widen the gap between their size and their index with respect to automorphisms arbitrarily. Fix any relational τ containing the symbol $<$ for a linear ordering. For a linearly ordered τ-structure $\mathfrak{A}$ we assume that its universe A is $\{1, \ldots, |A|\}$ with the natural ordering. For such $\mathfrak{A}$ of size $|A| = m$, let $\mathfrak{A}|^n$ be its trivial extension to size n, $n \geq m$:

$$\mathfrak{A}|^n := \big(\mathfrak{A} \,\dot\cup\, \{m+1, \ldots, n\}, A\big)$$

regarded as a $\tau \dot\cup \{U\}$-structure for a new unary predicate U to denote the domain of the original structure. U is only for notational convenience, since it is first-order definable from $<$ in $\mathfrak{A}|^n$. Denote by $\mathcal{K}^{\mathrm{ext}}$ the class of all such extensions of linearly ordered finite τ-structures. For $\mathfrak{A} \in \mathcal{K}^{\mathrm{ext}}$ there is a fixed-point definable isomorphic embedding of the ordered structure $\mathfrak{A}$ into the ordered second sort of $\mathfrak{A}^*$. Thus, over $\mathcal{K}^{\mathrm{ext}}$ the logic FP^* is as expressive as over linearly ordered representations, whence we get:

Observation 11 *On $\mathcal{K}^{\mathrm{ext}}$, FP^* captures PTime.*

The following observations and little lemmas show that in contrast, $\mathrm{FP}(\mathcal{Q})$ on $\mathcal{K}^{\mathrm{ext}}$ can almost be reduced to $\mathrm{FP}(\mathcal{Q})$ on the underlying linearly ordered structures $\mathfrak{A}$ themselves.

Observation 12 *Any global relation R on $\mathcal{K}^{\mathrm{ext}}$ is first-order interdefinable with a finite tuple of global relations on $\mathcal{K}^{\mathrm{ext}}$ whose interpretations fall within the U-part.*

To show this, split R into disjoint parts according to the equality type of that part of the tuple, that is situated outside the U-part. Let $p(\overline{x}^{(1)})$ be an equality type in variables $\overline{x}^{(1)}$ that form a subtuple of $\overline{x} = (x_1, \ldots, x_r)$, r the arity of R. Let $\overline{x}^{(2)}$ be the tuple consisting of those components not in $\overline{x}^{(1)}$. For each such p collect the U-internal parts of the according tuples into a relation within the U-part:

$$R_p := \Big\{\overline{x}^{(2)} \in U \;\Big|\; \forall\, \overline{x}^{(1)}\big(p(\overline{x}^{(1)}) \wedge \bigwedge_{x \in \overline{x}^{(1)}} \neg U x \rightarrow R\overline{x}\big)\Big\}.$$

By the obvious automorphism argument, R is faithfully represented by the tuple of these R_p for all p and the claim follows. Observe that the original R can be defined in a quantifier-free way from U and the R_p.

It follows that all first-order and fixed-point evaluations over $\mathcal{K}^{\text{ext}}$ can be simulated by evaluations entirely over the ground structures $\mathfrak{A}$ in a uniform way. The iteration is just performed over the first-order interdefinable presentations of the recursive predicate. The resulting process is best viewed in terms of simultaneous iteration on a system of predicates (which is known to be manageable within fixed-point logic by standard techniques). The same type of simulation almost goes through for the generalized quantifiers. Any quantifier application can at least be replaced by the application of another quantifier reducible to the original one, which only need be applied to formulae ranging over the U-part. The new quantifier is the one which expresses the structural property at hand of U and the representations R_p instead of R, in the spirit of the last observation. We therefore get:

Lemma 13 *Let $\mathcal{Q}$ be closed with respect to reduction. For every sentence φ in $\mathrm{FP}(\mathcal{Q})$ there is a sentence φ^* in $\mathrm{FP}(\mathcal{Q})$ which is equivalent with φ over the class $\mathcal{K}^{\text{ext}}$ and whose subformulae only range over the U-parts of the structures in $\mathcal{K}^{\text{ext}}$.*

We further restrict the class $\mathcal{K}^{\text{ext}}$ through fixing a dependence between the the size of $\mathfrak{A}$ and its extension: Let $f\colon \mathbf{N} \to \mathbf{N}$ be any function such that $f(n) \geq n$ for all n and put

$$\mathcal{K}^f \;:=\; \left\{ \mathfrak{A}|^{f(|A|)} \;\middle|\; \mathfrak{A} \text{ linearly ordered} \right\} \subset \mathcal{K}^{\text{ext}}.$$

Similarly for any class $\mathcal{K}_0$ of linearly ordered τ-structures let $\mathcal{K}_0^f$ denote the class of the corresponding extensions to size $f(|A|)$. Choosing f sufficiently fast growing and with PTime recognizable graph, we obtain:

Observation 14 *Let $\mathcal{K}_0$ be any recursive class of linearly ordered τ-structures. Then there is an f such that $\mathcal{K}_0^f$ is in PTime, and hence FP^*-definable by Observation 11.*

Over $\mathcal{K}^f$ we exploit the fact that we can restrict the application of quantifiers to trivial extensions. For $Q_K \in \mathcal{Q}$, let $K[f]$ be the class of those structures $\mathfrak{B}$ whose trivial extensions to size $f(|B|)$ are in K: $K[f] := \left\{ \mathfrak{B} \mid \mathfrak{B}|^{f(|B|)} \in K \right\}$. Let $\mathcal{Q}[f]$ be the class of quantifiers $Q_{K[f]}$ obtained from those Q_K in $\mathcal{Q}$ in this way. Then we have:

Lemma 15 *Let $\mathcal{Q}$ be closed with respect to reduction, $\mathcal{Q}[f]$ as just introduced. Over $\mathcal{K}^f$, any sentence of $\mathrm{FP}(\mathcal{Q})$ can equivalently be expressed by a sentence of $\mathrm{FP}(\mathcal{Q}[f])$ relativized to the U-part, i.e. for $\varphi \in \mathrm{FP}(\mathcal{Q})$ there is a $\varphi^* \in \mathrm{FP}(\mathcal{Q}[f])$ such that*

$$\left\{ \mathfrak{C} \in \mathcal{K}^f \mid \mathfrak{C} \models \varphi \right\} = \left\{ \mathfrak{A}|^{f(|A|)} \mid \mathfrak{A} \models \varphi^* \right\}.$$

In particular, if, for some class $\mathcal{K}_0$ of linearly ordered structures, $\mathcal{K}_0^f$ is $\mathrm{FP}(\mathcal{Q})$-definable over $\mathcal{K}^f$, then $\mathcal{K}_0$ itself is $\mathrm{FP}(\mathcal{Q}[f])$-definable.

It follows from the condition with respect to trivial extensions, that an invariant I for K directly provides an invariant for $K[f]$. I is PTime computable, hence FP-definable over the linearly ordered structures $\mathfrak{A}$. Think of quantifier evaluations for Q_K as based on an oracle query which upon input $I(\mathfrak{B})$ answers whether or not $\mathfrak{B}|^{f(|B|)}$ is in K. It is immediate from the above that for $\mathrm{FP}(\mathcal{Q})$-definable $\mathcal{K}_0^f$ the class $\mathcal{K}_0$ itself would be recognizable in PTime relative to such oracles. Since I is of sub-exponential variation, the oracle can essentially be provided by a table of sub-exponential size, in which the actual values for $I(\mathfrak{B})$ can be looked up. In order to finish the proof, we pass to boolean circuits and make this argument precise in terms of non-uniform complexity. Here we can think of the information in the table for the sub-exponentially many values of I as being wired in directly. What remains is fixed-point definable in terms of the base structure and comparisons with this table, whence we get:

Lemma 16 *Let $\mathcal{K}_0$ be a $\mathrm{FP}(\mathcal{Q}[f])$-definable class of linearly ordered structures. Then $\mathcal{K}_0$ is recognized by a sub-exponential size non-uniform sequence of boolean circuits.*

Note that we always refer to the notion of 'sub-exponential' of the simplicity criterion which is robust with respect to multiplication by a polynomial. This is needed here, since the simulation of the PTime core of the algorithm in boolean circuits contributes a polynomial factor to its size.

With Observation 14 and Lemma 15 we obtain a contradiction from the assumption that $\mathrm{FP}(\mathcal{Q}) \supset \mathrm{FP}^*$: Any recursive class $\mathcal{K}_0$ of linearly ordered structures would be recognized by some family of boolean circuits of sub-exponential size. This is known to be false, by virtue of a diagonalization argument due to Shannon, see e.g. [22]. In fact almost all boolean functions require size at least $2^n n^{-1}$, and from the counting argument leading to this result it is easy to obtain a recursive query on ordered structures, e.g. on words, which cannot be evaluated in sub-exponential size.

5 Applications and examples

The key examples are provided by structural properties related to cardinality invariants. These and another class of examples based on properties of rigid graphs are dealt with in this section. We give brief indications of direct proofs that the corresponding quantifiers are based on simple invariants.

5.1 Cardinality invariants

We want to establish counting functions as simple invariants. In order to allow for reductions and the passage to quotients in this treatment, consider the following general situation: Let $\mathfrak{A}$ be a finite τ-structure. Consider structures of type $\{R_1, \ldots, R_l\}$ which are definably interpreted by global relations R_i over $\mathfrak{A}|^n$. Suppose f is a counting function describing some cardinality related to the R_i. The crucial condition on $f = f(\mathfrak{A}|^n)$ as a simple invariant concerns its behaviour with respect to trivial extensions, i.e. its dependence on varying n. Fixing $\mathfrak{A}$, we use the following notation: Let $\underline{A} := \{\underline{a} \mid a \in A\}$ be a set of constant symbols, one for each element of A. Any predicate R that is defined over $\mathfrak{A}|^n$ can be described in terms of the equality types of its tuples in the language $\underline{A}$. Let always R^* denote the collection of these types, i.e. of all those equality types $p(\overline{x})$ with constant names for the elements of A, for which there is a tuple $\overline{c}$ in $\mathfrak{A}|^n$ (for sufficiently large n, $n \geq$ the arities of relations considered) such that $\overline{c} \in R$ and $\overline{c}$ realizes p. Note that R^* contains the complete information about R, since any two tuples $\overline{c}$ and $\overline{c}'$ that satisfy the same equality type over $\underline{A}$ are related by an automorphism of $\mathfrak{A}|^n$. The following is just an ad-hoc convention for the proofs:

Definition 17 *Consider a numerical function defined in terms of cardinalities of definable relations R_i in $\big(\mathfrak{A}|^n, R_1, \ldots, R_l\big)$. Call this function* good *if its representation as a function of n can be determined in polynomial time upon input $(\mathfrak{A}; R_1^*, \ldots, R_l^*)$, and if – with respect to varying $(\mathfrak{A}; R_1^*, \ldots, R_l^*)$ – there are only subexponentially many different possible functions thus obtained.*

The point is, of course, that cardinalities described by good functions provide simple invariants. The following is almost obvious:

Lemma 18 *The cardinality of any global relation R is good.*

Corollary 19 *For every cardinality quantifier the ordinary counting functions provide a simple invariant. The same is true of any quantifier which is reducible to a counting quantifier. Hence, $\mathcal{Q}_{\text{card}}$ is based on simple invariants.*

Lemma 20 *If R of arity $2r$ interprets an equivalence relation $\sim_R$ over the r-th power of $\mathfrak{A}|^n$, then the number of its equivalence classes, $\big|\, (A|^n)^r / \sim_R \big|$ is good.*

The claim can be treated in an inductive way: The universe is divided with respect to equality types over $\underline{A}$, and it is determined how many new classes contribute if one more of these slices is considered. The following auxiliary statements facilitate the proof:

i) For any two equality types p_1 and p_2 over $\underline{A}$ it can be determined from R^* whether there are any two r-tuples $\overline{c}_1$ and $\overline{c}_2$ inequivalent with respect to $\sim_R$, with $\overline{c}_i$ realizing p_i, $i = 1, 2$. Furthermore, for any two realizations of p_1 the number of realizations of p_2 equivalent (resp. inequivalent) with them, is the same. In particular, either every realization of p_1 is equivalent with some realization of p_2, or none is.

ii) For $\overline{c} \in A|^n$, the size of the equivalence class of $\overline{c}$ with respect to $\sim_R$, as well as the size of its restriction to realizations of any fixed p_2, only depends on the equality type p_1 of $\overline{c}$ over $\underline{A}$. For all p_1 (and p_2) these numbers are described by good functions. Similarly, the number of inequivalence edges between two fixed types is a good function.

iii) Fixing an enumeration of all equality types over $\underline{A}$ as $p_1, p_2, \ldots, p_N$, we prove inductively that for every k, $1 \leq k \leq N$, the number of equivalence classes w.r.t. $\sim_R$ over tuples which satisfy one of the first k types, is given by a good function f_k. In the inductive step from k to $k+1$ there are two cases: Either all realizations of p_{k+1} are equivalent to realizations of types p_i, $i \leq k$; in this case $f_{k+1} = f_k$. Otherwise, no realization of p_{k+1} is equivalent to any tuple of type treated earlier; in this case the difference between f_{k+1} and f_k is just the number of equivalence classes over realizations of p_{k+1}.

These considerations immediately generalize to the case of a congruence with respect to additional monadic predicates, and therefore to the counting of tuples in a definable predicate up to a definable congruence relation. As a consequence of this last lemma we obtain:

Proposition 21 *All classes definable in terms of cardinalities of quotients w.r.t. congruences have simple invariants. Hence also all classes of equivalence relations permit simple invariants. These facts are robust w.r.t. the passage to interpretations in powers and w.r.t. reductions, so that we conclude that $\mathcal{Q}^{\sim}_{\text{mon}}$ and $\mathcal{Q}^{\sim}_{\text{card}}$ are based on simple invariants.*

We can therefore apply the main theorem to obtain:

Theorem 22 *Let $\mathcal{Q} = \mathcal{Q}_{\text{card}}$, the class of all cardinality quantifiers, or $\mathcal{Q} = \mathcal{Q}_{\text{mon}}$, the class of all quantifiers of monadic type. Even the adjunction of all structural properties obtained from those via interpretations as quotients in arbitrary arity, does not comprise all* FP^**:* $\mathrm{FP}(\mathcal{Q}^{\sim}) \not\supseteq \mathrm{FP}^*$.

In particular of course $\mathrm{FP}(\mathcal{Q}^{\sim}_{\text{card}}) \not\supseteq$ PTime. We point out that this consequence also follows from the results in [3] concerning the non-uniform adjunction of all quantifiers $\exists^{\geq m}$ to infinitary logic with bounded number of variables, $L^{\omega}_{\infty\omega}(\exists^{\geq m})_{m\geq 1}$. It is not too hard to see that all $\mathrm{FP}(\mathcal{Q}^{\sim}_{\text{card}})$ is expressible in this infinitary counting logic. The interest in the present result lies in comparison with proper fixed-point logic with counting as introduced by Immerman. Since FP^* is a weak fragment of this logic, Theorem 22 shows that within the realm of cardinality properties at least, the Lindström extension approach falls short of what can be obtained with other natural means of extensions.

5.2 Other examples: Rigid classes

A rich class of examples not related to cardinality properties in the broadest sense, is provided by all PTime properties of *rigid* structures, i.e. structures without non-trivial automorphisms. Rigidity allows to deal with the crucial condition on trivial extensions in the simplicity criterion in a simple way, so that a PTime computable characteristic function of K essentially already yields a simple invariant for Q_K and its derived quantifiers over powers and quotients. In fact we obtain simple invariants of constant rather than sub-exponential variation. Consider a PTime recognizable class K of rigid graphs. Let χ be its characteristic function. Since χ is in PTime, and any trivial extension of a graph by more than one new element must have a non-trivial automorphism, the following provides a simple invariant for K itself:

$$\mathfrak{A} \longmapsto \big(\chi(\mathfrak{A}), \chi(\mathfrak{A}|^{|A|+1})\big).$$

In the following we write $\mathfrak{B}+m$ for the trivial extension of $\mathfrak{B}$ by m new elements, $\mathfrak{B}|^{|B|+m}$, and $B+m$ for its universe. Now suppose that K' reduces to K, i.e. that there is a quantifier-free formula $\varphi(x_1, x_2)$ such that $\mathfrak{B} \in K'$ if $(B, \varphi[\mathfrak{B}]) \in K$. For the same reason as above, the map $\mathfrak{B} \longmapsto \big(\chi(B, \varphi[\mathfrak{B}]), \chi(B+1, \varphi[\mathfrak{B}+1])\big)$ is a simple invariant for K'. Thus, Q_K is based on simple invariants. The case of the variants Q_{K_s} is entirely similar. We turn to the more general case of interpretations as quotients and consider a class K' that reduces to some $K^{\sim}_s$. For some quantifier-free formulae $\varphi(\overline{x}, \overline{x}')$ and $\psi(\overline{x}, \overline{x}')$, where $\overline{x}$ and $\overline{x}'$ are s-tuples, K' is the class of structures $\mathfrak{B}$ for which $\big(B^s, \varphi[\mathfrak{B}]\big)/\psi[\mathfrak{B}]$ is a well defined graph and in K. Consider trivial extensions of $\mathfrak{B}$ by $2s$ new elements. Let, as above, $\underline{B}$ be the collection of constant names for all elements of B. Observe that $2s$ new elements suffice to realize all possible equality types over $\underline{B}$ of any two s-tuples in any trivial extension of $\mathfrak{B}$, so that the behaviour of the equivalence relation induced by ψ on any trivial extension can be determined completely from its behaviour over the $2s$-extension. Suppose that $\big((B+2s)^s, \varphi[\mathfrak{B}+2s]\big)/\psi[\mathfrak{B}+2s]$ is well defined and still rigid. Then any permutation of the new elements must be confined within ψ-classes. This then is true of any extension by more than $2s$ elements, which implies that all further extensions lead to quotients isomorphic to the one obtained in the $2s$-extension. Therefore, in this case $\chi\big(\big((B+m)^s, \varphi[\mathfrak{B}+m]\big)/\psi[\mathfrak{B}+m]\big) = \chi\big(\big((B+2s)^s, \varphi[\mathfrak{B}+2s]\big)/\psi[\mathfrak{B}+2s]\big)$ for all $m \geq 2s$. If, on the other hand, the trivial extension by $2s$ elements either does not yield a well defined quotient or yields non-trivial automorphisms in this quotient, then this same defect extends to all further extensions (note that both φ and ψ are quantifier-free). Therefore, also in this case the value of χ on the interpreted structure becomes independent of m for all $m \geq 2s$. Thus, the following provides a simple invariant for K':

$$\mathfrak{B} \longmapsto \Big(\chi\big((B+m)^s, \varphi[\mathfrak{B}+m]\big)/\psi[\mathfrak{B}+m]\big)\Big)_{0\leq m\leq 2s}.$$

This gives the following

Proposition 23 *Let $\mathcal{Q}$ be the class of all PTime quantifiers whose defining classes consist of rigid structures. Then all quantifiers in $\mathcal{Q}^{\sim}$ are based on simple invariants.*

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