SO($\forall\exists^*$) sentences and their asymptotic probabilities

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Abstract

We prove a 0-1 law for the fragment of second order logic SO($\forall\exists^*$) over parametric classes of structures which allow only one unary atomic type, confirming partially a conjecture of Lacoste. This completes the investigation of 0-1 laws for fragments of second order logic (with equality) defined in terms of first order quantifier prefixes over, e.g., simple graphs and tournaments.

We also prove a low oscillation law, and establish the 0-1 law for $\Sigma^1_4(\forall\exists^*)$ without any restriction on the number of unary types.

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1 Introduction

Kolaitis and Vardi initiated the investigation of 0-1 laws for fragments of Σ_1^1 defined in terms of first order quantifier prefixes (see [3, 4, 5]). The classification of fragments with equality was completed by Pacholski and Szwast [7] who showed that there is a $\Sigma_1^1(\forall\forall\exists)$ sentence with no asymptotic probability. Very recently, Le Bars [1] proved that there is a sentence in $\Sigma_1^1(\forall\forall\exists)$ without equality that has no asymptotic probability, resolving the last open case. Lacoste [6] has suggested extending this analysis to full second order logic. He observed that the known results for Σ_1^1 imply that $\mathrm{SO}(\forall\exists^*)$ is the maximal fragment for which the existence of a 0-1 law is an open question. He conjectured that it does have a 0-1 law, and confirmed this for $\Sigma_2^1(\forall\exists^*)$, as well as for $\mathrm{SO}(\exists^*)$.

The main result in this paper establishes a 0-1 law for $\mathrm{SO}(\forall\exists^*)$ over parametric classes of structures which allow only one unary atomic type, partially confirming the conjecture. We also prove a more general, albeit weaker result, which states that over any parametric class the difference between limsup and liminf of the probabilities of any $\mathrm{SO}(\forall\exists^*)$ sentence is strictly smaller than 1.

Concerning the proof method in this paper, we note that all of the above mentioned 0-1 laws were (at least initially) proved by 'infinitary' methods. The proofs relied on the establishment of an appropriate *transfer property*, saying that a sentence has probability 1 in the finite if and only if it holds true in a certain fixed infinite structure, the so called *countable random structure.* In contrast, in the present paper we prove our results by strictly finitistic methods. In fact, we do not know whether or not the transfer property holds for $\mathrm{SO}(\forall\exists^*)$ and the classes for which we prove a 0-1 law. (Lacoste observed that transfer holds in at least one direction: if $\varphi \in \mathrm{SO}(\forall\exists^*)$ has probability 1 over finite structures, then it is true in the countable random structure.) A related observation is that our proof makes no use of extension axioms, which play a crucial role in virtually all of the other proofs.

Another peculiarity of our approach is that we succeed in establishing a 0-1 law without being able to say whether or not the almost sure theory is decidable, where the almost sure theory is the set of sentences with asymptotic probability 1. Again, the proof methods in the field typically yield decidability as a by product of the proof of the 0-1 law. This point is discussed a bit further in the final section.

1.1 Definitions

We give a general definition of fragments of second order logic (SO) defined in terms of first order quantifier prefixes. All sentences are assumed to be in prenex normal form, with SO quantifiers preceding first order ones, and may contain the equality symbol. Finite strings over the alphabet $\{\exists, \forall, \exists^*, \forall^*\}$

denote sets of prefixes in the obvious way.

Definition 1. Let w be a finite string over the alphabet $\{\exists, \forall, \exists^*, \forall^*\}$. Then $\mathrm{SO}(w)$ is the set of all SO sentences whose first order prefix belongs to the set denoted by w.

All structures are finite and relational. Given a signature σ, the class of all finite σ-structures is denoted $\mathcal{F}_\sigma$. We use $\mathbf{A}, \mathbf{B}$, etc., for structures, and A, B, etc., for their universes. When B is a subset of the universe of $\mathbf{A}$, $\mathbf{A}_{\restriction B}$ is the substructure of $\mathbf{A}$ with universe B. Given a structure $\mathbf{A}$ and a k-ary relation symbol R, $(\mathbf{A}, R^A)$ denotes the R-expansion of $\mathbf{A}$ obtained by adding the relation $R^A \subseteq A^k$ to $\mathbf{A}$. Given a signature σ, a (*complete atomic*) *k-ary σ-type*, τ, is a maximally consistent conjunction of literals (atomic and negated atomic formulas) with free variables from $x_0, \dots, x_{k-1}$. (We only consider complete atomic types.) A k-ary *hyperedge* is a maximally consistent conjunction of literals such that in each literal, the variables that occur are exactly $x_0, \dots, x_{k-1}$.

Definition 2 (Parametric classes). A *parametric condition* is a conjunction of sentences of the following form:

$$\forall x_1 \dots x_k((\bigwedge_{i \neq j} x_i \neq x_j) \to \varphi(x_1, \dots, x_k)), \tag{1}$$

where φ is a Boolean combination of atoms, each of which must involve *all* the variables $x_1, \dots, x_k$.

A *parametric class* is a class of finite structures axiomatised by a parametric condition. We will only consider parametric classes that contain structures of every finite cardinality.

The following well known classes of structures are parametric:

- Simple graphs, axiomatised by $\forall xy(x \neq y \to (Exy \leftrightarrow Eyx)) \wedge \forall x(\neg Exx)$.
- Tournaments, axiomatised by $\forall xy(x \neq y \to \neg(Exy \leftrightarrow Eyx)) \wedge \forall x(\neg Exx)$.

Definition 3. Let σ be a relational signature. A σ-structure $\mathbf{A}$ is *loopless* if for all relations $R \in \sigma$ and all elements $a \in \mathbf{A}$, $\mathbf{A} \models \neg R(a, \dots, a)$. That is, in a loopless structure every element realizes the same (atomic) unary type.

This is a natural generalization to arbitrary signatures of the notion of a loop-free graph. A class of structures is said to be *parametric loopless* if it is parametric and contains only loopless structures. The two examples of parametric classes given above are parametric loopless. Note that parametric conditions are perfectly suited to define loopless classes of structures and it is indeed easily verifiable whether a given parametric condition defines a loopless class or not.

For a sentence φ of signature σ and a class $\mathcal{C}$ of structures we write $\mu_n^{\mathcal{C}}(\varphi)$ for the quotient

$$\frac{\left|\{\mathbf{A} \in \mathcal{C} \ / \ A = \{0, \ldots, n-1\} \text{ and } \mathbf{A} \models \varphi\}\right|}{\left|\{\mathbf{A} \in \mathcal{C} \ / \ A = \{0, \ldots, n-1\}\}\right|}.$$

If the limit $\lim_{n\to\infty} \mu_n^{\mathcal{C}}(\varphi)$ exists, we call it the *asymptotic probability of* φ *in* $\mathcal{C}$ and denote it $\mu^{\mathcal{C}}(\varphi)$.

The following experiment yields a random n-element structure from a parametric class $\mathcal{C}$ of σ-structures. For that purpose, we assume the universe of the structure to be ordered in some way (the order is *not* a part of the relational vocabulary of the structure.) Let k be the maximum arity of any relation in σ. For each $j \leq k$ and each 'ascending' tuple of j distinct elements of the universe of the structure, we draw for that tuple at random one of the equiprobable j-hyperedges consistent with condition (1). It is easy to see that this produces a structure in $\mathcal{C}$.

2 The main result and overview of the proof

In order to state the main technical result in the paper, we introduce some notation.

Let $\mathcal{C}$ be a parametric class of σ-structures in which m unary types are realized. In the case $\mathcal{C} = \mathcal{F}_\sigma$, we have that $m = 2^{|\sigma|}$. Denote the unary types $t_1, \ldots, t_m$. We say that an ordering $o := t_{i_1} \leq t_{i_2} \leq \cdots \leq t_{i_m}$, where $\{i_1, \ldots, i_m\} = \{1, \ldots, m\}$, is *realized* in a structure $\mathbf{A}$, written $\mathbf{A} \models o$, iff for all $i, j \leq m$,

$$card\{a \in A \mid a \text{ has type } t_i\} \leq card\{a \in A \mid a \text{ has type } t_j\} \ \Leftrightarrow \ o \models t_i \leq t_j.$$

Proposition 4. *Let* $\mathcal{C}$ *be a parametric class and let* $\varphi \in \mathrm{SO}(\forall\exists^*)$.

If for every ordering of the types, φ *has a model in* $\mathcal{C}$ *that realizes that ordering, then* $\mu^{\mathcal{C}}(\varphi) = 1$.

The two main theorems follow from Proposition 4 immediately.

Theorem 5 (Low Oscillation Law). *Let* φ *be a* $\mathrm{SO}(\forall\exists^*)$ σ*-sentence and* $\mathcal{C}$ *be a parametric class. Then*

$$\limsup_{n\to\infty} \mu_n^{\mathcal{C}}(\varphi) - \liminf_{n\to\infty} \mu_n^{\mathcal{C}}(\varphi) \leq 1 - 1/m! \tag{2}$$

where m *is the number of unary* σ*-types realized in* $\mathcal{C}$.

Proof. Suppose that there is an ordering of types which is not realized in any model of φ. By the symmetry of unary types, all $m!$ possible orderings of unary types are equiprobable, so that

$$\limsup_{n\to\infty} \mu_n^{\mathcal{C}}(\varphi) \leq 1 - 1/m!.$$

Otherwise every ordering of types is realized in a model of φ, and hence $\mu^{\mathcal{C}}(\varphi) = 1$ by Proposition 4, so the difference in (2) is 0, which finishes the proof. □

Theorem 6 (0-1 Law). *Over any parametric loopless class of structures* $\mathcal{C}$ *the logic* SO($\forall\exists^*$) *has a 0-1 law, i.e.,* $\mu^{\mathcal{C}}(\varphi)$ *exists for every* $\varphi \in$ SO($\forall\exists^*$) *and is either* 0 *or* 1. *In particular, this is the case for simple graphs and for tournaments.*

Moreover, $\mu^{\mathcal{C}}(\varphi) = 1$ *if and only if* φ *is satisfiable by a finite member of* $\mathcal{C}$.

Proof. In a parametric loopless class $\mathcal{C}$, only one ordering of types is possible, because there is just one type. Hence, every sentence $\varphi \in$ SO($\forall\exists^*$) that is satisfied in $\mathcal{C}$ has a model that realizes every ordering of types. So by Proposition 4 we get $\mu^{\mathcal{C}}(\varphi) = 1$, as desired.

The other implication is trivial. □

Corollary 7. *Parity cannot be defined with probability* 1 *by any sentence of* SO($\forall\exists^*$) *over parametric classes.*

Proof. For any φ with $\mu^{\mathcal{C}}(\varphi \leftrightarrow \text{parity}) = 1$ the difference in (2) should be 1. □

It is interesting to compare our low oscillation law with *slow oscillation laws*, investigated by Shelah [9] under the name of *very weak* 0-1 *laws.* A slow oscillation law asserts that the difference

$$\mu_{n+1}(\varphi) - \mu_n(\varphi)$$

approaches 0 as n goes to ∞.

If we imagine the graph of the function $n \mapsto \mu_n(\varphi)$ as a wave, the slow oscillation law asserts that the frequency of that wave is decreasing with time (and, moreover, in quite a regular way). In contrast, our low oscillation law asserts that the amplitude of the wave is strictly smaller than 1. Both kinds of laws imply undefinability of parity even in the probabilistic sense (as in Corollary 7), because parity has simultaneously the highest possible frequency and amplitude.

However, we do not feel we have found a new kind of a limit law — it is our opinion that indeed SO($\forall\exists^*$) has a 0-1 law, as conjectured by Lacoste.

Proof of Proposition 4. The starting point for the proof is the following preservation property of SO($\forall\exists^*$), proven in Lacoste [6]. In order to formulate it, we introduce some notation. We write

$$\mathbf{B} \succcurlyeq \mathbf{A}_1 + \cdots + \mathbf{A}_s$$

if $B = A_1 \cup \cdots \cup A_s$ and $\mathbf{B}_{\restriction A_i} \cong \mathbf{A}_i$ for $i = 1, \ldots, s$. In this case, we say that $\mathbf{B}$ *can be covered by* $\mathbf{A}_1, \ldots, \mathbf{A}_s$.

If in addition $A_1, \ldots, A_s$ are pairwise disjoint, we say that $\mathbf{B}$ *can be partitioned into* $\mathbf{A}_1, \ldots, \mathbf{A}_s$ and denote it

$$\mathbf{B} \succcurlyeq \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_s.$$

In case $\mathbf{A}_1 \cong \mathbf{A}_2 \ldots \cong \mathbf{A}_n$, we write $\mathbf{B} \succcurlyeq \sum_{j=1}^{n} \mathbf{A}_1$ and $\mathbf{B} \succcurlyeq \bigoplus_{j=1}^{n} \mathbf{A}_1$, respectively. The reader should remember that this notation has its meaning only as a whole: we do not specify what $\succcurlyeq$, $\mathbf{A}_1 + \cdots + \mathbf{A}_s$, or $\mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_s$ mean separately.

Theorem 8 (Lacoste). *Let $\varphi \in$ SO($\forall\exists^*$) and let $\mathbf{B}$ be a structure. Suppose $\mathbf{B} \succcurlyeq \mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_s$ and $\mathbf{A}_i \models \varphi$ for $i = 1, \ldots, s$. Then $\mathbf{B} \models \varphi$.*

In other words, any structure which can be partitioned into models of φ is a model of φ itself.

Proof. The proof is presented here for the sake of completeness. We argue by induction on the number of second order quantifiers.

For $\varphi \in$ FO($\forall\exists^*$), it is both well-known and easy to show that the statement is true. For the induction hypothesis, assume that the theorem is true for all SO($\forall\exists^*$) sentences with at most n second order quantifiers. Now suppose that $\varphi \in$ SO($\forall\exists^*$) has $n+1$ second order quantifiers, and that $\mathbf{B}$ can be partitioned into $\mathbf{A}_i, i \leq s$, such that for all i, $\mathbf{A}_i \models \varphi$. We consider two cases, depending on whether the outermost quantifier is $\exists$ or $\forall$.

One, $\varphi = \exists R\theta$. By hypothesis, for each $i \leq s$, there is an R-expansion $\mathbf{A}'_i = (\mathbf{A}_i, R^{A_i})$, such that $\mathbf{A}_i \models \theta$. Let $\mathbf{B}'$ be the R-expansion of $\mathbf{B}$ such that $R^B = \bigcup_i R^{A_i}$. It is clear that $\mathbf{B}' \succcurlyeq \mathbf{A}'_1 \oplus \cdots \oplus \mathbf{A}'_s$. By the induction hypothesis, $\mathbf{B}' \models \theta$, so that $\mathbf{B} \models \varphi$, as desired.

Two, $\varphi = \forall R\theta$. Let $\mathbf{B}' = (\mathbf{B}, R^B)$ be an R-expansion of $\mathbf{B}$. By the induction hypothesis, for all $i \leq s, \mathbf{B}'_{\restriction A_i} \models \theta$, since each $\mathbf{B}'_{\restriction A_i}$ is an R-expansion of $\mathbf{A}_i$ and $\mathbf{A}_i \models \varphi$. Since $\mathbf{B}' \succcurlyeq \mathbf{B}'_{\restriction A_1} \oplus \cdots \oplus \mathbf{B}'_{\restriction A_s}$, we have that $\mathbf{B}' \models \theta$. Since R^B was arbitrary, $\mathbf{B} \models \varphi$, completing the proof. $\square$

Note that the above proof does not extend in any obvious way to coverings. In fact, the Observation in Section 6 shows that the analagous statement for coverings rather than partitions is false.

One of our main concerns later, in Section 3, will be to extend Theorem 8 to a very special kind of coverings. It is easy to see that if $\mathcal{C}$ is parametric and $\mathbf{A} \in \mathcal{C}$, then any substructure of the form $\mathbf{A}_{\restriction B}$ is again in $\mathcal{C}$. Indeed, the parametric condition defining $\mathcal{C}$ is a universal sentence and its truth is therefore preserved under substructures. A kind of a converse of the above fact holds, too, and will be important for us.

Proposition 9. *Let $\mathcal{C}$ be a parametric class. If $\mathbf{A}_1, \ldots, \mathbf{A}_s \in \mathcal{C}$ and $B = A_1 \cup \cdots \cup A_s$, then there exists $\mathbf{B} \in \mathcal{C}$ with universe B such that $\mathbf{B}_{\restriction A_i} = \mathbf{A}_i$, for $i = 1, \ldots, s$, and the the substructures $\mathbf{B}_{\restriction A_1}, \ldots, \mathbf{B}_{\restriction A_s}$ constitute a cover for $\mathbf{B}$.*

Proof. We begin with $\mathbf{B}_0 = \mathbf{A}_1 \cup \cdots \cup \mathbf{A}_s$, i.e., with a structure whose universe and relations are unions of the respective universes and relations of the $\mathbf{A}_i$'s. $\mathbf{B}_0$ need not be a member of $\mathcal{C}$. Each of the conjuncts from (1) holds true in $\mathbf{B}_0$ if the variables x_i are restricted to come from some single $\mathbf{A}_j$, because $\mathbf{A}_j \in \mathcal{C}$. Now for every k and every tuple of l distinct elements from B which is not contained in any single A_i, we can assign to that tuple any hyperedge consistent with (1). This yields the desired structure $\mathbf{B} \in \mathcal{C}$. $\square$

From now on, when we form a structure which can be covered by structures from $\mathcal{C}$, we tacitly assume it to be enriched by additional hyperedges so that it too is a member of $\mathcal{C}$. E.g., if we say that $\mathbf{B}$ is a disjoint union of structures from $\mathcal{C}$, we mean a disjoint union with those extra hyperedges added, and which therefore is in $\mathcal{C}$.

The main part of the proof consists in establishing the following lemmas, which will be proven in the next two sections of the paper.

Lemma 10. *Let $\varphi \in \mathrm{SO}(\forall\exists^*)$ be a sentence which has a finite model $\mathbf{A} \in \mathcal{C}$ of cardinality k, where $\mathcal{C}$ is a parametric class. There exists an $n \in \mathbb{N}$ and a model $\mathbf{B} \succcurlyeq \bigoplus_{j=1}^{n} \mathbf{A}$ such that for every unary type t which is realized in $\mathbf{A}$, there exists an extension $\mathbf{C}$ of $\mathbf{B}$, $\mathbf{C} \in \mathcal{C}$, by one element of type t, which is a model of φ.*

The other lemma might be interesting for random graph theorists.

Lemma 11. *Let $\mathbf{A} \in \mathcal{C}$ be a finite structure of cardinality k with u_j elements of type t_j for $j = 1, \ldots, m$. Suppose that we have a set B of kn elements, to which unary types have already been assigned, so that there are $u_i n$ elements of type t_i, for $i = 1, \ldots, m$. With probability approaching 1 as n grows to infinity, if we draw hyperedges among elements of B, we get a structure $\mathbf{B}$ satisfying*

$$\mathbf{B} \succcurlyeq \bigoplus_{i=1}^{n} \mathbf{A}. \tag{3}$$

To give the reader an intuitive hint what the lemma asserts, first note that if (3) holds true, i.e., if $\mathbf{B}$ can be partitioned into a copies of $\mathbf{A}$, then the numbers of realizations of each of the unary types in $\mathbf{B}$ must agree with the total number of realizations of the same type in the blocks of the partition. So this agreement is a necessary condition for (3) to hold. The lemma asserts that with high probability it is also sufficient.

We now show that Proposition 4 follows in a relatively straightforward manner from these lemmas and the preservation theorem of Lacoste. Below we assume that $\mathcal{C}$ is a fixed parametric class. A model always means a model from $\mathcal{C}$.

First we show that if every ordering of unary types is realized in a model of φ, then φ has a model in which all unary types are realized precisely the same number of times.

Lemma 12. *Let o_1, o_2 be two orderings which differ only by a transposition of two consecutive types. If φ has two models $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{C}$ with $\mathbf{A}_i \models o_i$, $i = 1, 2$, then φ has a model in $\mathcal{C}$ which realizes these two orderings simultaneously. Moreover, the new model satisfies all orderings which are realized by both $\mathbf{A}_1$ and $\mathbf{A}_2$.*

Proof. Let s, t be the types which are transposed in o_1 and o_2. If the number of realizations of s and t are equal in either $\mathbf{A}_1$ or $\mathbf{A}_2$, then we are already done.

Otherwise let the numbers of their realizations be w.l.o.g. $a_1 < b_1$ in $\mathbf{A}_1$ and $a_2 > b_2$ in $\mathbf{A}_2$. Consider the disjoint union of $a_2 - b_2$ copies of $\mathbf{A}_1$ and $b_1 - a_1$ copies of $\mathbf{A}_2$, which is by Theorem 8 a model of φ. (This model can, by Proposition 9, be assumed to be from $\mathcal{C}$. The same comment applies in many places below.) It has

- $(a_2 - b_2)a_1 + (b_1 - a_1)a_2 = a_2 b_1 - a_1 b_2$ realizations of s and
- $(a_2 - b_2)b_1 + (b_1 - a_1)b_2 = a_2 b_1 - a_1 b_2$ realizations of t,

as required. Observe that the new structure obviously satisfies all the orderings satisfied simultaneously by both of the initial structures. $\square$

Starting with a set of models of φ in which all of the orderings are realized, by repeatedly applying Lemma 12, we can produce a single model of φ which satisfies all the orderings simultaneously, i.e., it has an equal number of realizations of all unary types. Denote this model $\mathbf{A}_\emptyset$.

By Lemma 10, there is a model $\mathbf{B}$ with $\mathbf{B} \succcurlyeq \bigoplus_{j=1}^{n} \mathbf{A}_\emptyset$ with the following property. For each type t_i, there is a model $\mathbf{A}_{t_i}$ of φ which is a one element extension of $\mathbf{B}$ and has an equal number of realizations of all unary types except for t_i, which is realized once more than any of the other types. Finally, let $\mathbf{A}'_\emptyset$ be the disjoint union of all $\mathbf{A}_{t_i}$.

The models $\mathbf{A}_\emptyset, \mathbf{A}'_\emptyset, \mathbf{A}_{t_1}, \ldots, \mathbf{A}_{t_m}$ are the puzzle elements from which we will assemble random structures. Let $k = |A'_\emptyset|$.

Suppose we are given a set B of n elements, where n is large. Our task is to show that a random element of $\mathcal{C}$ with universe B can be partitioned into copies of $\mathbf{A}_\emptyset, \mathbf{A}'_\emptyset, \mathbf{A}_{t_1}, \ldots, \mathbf{A}_{t_m}$.

We begin with some preparatory work, taking care of the unary types first. We start by drawing only the unary types over B, i.e., we draw (hyper)edges consisting of tuples of identical elements and, depending on the outcome, we construct a pre-partition of the universe, consisting of subsets which will later be partitioned into copies of models of φ. The only constraint we have to satisfy now is that the cardinalities of unary types agree.

W.l.o.g., we get a_i elements of type t_i for $i = 1, \ldots, m$, with $a_i \leq a_j$ for $i \leq j$. By the Law of Large Numbers, with asymptotic probability 1 we have $a_m - a_1 \leq \frac{1}{2k} n$.

First, for each $i = 2, \ldots, m$, we reserve one piece of the pre-partition for $a_i - a_1$ copies of $\mathbf{A}_{t_i}$, respecting the unary types, of course. Thus, the leftover now has an equal number of elements of each of the unary types and is therefore of cardinality lm for some l.

The total cardinality of the reserved subsets does not exceed $k(a_m - a_1) \leq n/2$, which guarantees we still have enough space to put all the additional blocks we need. Now we use $\mathbf{A}_\emptyset$ and $\mathbf{A}'_\emptyset$. Because $|\mathbf{A}'_\emptyset| = (p|\mathbf{A}_\emptyset| + 1)m$ for some p, the quotients $|\mathbf{A}_\emptyset|/m$ and $|\mathbf{A}'_\emptyset|/m$ are relatively prime. Therefore by the Chinese Reminder Theorem $l = a|\mathbf{A}_\emptyset|/m + b|\mathbf{A}'_\emptyset|/m$ for some a and b, and consequently $lm = a|\mathbf{A}_\emptyset| + b|\mathbf{A}'_\emptyset|$. We can choose both a and b to be positive, since l is large. It follows that we can create two additional blocks of our pre-partition, one for a copies of $\mathbf{A}_\emptyset$ and one for b copies of $\mathbf{A}'_\emptyset$. If we decompose some of the copies of $\mathbf{A}_\emptyset$ into copies of $\mathbf{A}_j$, we can assure the number of each of the structures we use as puzzle elements appears many times in the partition to be created, so the premise of Lemma 11 is satisfied almost surely, and consequently a random $\mathbf{B} \in \mathcal{C}$ of size n can almost surely be partitioned into isomorphic copies of models of φ. Now Theorem 8 implies that $\mathbf{B}$ almost surely is a model of φ. $\square$

3 Logical part of the proof

In this section we prove Lemma 10.

Proof. Fix a sentence φ in SO($\forall\exists^*$) and a parametric class $\mathcal{C}$.

We first want to describe a general construction that will produce, given a structure $\mathbf{A} \in \mathcal{C}$, a highly symmetric structure in $\mathcal{C}$ that can be partitioned into many copies of $\mathbf{A}$. In the special case where $\mathcal{C} = \mathcal{F}_\sigma$, this can be done as follows. Let $\mathbf{A}$ be a structure with universe $a_0, \ldots, a_k$. For $n \in \mathbb{N}$, the

n-*unfolding* of $\mathbf{A}$, $F_n(\mathbf{A})$, is the direct product $\mathbf{A} \times \mathbf{K}_n$, where $\mathbf{K}_n$ is the complete structure of size n; the universe of $F_n(\mathbf{A})$ is $\{a_{ij} | i \leq k, j < n\}$. In the general case when $\mathcal{C} \neq \mathcal{F}_\sigma$, the definition is more complicated, and will be given below. (This is because, for an arbitrary $\mathcal{C}$ and $\mathbf{A} \in \mathcal{C}$, $\mathbf{A} \times \mathbf{K}_n$ may not also be in $\mathcal{C}$.)

We observe that, for $\mathcal{C} = \mathcal{F}_\sigma$, $F_n(\mathbf{A})$ has the following properties. For all $\mathbf{A}$, $F_1(\mathbf{A}) \cong \mathbf{A}$, and for each $n \in \mathbb{N}$, $F_n(\mathbf{A}) \succcurlyeq \bigoplus_{j=1}^n \mathbf{A}$. Indeed, the substructures of $F_n(\mathbf{A})$ with universes $\{a_{0m}, \dots, a_{km}\}$, $m < n$, yield the desired partition. (Thus by Theorem 8, if $\varphi \in \mathrm{SO}(\forall\exists^*)$ and $\mathbf{A} \models \varphi$, then $F_n(\mathbf{A}) \models \varphi$, for all n.) More generally, for any set $C \subseteq F_n(A)$, $C = \{a_{0m_0}, \dots, a_{km_k}\}$, with each $m_i < n$, the structure $F_n(\mathbf{A})_{\restriction C} \cong \mathbf{A}$. Thus, $F_n(\mathbf{A})$ can be partitioned in very many ways into copies of $\mathbf{A}$.

We now give the general definition of n-unfoldings for an arbitrary parametric class $\mathcal{C}$, in such a way that they will also have the properties specified in the preceding paragraph. Although it is slightly technical, the basic idea is simply to define a highly symmetric structure much like $\mathbf{A} \times \mathbf{K}_n$ that is also in $\mathcal{C}$. Let $\mathcal{C}$ be a class of σ-structures, and t be the maximum arity of a relation in σ. For each $s, 2 \leq s \leq t$, we choose an s-hyperedge θ_s, which is consistent with $\mathcal{C}$. (That is, there is an $\mathbf{A} \in \mathcal{C}$, and an s-tuple $\vec{a} \subseteq A$ such that $\mathbf{A} \models \theta_j[\vec{a}]$.) The n-unfolding is relative to both $\mathcal{C}$ and the choice of hyperedges $\theta_s, 2 \leq s \leq t$, though this will be supressed in the notation.

Let $\mathbf{A} \in \mathcal{C}$ be a structure with universe $a_0, \dots, a_k$. For $n \in \omega$, the n-*unfolding* of $\mathbf{A}$ will be a structure $F_n(\mathbf{A})$, with universe $\{a_{ij} | i \leq k, j < n\}$, and the properties proscribed below. We lexicographically order the universe, so that $a_{ij} < a_{lm}$ iff $i < l$ or $i = l$ and $j < m$. (This is *not* added as part of the logical vocabulary). Say that two elements a_{ij} and a_{lm} are in the same row iff $i = l$. An s-tuple $(b_1, \dots, b_s) \in A$ is monotone iff for all $q < r \leq s, b_q < b_r$. It is *horizontal* if it contains two elements in the same row. Observe that $F_n(\mathbf{A})$ can be completely described by assigning, to each monotone s-tuple $\vec{a} \in F_n(A)$, $2 \leq s \leq min(k,t)$, an s-hyperedge consistent with $\mathcal{C}$.

Definition 13. Let $\mathbf{A} \in \mathcal{C}$ be a structure with universe $a_0, \dots, a_k$. The n-*unfolding* of $\mathbf{A}$ is the structure $F_n(\mathbf{A})$ with universe $\{a_{ij} | i \leq k, j < n\}$, and with hyperedges defined as follows.

1. Let $\vec{a} = (a_{i_1 j_1}, \dots, a_{i_s j_s})$ be a monotone, not horizontal s-tuple of elements. Then $\vec{a}$ realizes the same s-hyperedge type as $(a_{i_1}, \dots, a_{i_s})$ in $\mathbf{A}$.

2. Let $\vec{a} = (a_{i_1 j_1}, \dots, a_{i_s j_s})$ be a monotone and horizontal s-tuple of elements. Then $\vec{a}$ realizes the hyperedge type θ_s.

It is clear that $F_n(\mathbf{A}) \in \mathcal{C}$. One can easily verify that it also has the same properties of $\mathbf{A} \times \mathbf{K}_n$ stated above.

We now define a slightly modified unfolding. Given $\mathbf{A}$ with universe $\{a_0, \ldots, a_k\}$, let $F_n^+(\mathbf{A}, a_i)$ be the substructure of $F_{n+1}(\mathbf{A})$ with universe $F_n(\mathbf{A}) \cup \{a_{in}\}$. In other words, to obtain $F_n^+(\mathbf{A}, a_i)$ we extend $F_n(\mathbf{A})$ by adding one extra 'copy' of the element a_i. $F_n^+(\mathbf{A}, a_0) \succcurlyeq \bigoplus_{j=1}^n \mathbf{A}$ does not hold any more, but still $F_n^+(\mathbf{A}, a_0) \succcurlyeq \sum_{j=1}^{n+1} \mathbf{A}$.

Note that for l, m, with $l + m = n$, and for each $\mathbf{A}$ and $a_0 \in A$, $F_n^+(\mathbf{A}, a_0) \succcurlyeq F_l(\mathbf{A}) \oplus F_m^+(\mathbf{A}, a_0)$. Thus we have the following monotonicity property. Let $\mathbf{A}$ be a model and $\varphi \in \mathrm{SO}(\forall\exists^*)$ a sentence such that $\mathbf{A} \models \varphi$. If $F_m^+(\mathbf{A}) \models \varphi$, for some $m \in \mathbb{N}$, then for all $n \geq m$, also $F_n^+(\mathbf{A}) \models \varphi$.

Lemma 10 will follow easily from the following claim. It can be seen as an extension of Theorem 8 to a certain kind of coverings, announced after the proof of Theorem 8.

Claim 1. *Let $\mathbf{A}$ be a structure with universe $\{a_0, \ldots, a_k\}$. Then for all sentences $\varphi \in \mathrm{SO}(\forall\exists^*)$ such that $\mathbf{A} \models \varphi$, there is an $m \in \mathbb{N}$ such that for all $n \geq m$, $F_n^+(\mathbf{A}, a_0) \models \varphi$.*

Proof of claim. By the monotonicity property mentioned above, it suffices to show that for all φ, if $\mathbf{A} \models \varphi$, there is some m such that $F_m^+(\mathbf{A}, a_0) \models \varphi$. In this case, we define $g(\varphi, \mathbf{A}, a_0)$ to be the least such m. (We can stipulate $g(\varphi, \mathbf{A}, a_0) = 0$, for example, if $\mathbf{A} \not\models \varphi$.) Thus, we want to show that g is total. We argue by induction on the number of second order quantifiers.

Suppose that $\mathbf{A} \models \varphi$. For the base case, φ first order, it is easy to verify directly that $F_1^+(\mathbf{A}, a_0) \models \varphi$.

Consider the case $\varphi = \exists R\psi, \psi \in \mathrm{SO}(\forall\exists^*)$. $\mathbf{A} \models \varphi$, so there is an expansion $\mathbf{A}' = (\mathbf{A}, R^A)$ such that $\mathbf{A}' \models \psi$. By the induction hypothesis, $g(\psi, \mathbf{A}', a_0)$ is well-defined and > 0. We claim that for all $m \geq g(\psi, \mathbf{A}', a_0)$, $F_m^+(\mathbf{A}, a_0) \models \varphi$. This follows immediately from the fact that for all m, $F_m^+(\mathbf{A}', a_0)$ is an R-expansion of $F_m^+(\mathbf{A}, a_0)$.

The last case is $\varphi = \forall R\psi, \psi \in \mathrm{SO}(\forall\exists^*)$, and $\mathbf{A} \models \varphi$. Let $\{\mathbf{A}_0, \ldots, \mathbf{A}_t\}$ be the set of all R-expansions of $\mathbf{A}$, up to isomorphism. (Clearly there are only finitely many.) Observe that for all $s \leq t, \mathbf{A}_s \models \psi$; in particular, by the induction hypothesis, $g(\psi, \mathbf{A}_s, a_0)$ is well-defined and > 0. Let $M = \max\{g(\psi, \mathbf{A}_s, a_0) | s \leq t\}$.

We claim that there is some (large) N such that $F_N^+(\mathbf{A}, a_0) \models \varphi$. (Later we describe how big N must be.) Let $\vec{a}_m$ denote the tuple $(a_{1m}, \ldots, a_{km})$, so that the sets $\{a_{0m} | m \leq N\} \cup \{\vec{a}_m | m < N\}$ partition the universe of $F_N^+(\mathbf{A}, a_0)$. Observe that for each $l \leq N, m < N$, the substructure of $F_N^+(\mathbf{A}, a_0)$ on $\{a_{0l}\} \cup \vec{a}_m$ is isomorphic to the original structure $\mathbf{A}$.

It remains to show that every R-expansion $\mathbf{A}'$ of $F_N^+(\mathbf{A}, a_0)$ is a model of ψ. First, we claim that it suffices to show that there are sets $\{a_{0\alpha(i)} | i \leq M\}$ and $\{\vec{a}_{\beta(j)} | j < M\}$, α and β injective functions, so that for all $i, i' \leq M, j, j' < M - 1$, the substructures $\mathbf{A}'_{\restriction\{a_{\alpha(i)}\} \cup \vec{a}_{\beta(j)}}$ and $\mathbf{A}'_{\restriction\{a_{\alpha(i')}\} \cup \vec{a}_{\beta(j')}}$ are isomorphic. In this case, we partition $\mathbf{A}'$ into two substructures, with universes $B_1 = \{a_{0\alpha(i)} | i \leq M\} \cup \{\vec{a}_{\beta(j)} | j < M\}$ and $B_2 = F_N^+(A, a_0) - B_1$, and show that each substructure satisfies ψ. By Theorem 8, this will imply that $\mathbf{A}' \models \psi$, as desired. $\mathbf{A}'_{\upharpoonright B_2}$ is isomorphic to an R-expansion of $F_{N-M}(\mathbf{A})$, and $F_{N-M}(\mathbf{A}) \models \varphi$, so clearly $\mathbf{A}'_{\upharpoonright B_2} \models \psi$. It is easy to check that $\mathbf{A}'_{\upharpoonright B_1}$ is isomorphic to $F_M^+(\mathbf{A}_s, a_0)$, for some $s \leq t$. By the choice of M, we also have that $F_M^+(\mathbf{A}_s, a_0) \models \psi$. This establishes this claim.

Let q be the number of $k+1$-ary R-types. View the substructure $\mathbf{A}'$ as a complete bipartite graph, whose vertices are the elements $a_{0i}, i \leq N$ and the tuples $\vec{a}_j, j < N$. Let $V_1 = \{a_{0i} \mid 0 \leq i \leq N\}$ and $V_2 = \{\vec{a}_j \mid 0 \leq j < N\}$. Each 'edge' from $a_{0i} \in V_1$ to $\vec{a}_j \in V_2$ is 'colored' one of q colors, depending on the $k+1$-ary R-type of $(a_{0i}, \vec{a}_j)$. By the Ramsey's theorem for bipartite graphs, for N large enough, there is an induced bipartite subgraph, with vertex sets $V_1' \subset V_1$ and $V_2' \subset V_2$, $|V_1'| = M, |V_2'| = M - 1$, all of whose edges are the same color. It is easy to see that the substructure of $\mathbf{A}'$ with universe $V_1' \cup V_2'$ satisfies the property that, for all $a, a' \in V_1'$ and all $\vec{a}, \vec{a}' \in V_2'$, the structures $\mathbf{A}'_{\upharpoonright\{a\}\cup\vec{a}}$ and $\mathbf{A}'_{\upharpoonright\{a'\}\cup\vec{a}'}$ are isomorphic, as desired. See Figure 1 below for a pictorial representation.

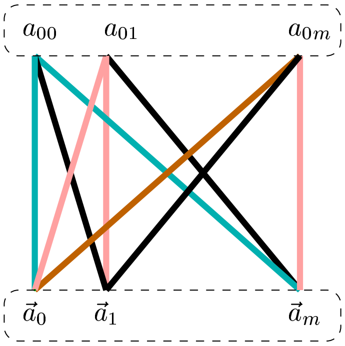

Figure 1: $\mathbf{A}'_{\upharpoonright V_1 \cup V_2}$ viewed as a colored complete bipartite graph.

We now describe how big N must be. Let N be large enough so that every complete bipartite graph $(V_1, V_2; E)$, with $|V_1|, |V_2| \geq N$, whose edges are q-colored, contains a monochromatic induced subgraph $(V_1', V_2'; E')$ with $|V_1'| = M$ and $|V_2'| = M - 1$. This completes the proof of the claim. $\square$

Lemma 10 now follows by letting the number n from the statement of the lemma be $\max\{g(\varphi, \mathbf{A}, a) \mid a \in A\}$.

4 Combinatorial part of the proof

In this section we prove Lemma 11.

Proof. The general idea is to draw the (hyper)edges between the vertices of the structure sequentially, considering k vertices at a time. If at some point they yield a copy of $\mathbf{A}$, we add them as a block of the partition we are constructing. If they fail, we rearrange them with other elements and try again, and so on. The main difficulty is to find a way of drawing edges so that we can rearrange vertices easily if we fail to get new blocks of the partition.

Our method consists of three phases. In the first one we create all but $O(\log n)$ blocks of the partition; it is based on simple cyclic rearrangements. The second phase is similar to the first one and yields at the end all but $O(\log \log n)$ blocks of the partition. It requires some backtracking, however, as the rearrangements require decomposition of some blocks created in the first phase. In the third and last phase, again using vertices from blocks created before, we finally put the remaining elements into copies of $\mathbf{A}$.

We begin by dividing B into k sets B_i, $i = 1, \ldots, k$, of cardinality n, each consisting of elements of identical unary types. We will be looking for copies of $\mathbf{A}$ that contain exactly one element from each of the sets $B_i, i \leq k$. W.l.o.g., let $B = \{1, \ldots, nk\}$ and $B_i = \{(i-1)n+1, \ldots, in\}$. We imagine the blocks of the partition to be located on k coaxial discs, which can be rotated independently. At each step of the procedure whole columns can be deleted from the discs (which occurs when the column contains an isomorphic copy of $\mathbf{A}$). We call these discs a *cylinder*. Figure 2 gives an example with $k = 3$.

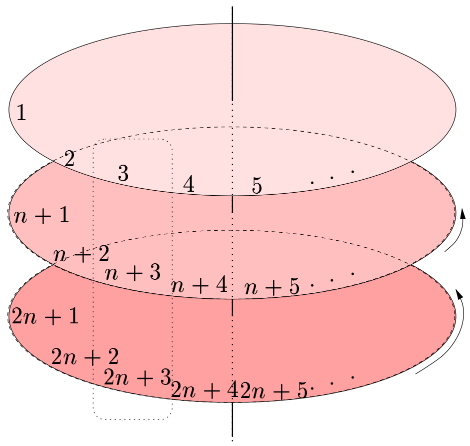

Figure 2: Decomposition of B into a cylinder made of 3 discs. One column is marked. The numbered elements move as the discs rotate.

Single step. A single step consists of the following sequence of actions applied to a cylinder. First we draw all the hyperedges joining vertices located in the same column on the discs. Then we examine the k-element substructures so obtained. Any column which yields an isomorphic copy of $\mathbf{A}$ is deleted from the cylinder and set aside as a prospective block of the partition to be created. The discs are then rotated: the second disc by 1, the third by 2, ... , the k-th disc by $k-1$. (More precisely, rotation means that we move the elements counter-clockwise to the first, second, ... , $k-1$-st, respectively, undeleted column.)

Let us analyze this single step. The probability that we get a copy of $\mathbf{A}$ within a fixed column is some fixed $p > 0$; let $q = 1 - p/2$. We use the following well known inequality of Chernoff [2], concerning the number $S_{n,p}$ of heads in n coin flips, where the probability of heads is p.

$$Pr(|S_{n,p} - np| \geq h) < 2e^{-2h^2/n}$$

Suppose that there were initially $c \geq \log_{1/q} n$ undeleted columns. We use this inequality to bound the probability that we get copies of $\mathbf{A}$ in at least $cp/2$ columns, or equivalently that at most cq columns are left after the copies of $\mathbf{A}$ are deleted. In our case, n is c, h is $cp/2$, so the probability of failure is at most

$$2e^{-2(c^2p^2/4)/c} = 2e^{-p^2c/2} \leq n^{-\delta},$$

for some $\delta > 0$, independent of c.

Phase I. Starting with initially full discs, i.e., with n columns, we repeat the above step until either some column contains two elements which have been already considered together before (so one of the discs has made a whole turn), or there are less than $\log_{1/q} n$ undeleted columns left.

Consider the first $\log_{1/q} n$ steps. By the probability bound calculated above, with asymptotic probability 1 after no step do we have more than qc undeleted columns, if we started with c undeleted columns. That is, the number of undeleted columns decreases at least as fast as nq^m, where m is the number of steps. It follows that after at most $1+\log_{1/q} n$ steps we would have no columns left at all, and consequently we stop almost surely after at most $\log_{1/q} n$ steps.

We now claim that no matter what the reason to stop iterating this procedure, with probability $1-o(1)$ we are left with at most $(k-1)\log_{1/q} n$ undeleted columns. Indeed, either we stop because the number of remaining columns is smaller than $\log_{1/q} n \leq (k-1)\log_{1/q} n$, or else because some disc has made a whole turn. Fix an element on that disc. In each step it "jumps" over at most $k-1$ of the columns which appear undeleted at the end, because the fastest of the discs rotates with velocity $k-1$ columns/step, where only undeleted columns count. That element has "jumped" over all of them in $\log_{1/q} n$ steps, so there can be at most $(k-1)\log_{1/q} n$ undeleted columns.

Summing up, with probability $1-o(1)$ we have partitioned all but $kr \leq k(k-1)\log_{1/q} n$ elements of B into copies of $\mathbf{A}$.

Phase II. To begin, we want to make k new cylinders. These will contain all the elements remaining on the cylinder at the end of Phase I, together with some elements from the copies of $\mathbf{A}$ created in the very first step of Phase I. With probability $1-o(1)$, there are at least $(1-q)n$ of these latter elements.

We take $k-1$ groups of elements from these copies of $\mathbf{A}$, each group consisting of r blocks of the partition. The blocks are disassembled and are combined with the vertices remaining after Phase I into k new cylinders, of k discs each, in the following way. Each cylinder has one disc made of elements taken from one disc of the old Phase I cylinder; the remaining discs are filled with elements taken from the disassembled copies of $\mathbf{A}$. The method we use is shown on Fig. 3. The principle is that each cylinder contains exactly one disc created from elements taken from one group, where we also consider the undeleted columns as a group. Also, each element from the k new cylinders is located on a disc of the same height as it was in Phase I, thereby ensuring compatability of the atomic types. This construction guarantees that we will never consider any tuple of vertices containing elements that were already considered together in the previous phase.

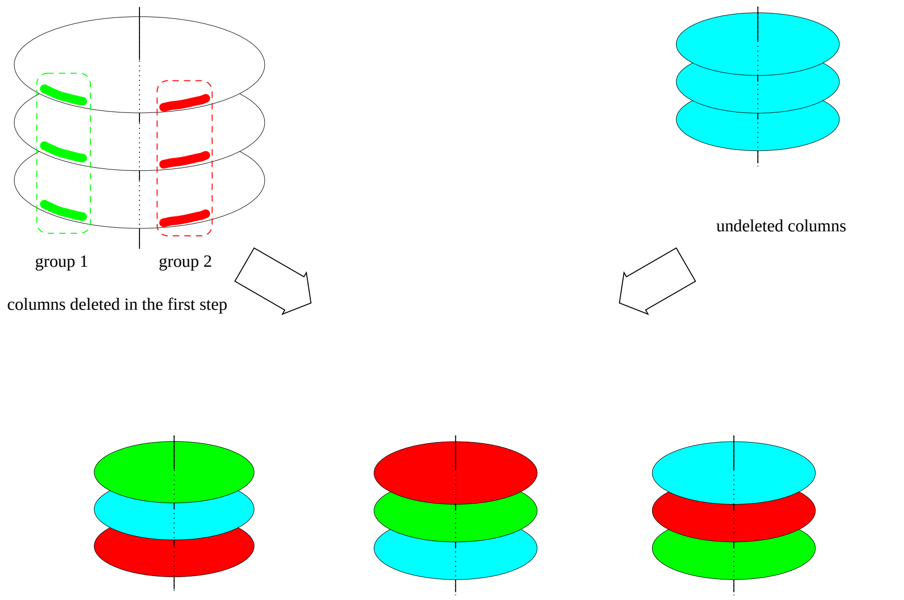

Figure 3: Reorganization of the old disc with groups of disassembled copies of $\mathbf{A}$.

Applying the previous argument to each of the new cylinders separately, where the initial number of columns n is now replaced by r, with probability $1-o(1)$ finally we are left with

$$R \leq k(k-1)\log_{1/q} r \leq k(k-1)\log_{1/q}[(k-1)\log_{1/q} n] = O(\log_{1/q}\log_{1/q} n)$$

undeleted columns (the total number in all the cylinders). All remaining vertices are organized into copies of $\mathbf{A}$.

Phase III. In the final part of the procedure, we always dissolve $R(k-1)$ blocks from those obtained in the very first step of the first phase, organize with the remaining $R(k-1)$ vertices into discs and draw edges. But now we play "everything or nothing". If we do not get copies of $\mathbf{A}$ in all columns, we put the vertices taken from disassembled copies of $\mathbf{A}$ back in their old configurations. We then try with new ones, until we get copies of $\mathbf{A}$ in all the columns, in which case we have partitioned all of B into copies of $\mathbf{A}$. The probability of such a success in one attempt is $p^{R(k-1)} = p^{O(\log_{1/q}\log_{1/q} n)}$. Then the probability we fail in all the attempts is

$$\left(1 - p^{O(\log_{1/q}\log_{1/q} n)}\right)^{\#\text{ of attempts}} \leq \left(1 - (1/\log n)^{O(1)}\right)^{\#\text{ of attempts}}.$$

Consequently, in order to show that we almost surely succeed, it is enough to show that $(\log n)^{O(1)} = o(\#\text{ of attempts})$. But this is obvious, since we can try at least $(pn/2 - O(\log n))/O(\log_{1/q}\log_{1/q} n)$ times.

We have shown that after the three phases with probability $1-o(1)$ we have a partition of $\mathbf{B}$ into disjoint copies of $\mathbf{A}$. This completes the proof of the lemma. $\square$

5 $\Sigma^1_4(\forall\exists^*)$

In this section, we prove that $\Sigma^1_4(\forall\exists^*)$ has a 0-1 law, with no restriction on unary types. This improves Lacoste's result who showed the same for $\Sigma^1_2(\forall\exists^*)$.

Theorem 14. *The logic $\Sigma^1_4(\forall\exists^*)$ has a* 0-1 *law.*

Note that we do not have here the nice additional property we had in the 0-1 law of Corollary 6. Indeed, there are satisfiable sentences of $\Sigma^1_4(\forall\exists^*)$ which nevertheless have asymptotic probability 0. A simple example is the sentence $\forall x \neg Exx$, asserting that there are no loops.

We prove the theorem over classes of the form $\mathcal{F}_\sigma$, but the proof extends immediately to arbitrary parametric classes. By remarks in Section 2 following Theorem 8, it suffices to prove the next proposition.

Proposition 15. *Let $\varphi \in \Sigma_4^1(\forall\exists^*)$ be a sentence with signature σ. If φ has a finite model $\mathbf{A}$ in which every unary atomic σ-type is realized, then for every ordering o of σ-types, there is a finite structure $\mathbf{B}$ such that $\mathbf{B} \models \varphi$ and $\mathbf{B} \models o$.*

The proposition follows immediately from (iterated applications of) the next lemma.

Lemma 16. *Let $\varphi \in \Sigma_4^1(\forall\exists^*)$ be a σ sentence, $\{t_1, \ldots, t_m\}$ the set of σ-types, and $o := t_1 \leq t_2 \leq \ldots \leq t_m$. Suppose that there is a finite structure $\mathbf{A}$ such that $\mathbf{A} \models \varphi$, every type $t_j, j \leq m$, is realized in $\mathbf{A}$, and $\mathbf{A} \models o$. Then for each $l < m$, there is a finite structure $\mathbf{B}$ such that $\mathbf{B} \models \varphi$, every type $t_j, j \leq m$, is realized in $\mathbf{B}$, and $\mathbf{B} \models o'$, where $o' := t_1 \leq \ldots \leq t_{l-1} \leq t_{l+1} \ldots \leq t_m \leq t_l$.*

As in the proof of Lemma 10, we require a modification of the unfolding of a structure, which permits the addition of arbitrarily many extra copies of a single element. Given a finite structure $\mathbf{A}$ with universe $\{a_0, \ldots, a_k\}$, and numbers $n, p, n < p$, let $F_n^p(\mathbf{A}, a_0)$ be the substructure of $F_p(\mathbf{A})$ with universe $F_n(A) \cup \{a_{0n}, \ldots, a_{0(p-1)}\}$. Observe that $F_n^{n+1}(\mathbf{A}, a_0) = F_n^+(\mathbf{A}, a_0)$.

Lemma 16 is a consequence of the next lemma.

Lemma 17. *Let φ and $\mathbf{A}$ be as above, $|A| = k$, and let t_l be a σ-type. Let $\{a_0, \ldots, a_{k-1}\}$ be the universe of $\mathbf{A}$, such that a_0 realizes t_l. Then there are n, p such that $p \geq nk$ and $F_n^p(\mathbf{A}, a_0) \models \varphi$.*

This lemma now follows from the sequence of sublemmas given below.

Sublemma 18. *Let $\varphi \in \Pi_1^1(\forall\exists^*)$, $\mathbf{A} \models \varphi$, and $a_0 \in A$. Then for all p, $F_1^p(\mathbf{A}, a_0) \models \varphi$.*

This is an easy consequence of the fact that for all $\theta \in \mathrm{FO}(\forall\exists^*)$ (and hence all $\theta \in \Pi_1^1(\forall\exists^*)$) and all $\mathbf{B}$, if $\mathbf{B}$ is the (not necessarily disjoint) union of structures $\mathbf{B}_i, i \in I$, such that for all i, $\mathbf{B}_i \models \theta$, then $\mathbf{B} \models \theta$.

Sublemma 19. *Let $\varphi \in \Sigma_2^1(\forall\exists^*)$, $\mathbf{A} \models \varphi$, and $a_0 \in \mathbf{A}$. Then for all p, $F_1^p(\mathbf{A}, a_0) \models \varphi$.*

Let $\varphi = \exists \bar{R}_1 \forall \bar{R}_2 \theta, \theta \in \mathrm{FO}(\forall\exists^*)$. By hypothesis, there is an $\bar{R}_1$-expansion $\mathbf{A}' = (\mathbf{A}, \bar{R}^A)$ such that $\mathbf{A}' \models \forall \bar{R}_2 \theta$. By Sublemma 18, $F_1^p(\mathbf{A}', a_0) \models \forall \bar{R}_2 \theta$. Since $F_1^p(\mathbf{A}', a_0)$ is an $\bar{R}_1$-expansion of $F_1^p(\mathbf{A}, a_0)$, $F_1^p(\mathbf{A}, a_0) \models \varphi$.

To prove the next sublemma, we need a slight generalization of the previous one. Let $\mathbf{A}$ have universe $a_0, \ldots, a_{k-1}$, and let $\mathbf{B}$ have universe $\{b_{00}, \ldots, b_{0(p-1)}\} \cup \{b_1, \ldots, b_{k-1}\}$. Say that $\mathbf{B}$ is *near to* $F_1^p(\mathbf{A})$ iff for all $q \leq p-1$, $\mathit{type}(b_{0q}, b_1, \ldots, b_{k-1}) = \mathit{type}(a_0, a_1, \ldots, a_{k-1})$. This implies that $\mathbf{B}$ can be covered with copies of $\mathbf{A}$, in a uniform way, though it may not actually be isomorphic to $F_1^p(\mathbf{A})$, because, for example, $type(b_{0m}, b_{0n})$ need not equal $type(a_{0m}, a_{0n})$.

The proof of the following claim is essentially that of the preceding sublemma. For all $\varphi \in \Sigma_2^1(\forall\exists^*)$, and all $\mathbf{A}$, if $\mathbf{A} \models \varphi$ and $\mathbf{B}$ is near to $F_1^p(\mathbf{A})$, then also $\mathbf{B} \models \varphi$.

Sublemma 20. *Let $\varphi \in \Pi_3^1(\forall\exists^*)$, $\mathbf{A} \models \varphi$, $|A| = k$, and $a_0 \in \mathbf{A}$. Then there are $n, p, p \geq nk$, such that $F_n^p(\mathbf{A}, a_0) \models \varphi$.*

This is the only Sublemma with a non-trivial proof; it involves a straightforward counting argument. Let $\varphi = \forall \bar{R}_1 \theta, \theta \in \Sigma_2^1(\forall\exists^*)$. We want to show that there are $n, p, p \geq nk$, such that for every $\bar{R}_1$-expansion $\mathbf{A}' = \left(F_n^p(\mathbf{A}, a_0), \bar{R}_1^A\right)$ of $F_n^p(\mathbf{A}, a_0)$, there are $\mathbf{B}_1, \ldots, \mathbf{B}_t$, such that $\mathbf{A}' \succcurlyeq \mathbf{B}_1 \oplus \ldots \oplus \mathbf{B}_t$ — where $\mathbf{B}_t \cong (F_m(\mathbf{A}), \bar{R}_1)$, $m \leq n$, and for all $s < t$, $\mathbf{B}_s$ is near to $F_1^{p_s}(\mathbf{A}_s)$, $p_s \leq p$, $\mathbf{A}_s$ an $\bar{R}_1$-expansion of $\mathbf{A}$. By the preceding claim and Theorem 8, this implies that $F_n^p(\mathbf{A}) \models \varphi$.

Let q be the number of complete atomic k-ary $\bar{R}_1$-types, and set $r = (q-1)/q$. Choose n to be large enough (and even) so that $r^{n/2} \leq 1/2k$, and let $p = nk$. As in the proof of Lemma 10, for $j \leq n-1$, let $\vec{a}_j = \{a_{1j}, \ldots, a_{(k-1)j}\}$. Define $C = \{a_{00}, \ldots, a_{0(p-1)}\} \subseteq A$.

Now let $\mathbf{A}'$ be an arbitrary $\bar{R}_1$-expansion of $F_n^p(\mathbf{A})$. We will define the structures $\mathbf{B}_s \subseteq \mathbf{A}', s < t$, inductively. Because there are q many k-ary $\bar{R}_1$-types, there is a set $C_1 \subseteq C$ of size $\geq p/q$ such that for all $c, c' \in C$, $(c, \vec{a}_0)$ and $(c', \vec{a}_0)$ realize the same type in $\mathbf{A}'$.

We consider two cases. One, $|C_1| \geq p - (n-1)$. Let D be an arbitrary subset of C of size $p - (n-1)$, and let $\mathbf{B}_1$ be the substructure of $\mathbf{A}'$ with universe $B_1 = D \cup \vec{a}_0$, $\mathbf{B}_2$ the substructure with universe $B_2 = A' \setminus B_1$. Clearly $\mathbf{B}_1$ is near to $F_1^{p-(n-1)}(\mathbf{A}_1)$, where $\mathbf{A}_1$ is an $\bar{R}_1$-expansion of $\mathbf{A}$, and $\mathbf{B}_2$ is (isomorphic to) an $\bar{R}_1$-expansion of $F_{n-1}(\mathbf{A})$. Thus $\mathbf{A}' \succcurlyeq \mathbf{B}_1 \oplus \mathbf{B}_2$, and $\mathbf{B}_1, \mathbf{B}_2$ satisfy the conditions specified at the beginning of the argument.

Otherwise, $|C_1| < p - (n-1)$. In this case, we define $\mathbf{B}_1$ to be the substructure with universe $B_1 = C_1 \cup \vec{a}_0$. Observe again that $\mathbf{B}_1$ is near to $F_1^{|C_1|}(\mathbf{A}_1)$, $\mathbf{A}_1$ an $\bar{R}_1$-expansion of $\mathbf{A}$.

We now iterate this construction. Suppose that we have defined disjoint structures $\mathbf{B}_1, \ldots, \mathbf{B}_d$, such that for each $\mathbf{B}_i, i \leq d$, $B_i = C_i \cup \vec{a}_i$, $C_i \subseteq C$, and $\mathbf{B}_i$ near to $F_1^{|C_i|}(\mathbf{A}_i, a_0)$, $\mathbf{A}_i$ an $\bar{R}_1$-expansion of $\mathbf{A}$. Moreover, for each $i \leq d$, $|C_i| \geq \frac{1}{q}(p - |\bigcup_{j<i} C_j|)$ and $|\bigcup_{j<i} C_j| < p - (n-d)$. In particular, observe that for all $i \leq d$, $|C \setminus (\bigcup_{j \leq i} C_j)| \leq r^i p$. As above, there is a set $C_{d+1} \subseteq C \setminus (\bigcup_{j \leq d} C_j)$ of size $\geq \frac{1}{q}|C \setminus (\bigcup_{j \leq d} C_j)|$ such that for all $c, c' \in C_{d+1}$, $(c, \vec{a}_0)$ and $(c', \vec{a}_0)$ realize the same type in $\mathbf{A}'$. The construction divides again into two cases, depending on whether or not $|\bigcup_{j \leq d+1} C_j| \leq p - (n - (d+1))$.

Case 1 Suppose that $|\bigcup_{j \leq d+1} C_j| \geq p - (n - (d+1))$. In this case, we choose some subset $D \subseteq C_{d+1}$ such that $|D \cup \bigcup_{j \leq d} C_j| = p - (n - (d+1))$, and let $\mathbf{B}_{d+1}$ be the substructure of $\mathbf{A}'$ with universe $D \cup \vec{a}_d$. It is then clear that, as in the base case, letting $\mathbf{B}_{d+2}$ be the substructure with universe $A' \setminus \bigcup_{j \leq d+1} B_j$, the structures $\mathbf{B}_i, i \leq d+2$, satisfy the conditions specified above.

Case 2 Otherwise, $|\bigcup_{j \leq d+1} C_j| < p - (n - (d+1))$. We let $\mathbf{B}_{d+1}$ be the substructure with universe $C_{d+1} \cup \vec{a}_d$. Observe that the set of structures $\mathbf{B}_1, \ldots, \mathbf{B}_{d+1}$ again satisfy the inductive hypothesis, so we can repeat this procedure.

Finally, we claim that there is some $l \leq n/2$ such that the structures $\mathbf{B}_1, \ldots, \mathbf{B}_{l+1}$ satisfy the conditions stated at the beginning of the proof. This is because for all i, $|C \setminus (\bigcup_{j \leq i} C_j)| \leq r^i p$, so that for some $l \leq n/2$,

$$p - |\bigcup_{j \leq l} C_j| = |C \setminus (\bigcup_{j \leq l} C_j)| \leq r^l p \leq \frac{1}{2k} p = \frac{n}{2} \leq n - l$$

(by the choice of n). Thus, $|\bigcup_{j \leq l} C_j| \geq p - (n-l)$. So the inductive process described above will stop for this l, as we will be in the Case 1.

Sublemma 21. *Let $\varphi \in \Sigma^1_4(\forall\exists^*)$, $\mathbf{A} \models \varphi$, $|A| = k$, and $a_0 \in \mathbf{A}$. Then there are $n, p, p \geq nk$, such that $F^p_n(\mathbf{A}, a_0) \models \varphi$.*

This Sublemma follows from Sublemma 20 as Sublemma 19 did from Sublemma 18. Let $\varphi = \exists \bar{R}_1 \theta$, $\theta \in \Pi^1_3(\forall\exists^*)$. By hypothesis, there is an R_1-expansion $\mathbf{A}'$ of $\mathbf{A}$ such that $\mathbf{A}' \models \theta$. By Sublemma 20, there are $n, p, p \geq nk$, such that $F^p_n(\mathbf{A}', a_0) \models \theta$. Therefore $F^p_n(\mathbf{A}, a_0) \models \varphi$.

This completes the proof of Lemma 17.

6 Further remarks

Spectra. Recall that the *spectrum* of a sentence φ is the set of natural numbers n such that φ has a model of cardinality n. The next observation follows easily from Lemma 10, using the Chinese Remainder Theorem.

Corollary 22. *The spectrum of an* SO($\forall\exists^*$) *sentence is either empty or cofinite.*

We observe that not every spectrum of an SO($\forall\exists^*$) sentence is a continuous interval of the natural numbers.

Observation. *There is a (purely monadic)* SO($\forall\exists^*$) *sentence that has a model of cardinality* 2 *but not of cardinality* 3.

Let P be a unary predicate. Consider the following sentence.

$$\forall R \exists S \forall x \exists y \big((Px \to (Rx \leftrightarrow (\neg Py \wedge Sy))) \wedge (\neg Px \to (Rx \leftrightarrow (Py \wedge Sy))) \big)$$

It is easy to verify that this sentence has the desired property.

On the other hand, the spectrum of any $\Sigma^1_2(\forall\exists^*)$ sentence is continuous. For example, if $\varphi \in \Sigma^1_2(\forall\exists^*)$ and $\mathbf{A} \models \varphi$, then for every $a \in A$, the structure $F_1^+(\mathbf{A}, a)$, (using the notation from the proof of Lemma 10), has cardinality $|\mathbf{A}|+1$ and $F_1^+(\mathbf{A}, a) \models \varphi$. Therefore the above example is optimal in terms of quantifier prefix.

Easy observations about $\mathrm{SO}(\exists\forall^*)$ It follows immediately from Theorem 8 that an $\mathrm{SO}(\exists\forall^*)$ sentence φ has a model iff it has a model of cardinality 1. Therefore this logic has a decidable satisfiability problem and has the finite model property.

Observe that as $\mathrm{SO}(\forall\exists^*)$ and $\mathrm{SO}(\exists\forall^*)$ are closed under SO quantification, every spectrum is the spectrum of a sentence over the empty signature. Thus the spectra of $\mathrm{SO}(\exists\forall^*)$ sentences are exactly the complements of spectra of $\mathrm{SO}(\forall\exists^*)$ sentences. By Corollary 22, every $\mathrm{SO}(\exists\forall^*)$ spectrum is either finite or $\mathbb{N}$.

We also observe that if $\varphi \in \mathrm{SO}(\exists\forall^*)$ has an infinite model $\mathbf{A}$, then $\mathit{spectrum}(\varphi) = \mathbb{N}$. For each $n \in \mathbb{N}$, $\mathbf{A}$ can be partitioned into infinitely many disjoint substructures, each of size n. If every such substructure were a model of $\neg\varphi$, then by Theorem 8, $\mathbf{A} \models \neg\varphi$ also. Therefore φ has a model of size n.

7 Final remarks

The most obvious question left open is Lacoste's original one: does $\mathrm{SO}(\forall\exists^*)$ have a 0-1 law over arbitrary (parametric) classes of structures? Also, as mentioned in the introduction, even where we have established a 0-1 law, we do not know whether the almost sure theory is decidable. In fact, it is open whether the satisfiability problem for $\mathrm{SO}(\forall\exists^*)$ is decidable, and whether the logic has the finite model property. It would be interesting to know the answer to these questions since it has been observed that for fragments L of Σ^1_1 with equality, defined in terms of first order prefix, L has a 0-1 law iff the satisfaction problem for L is decidable iff L has the finite model property. Furthermore, for every such L that has a 0-1 law, the almost sure theory of L is decidable. Without equality there is an exception — the class $\Sigma^1_1(\forall\forall\exists)$, mentioned in the introduction.

Beyond Σ^1_1, Lacoste observed that $\mathrm{SO}(\exists^*)$ has a 0-1 law: this follows from the fact that every sentence in $\mathrm{SO}(\exists^*)$ defines a class that is closed under extensions. It is well-known that $\mathrm{SO}(\exists^*)$ has the finite model property; more recently it was shown in [8] that its satisfiability problem is decidable. Because a sentence in $\mathrm{SO}(\exists^*)$ has probability one iff it has a finite model, together these results also imply the decidability of the almost sure theory of $\mathrm{SO}(\exists^*)$.

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