Fragments of Existential Second-Order Logic and Logics with Team Semantics

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Teams

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An assignment is a function $s : \{x_1, x_2, \dots\} \longrightarrow A$.

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Team: a *set* X of assignments over the same domain

$$\begin{array}{c|ccc} X & x_1 & x_2 & x_3 \\ \hline s_1 & s_1(x_1) & s_1(x_2) & s_1(x_3) \\ s_2 & s_2(x_1) & s_2(x_2) & s_2(x_3) \\ s_3 & s_3(x_1) & s_3(x_2) & s_3(x_3) \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

A team X can be viewed as a relation $X(x_1, x_2, x_3)$.

Dependency Concepts

Let X be a team.

Dependence atoms:

$\mathfrak{A} \models_X \mathrm{dep}(\bar{x}, y) \iff$ for all $s, s' \in X$, $s(\bar{x}) = s'(\bar{x})$ entails $s(y) = s'(y)$

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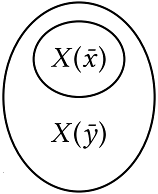

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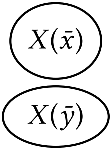
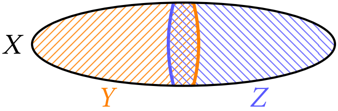


Team Semantics

It is possible to evaluate FO-formulae (in NNF) with teams.

Disjunctions in Team Semantics

$$\mathfrak{A} \models_X \varphi_1 \vee \varphi_2 \iff \mathfrak{A} \models_Y \varphi_1 \text{ and } \mathfrak{A} \models_Z \varphi_2 \text{ for some } Y \cup Z = X$$



Definitions for $\forall, \exists, \wedge$ are without big surprises!
FO-Literals are checked against all assignments.

FO($\mathcal{D}$) is FO extended with $\mathcal{D}$-atoms.

Known Connections between these Logics

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Existential Second-Order Logic

Σ^1_1 = FO (in negation normal form) + the following quantifiers:

$\exists R \varphi(R)$ where R is a relation symbol

Normalform: $\exists \bar{R} \varphi(\bar{R})$ where $\varphi(\bar{R}) \in$ FO

Comparing Team-Semantics-Logics with Tarski-Logics:

$\varphi(\bar{x})$ is *equivalent* to $\psi(X)$, if and only if

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Closure Properties

Let $\varphi(\bar{x})$ be a formula of a logic with team semantics.
Let $\psi(X)$ be a sentence with Tarski semantics.
Downwards Closure: Formula is downwards closed, if

$$\mathfrak{A} \models_X \varphi, Y \subseteq X \implies \mathfrak{A} \models_Y \varphi.$$

$$\mathfrak{A} \models \psi(X), Y \subseteq X \implies \mathfrak{A} \models \psi(Y).$$

Union Closure: Formula is closed under unions, if

$$\mathfrak{A} \models_{X_i} \varphi \text{ for all } i \in I \implies \mathfrak{A} \models_{\bigcup_{i \in I} X_i} \varphi.$$

$$\mathfrak{A} \models \psi(X_i) \text{ for all } i \in I \implies \mathfrak{A} \models \psi(\bigcup_{i \in I} X_i).$$

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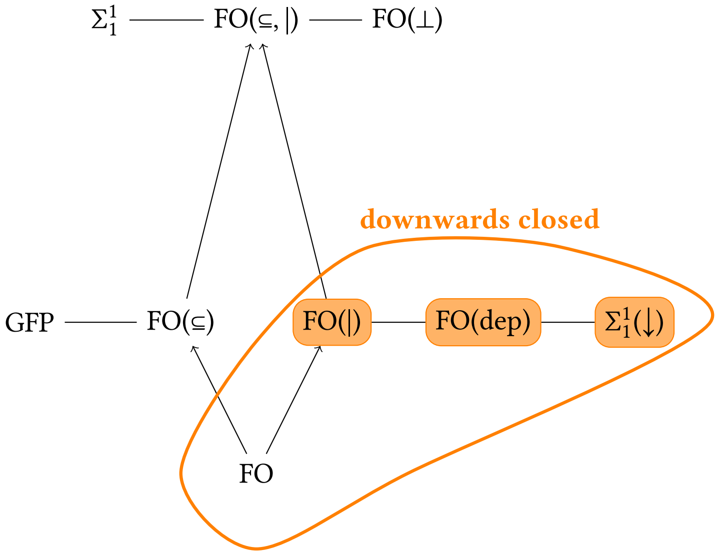

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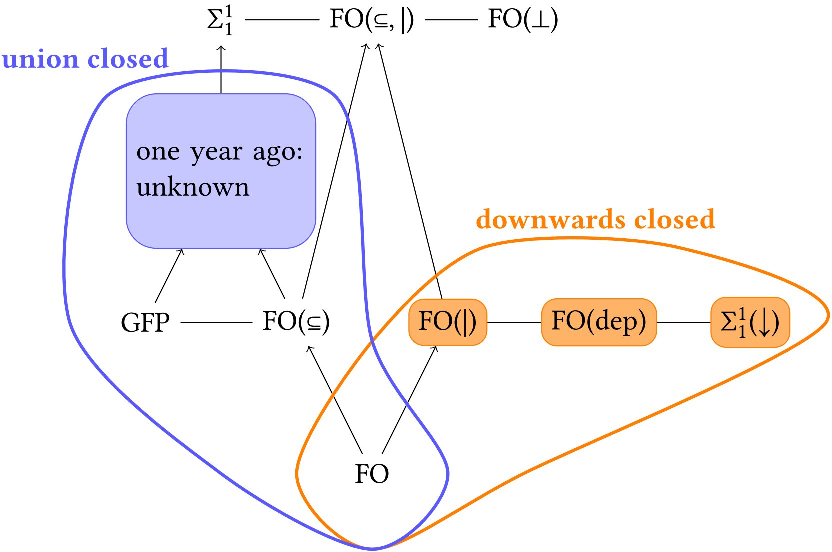

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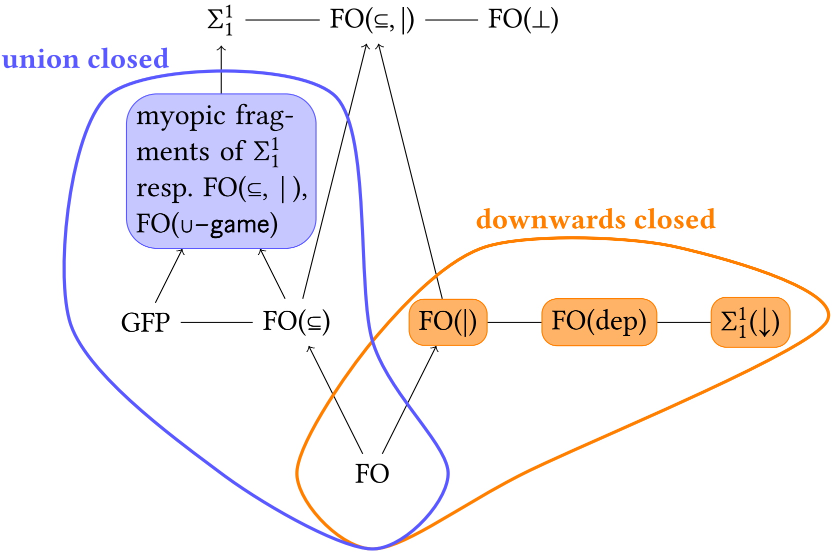

Contributions

1. Syntactic characterisations for closure properties and model-checking games for Σ_1^1
2. Rönnholm's question regarding the connection between inclusion logic of bounded arity and greatest fixed-point logics
3. Logics with dependency concepts up to a given equivalence

Characterisation of the Union Closed Fragment

Joint work with Richard Wilke

Let $\varphi(X) \in \Sigma_1^1$. Then the following are equivalent:

1. $\varphi(X)$ is union closed.
2. $\varphi(X)$ is equivalent to some myopic Σ_1^1-sentence.
3. $\varphi(X)$ is equivalent to some $\bar{x}$-myopic $\mathrm{FO}(\subseteq, \mid)$-formula.
4. $\varphi(X)$ is equivalent to some $\mathrm{FO}(\cup\!-\!\mathsf{game})$-formula.

Myopic Σ_1^1-Sentences

Myopic Σ_1^1-sentences are of the form

$$\forall \bar{x}(X\bar{x} \longrightarrow \exists \bar{R}\psi(X, \bar{R}, \bar{x}))$$

where X occurs only positively in $\psi \in \mathrm{FO}$.

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It is easy to prove that Σ_1^1-myopic sentences are closed under unions. If $\varphi(X)$ is closed under unions, then $\varphi(X)$ is equivalent to

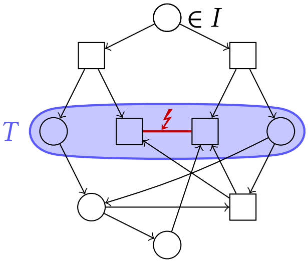
$$\forall \bar{x}(X\bar{x} \longrightarrow \exists Y(Y \subseteq X \wedge Y\bar{x} \wedge \varphi(Y))).$$

Inclusion-Exclusion Games

An **inclusion-exclusion game** is a structure

$$\mathcal{G} = (V, V_0, V_1, I, E, T, E_{ex})$$

where $V = V_0 \uplus V_1$.



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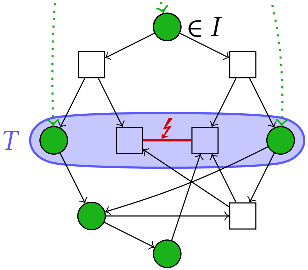

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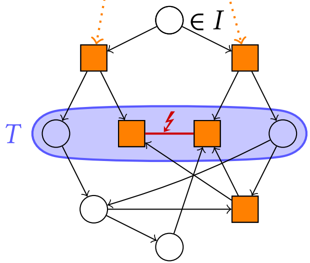

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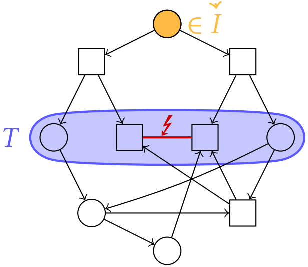

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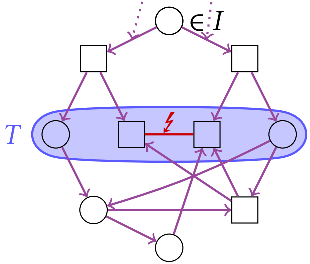

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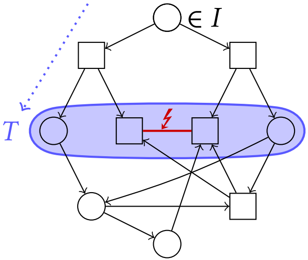

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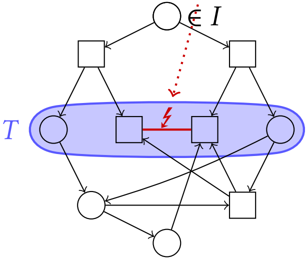

Winning Strategies and Target Sets

A **winning strategy** in $\mathcal{G}$ for 0 is a subgraph $\mathcal{S} := (W, F) \subseteq (V, E)$ s.t.

1. For every $v \in W \cap V_0$, $\mathcal{S}$ plays **at least one** outgoing edge of v.
2. For every $v \in W \cap V_1$, $\mathcal{S}$ plays **all** outgoing edges of v.
3. $I \subseteq W$
4. $(W \times W) \cap E_{\text{ex}} = \emptyset$

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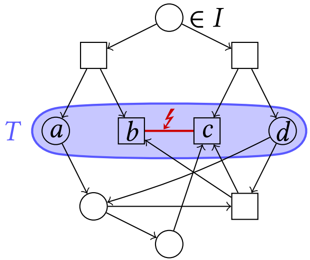

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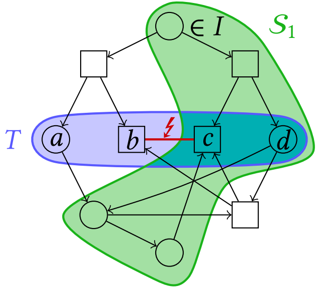

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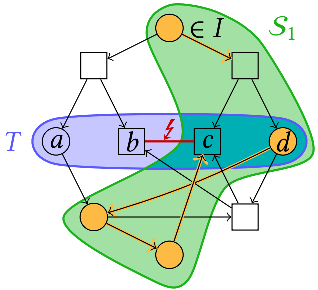

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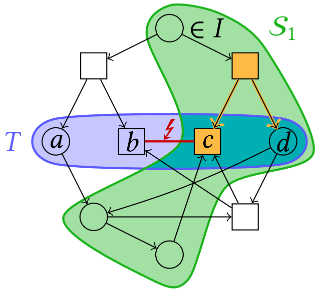

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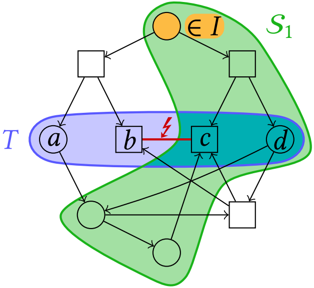

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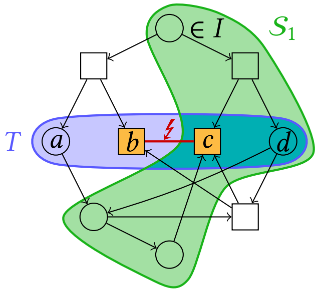

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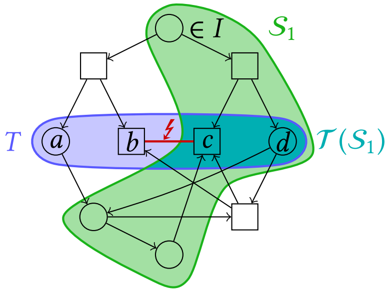

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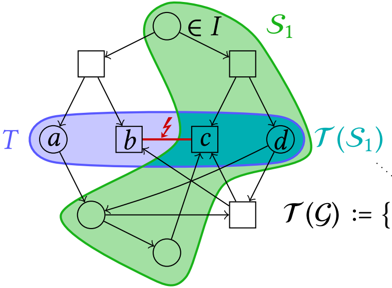

$\mathcal{T}(\mathcal{G}) := \{\{c, d\},$

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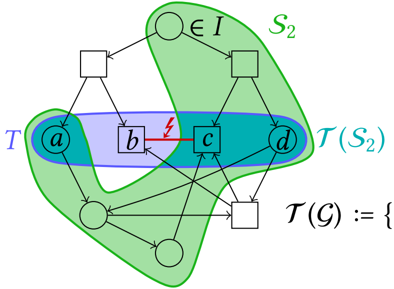
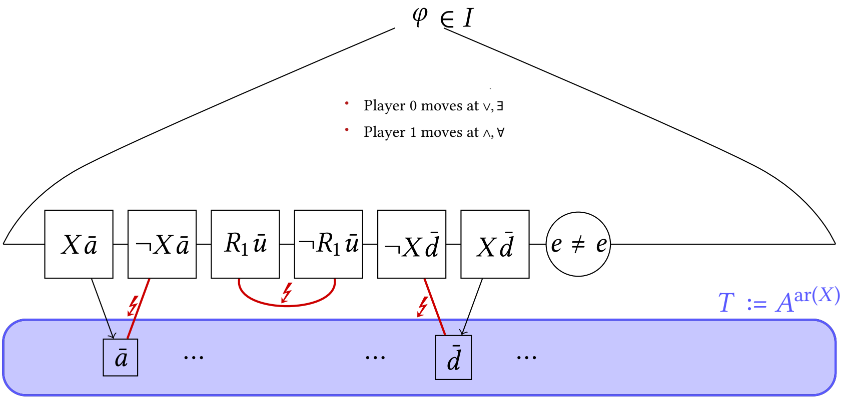
We define $\mathcal{T}(\mathcal{S}) := W \cap T$ and $\mathcal{T}(\mathcal{G}) := \{\mathcal{T}(\mathcal{S}) : \mathcal{S} \text{ winning strategy}\}$.



$\mathcal{T}(\mathcal{G}) := \{\{c, d\}, \{a, c, d\}\}$

Model-Checking Games for Σ^1_1

Let $\psi(X) := \exists \bar{R}\varphi(X, \bar{R}) \in \Sigma^1_1$ where $\varphi(X, \bar{R}) \in \mathrm{FO}$ (is in NNF).
The game $\mathcal{G}_X(\mathfrak{A}, \psi)$ is defined as in the following picture:



$$\mathfrak{A} \models \psi(Y) \iff Y \in \mathcal{T}(\mathcal{G}_X(\mathfrak{A}, \psi))$$

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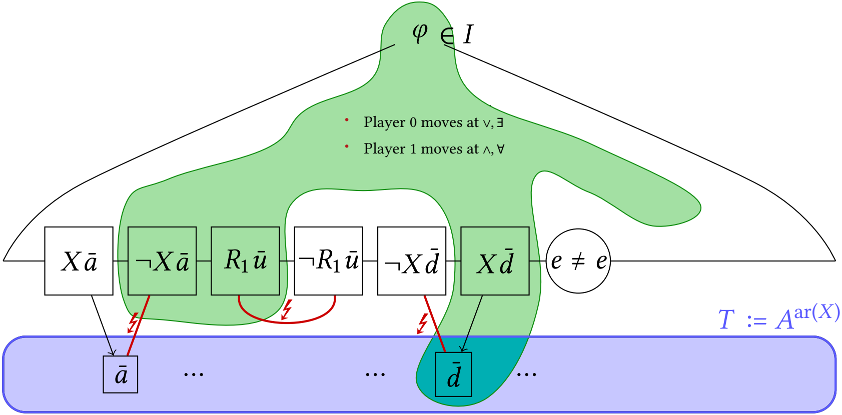

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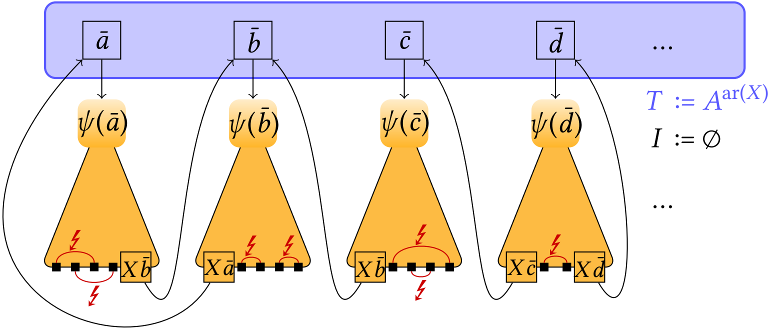
Model-Checking Games for Myopic Σ_1^1-Sentences

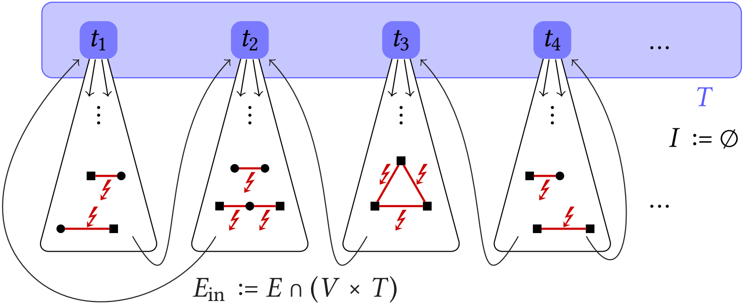
Let $\varphi(X) := \forall \bar{x}(X\bar{x} \longrightarrow \exists \bar{R}\psi(X, \bar{R}, \bar{x}))$ be a **myopic** Σ_1^1-sentence.
The model-checking game $\mathcal{G}(\mathfrak{A}, \varphi)$ has the following form:



$$\mathfrak{A} \models \varphi(Y) \iff Y \in \mathcal{T}(\mathcal{G}(\mathfrak{A}, \varphi))$$

Union Games

An inclusion-exclusion game is a **union game**, if it can be decomposed into components like in the following picture:



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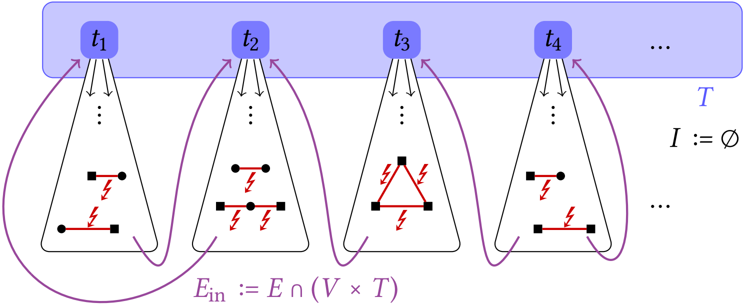

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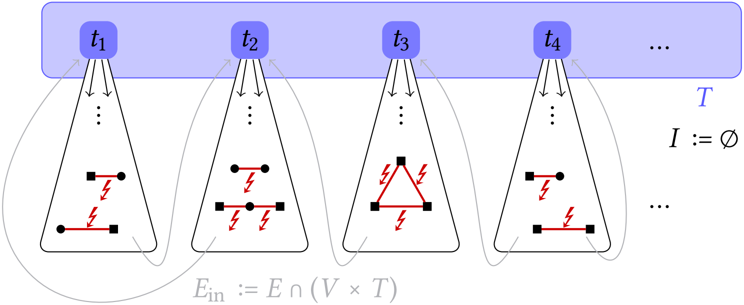

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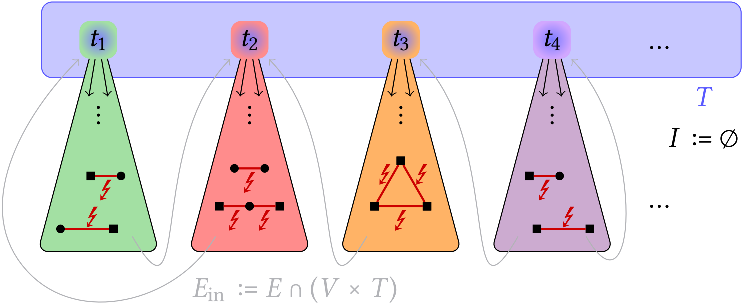

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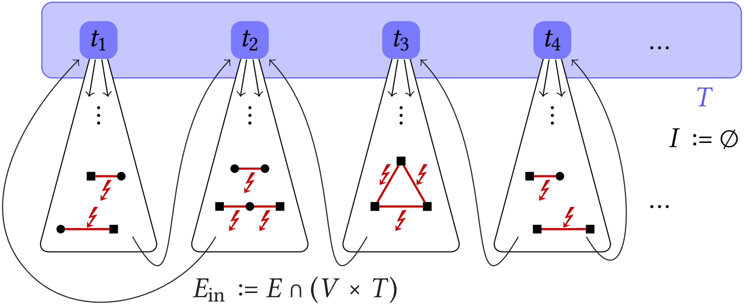

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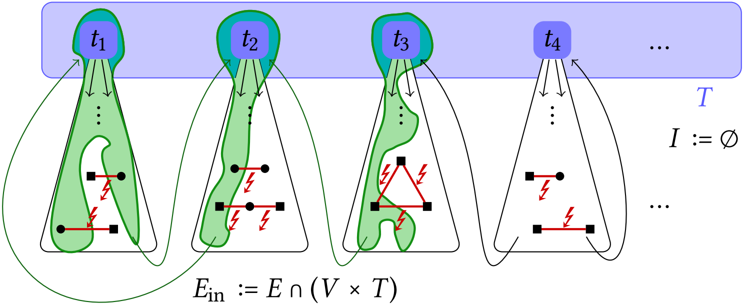

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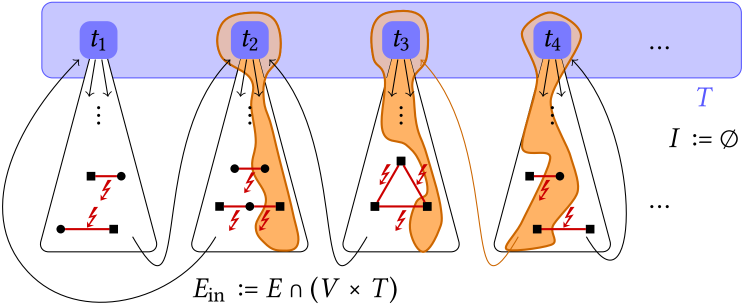

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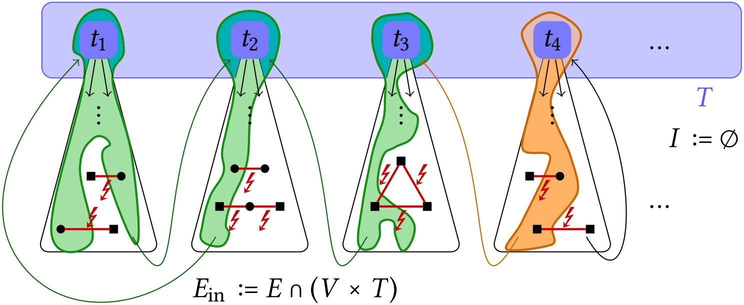

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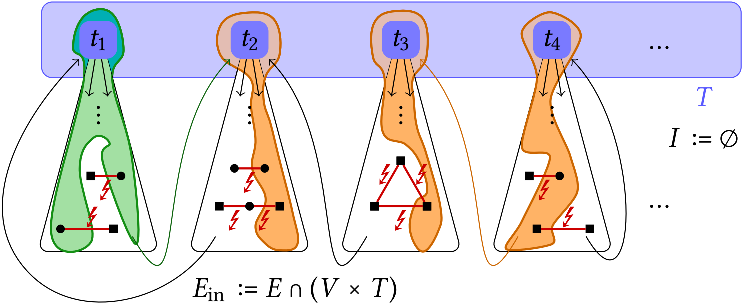

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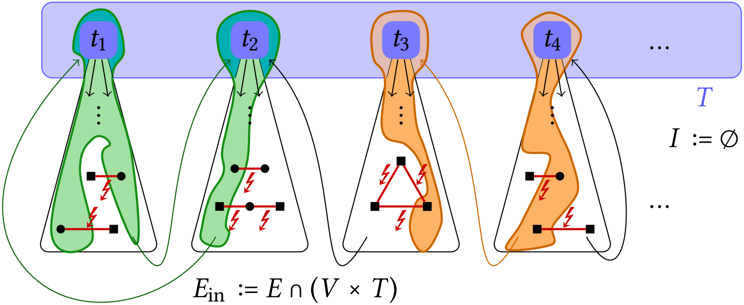

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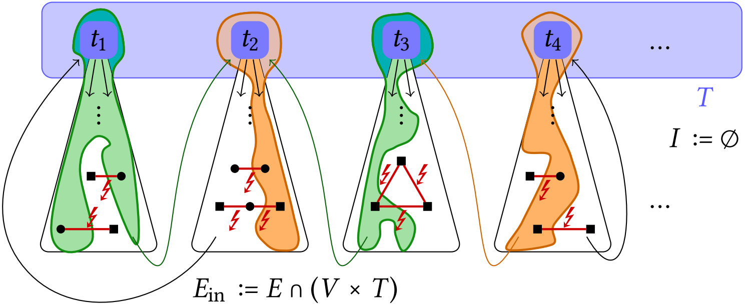

Observation: $\mathcal{T}(\mathcal{G})$ is closed under unions, because we can reassemble the components of winning strategies.

From Union Games to Inclusion-Exclusion-Logic

Associate

- Game $\leftrightsquigarrow$ Formula
- Strategy $\leftrightsquigarrow$ Team

This association leads to the following questions:

- What are the components of a team?
- How can we restrict a formula "to these components"?

Components of a Team

Let X be a team with $\mathrm{dom}(X) = \{\bar{x}, \bar{y}\}$.

X	$\bar{x}$	$\bar{y}$
s_1	$\bar{a}$	$\bar{v}_1$
s_2	$\bar{a}$	$\bar{v}_2$
s_3	$\bar{a}$	$\bar{v}_3$
s_4	$\bar{b}$	$\bar{v}_4$
s_5	$\bar{b}$	$\bar{v}_5$
s_6	$\bar{c}$	$\bar{v}_6$
s_7	$\bar{c}$	$\bar{v}_7$
s_8	$\bar{c}$	$\bar{v}_8$

The $\bar{x}$-**components** of X are subteams of the form

$$X\restriction_{\bar{x}=\bar{v}} := \{s \in X \,:\, s(\bar{x}) = \bar{v}\}.$$

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s_5	$\bar{b}$	$\bar{v}_5$	
s_6	$\bar{c}$	$\bar{v}_6$	
s_7	$\bar{c}$	$\bar{v}_7$	$X\restriction_{\bar{x}=\bar{c}}$
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The $\bar{x}$-**components** of X are subteams of the form

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$$\mathfrak{A} \models_X \bar{v} \mid \bar{w} \iff \text{for all } s, s' \in X, s(\bar{v}) \neq s'(\bar{w}).$$

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$$\mathfrak{A} \models_X \bar{x}\bar{v} \mid \bar{x}\bar{w} \iff \mathfrak{A} \models_{X\upharpoonright_{\bar{x}=\bar{a}}} \bar{v} \mid \bar{w} \text{ for all } \bar{a} \in X(\bar{x})$$

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Similarly, we have

$$\mathfrak{A} \models_X \bar{x}\bar{v} \subseteq \bar{x}\bar{w} \iff \mathfrak{A} \models_{X\upharpoonright_{\bar{x}=\bar{a}}} \bar{v} \subseteq \bar{w} \text{ for all } \bar{a} \in X(\bar{x}).$$

Guarded Formulae

A formula $\varphi(\bar{x}, \bar{y}) \in \mathrm{FO}(\subseteq, \mid)$ is $\bar{x}$-**guarded**, if

1. Inclusion-exclusion atoms in φ are always of the form
   - $\bar{x}\bar{v} \mid \bar{x}\bar{w}$ or
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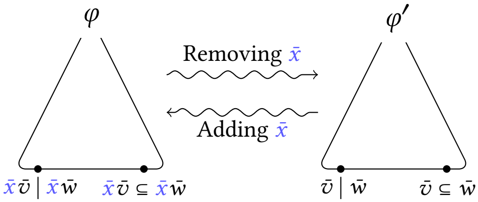
The (un)guarded version of a formula is obtained as follows:



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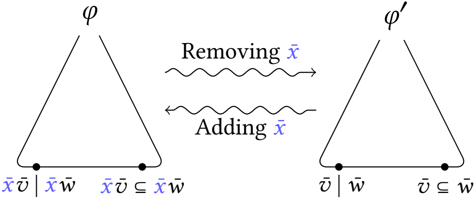

Lemma: $\mathfrak{A} \models_X \varphi \iff \mathfrak{A} \models_Y \varphi'$ for every $\bar{x}$-component Y of X.

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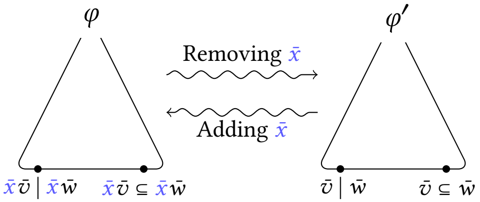

Lemma: $\mathfrak{A} \models_X \varphi \iff \mathfrak{A} \models_Y \varphi'$ for every $\bar{x}$-component Y of X.

Problem: Deleting $\bar{x}$-components preserves satisfaction.

Myopic Fragment of FO($\subseteq$, $|$)

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Let ψ be a subformula of φ and $\mathfrak{A} \models_{X_i} \psi$ for every i.

X_1	$\bar{x}$	$\bar{y}$
s_1	$\bar{a}$	$\bar{u}_1$
s_2	$\bar{a}$	$\bar{u}_2$
s_3	$\bar{b}$	$\bar{u}_3$
s_4	$\bar{b}$	$\bar{u}_4$
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X_2	$\bar{x}$	$\bar{y}$
s'_1	$\bar{a}$	$\bar{v}_1$
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$\rightsquigarrow$

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Capturing the Union-Closed Fragment

Let $\varphi(X) \in \Sigma_1^1$ be a myopic sentence.

Task: Construct equivalent, $\bar{x}$-myopic $\mu(\bar{x}) \in \mathrm{FO}(\subseteq, \mid)$.

Let $\mathfrak{A}$ be a structure and Y be a team with $\mathrm{dom}(Y) = \{\bar{x}\}$.

$$\begin{tikzcd}
 & \mathfrak{A} \models \varphi(Y(\bar{x})) \arrow[dl, Leftrightarrow] & & \mathfrak{A} \models_Y \mu(\bar{x}) \arrow[dr, Leftrightarrow] & \\
Y(\bar{x}) \in \mathcal{T}(\mathcal{G}(\mathfrak{A}, \varphi)) \arrow[drr, Leftrightarrow] & & & & \mathfrak{A} \models_{h^{-1}(Y_{\mathcal{G}})} \vartheta_{\mathcal{T}}^{\mathcal{I}}(\bar{y}) \arrow[dll, Leftrightarrow] \\
 & & \mathcal{G}(\mathfrak{A}, \varphi) \models_{Y_{\mathcal{G}}} \vartheta_{\mathcal{T}}(y) & &
\end{tikzcd}$$

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Let $\varphi(X) \in \Sigma^1_1$ be a myopic sentence.

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By **construction** of the union game $\mathcal{G}(\mathfrak{A}, \varphi)$.

$$
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Capturing the Union-Closed Fragment

Let $\varphi(X) \in \Sigma^1_1$ be a myopic sentence.

Task: Construct equivalent, $\bar{x}$-myopic $\mu(\bar{x}) \in \mathrm{FO}(\subseteq, \mid)$.

Let $\mathfrak{A}$ be a structure and Y be a team with $\mathrm{dom}(Y) = \{\bar{x}\}$.

By **construction** of the union game $\mathcal{G}(\mathfrak{A}, \varphi)$.

$$\begin{tikzcd}
 & \mathfrak{A} \models \varphi(Y(\bar{x})) \arrow[dl, Leftrightarrow] & & \mathfrak{A} \models_Y \mu(\bar{x}) \arrow[dr, Leftrightarrow] & \\
Y(\bar{x}) \in \mathcal{T}(\mathcal{G}(\mathfrak{A}, \varphi)) \arrow[drr, Leftrightarrow] & & & & \mathfrak{A} \models_{h^{-1}(Y_\mathcal{G})} \vartheta^{\mathcal{I}}_{\mathcal{T}}(\bar{y}) \\
 & & \mathcal{G}(\mathfrak{A}, \varphi) \models_{Y_\mathcal{G}} \vartheta_{\mathcal{T}}(y) \arrow[urr, Leftrightarrow] & &
\end{tikzcd}$$

$\vartheta_{\mathcal{T}}(y) \in \mathrm{FO}(\subseteq, \mid)$ is a *y-myopic* formula that **defines** $\mathcal{T}(\mathcal{G}(\mathfrak{A}, \varphi))$.

$Y_G(y) := Y(\bar{x})$

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Use a **logical interpretation** $\mathcal{I}$ with $\mathcal{I}(\mathfrak{A}) \cong \mathcal{G}(\mathfrak{A}, \varphi)$ for some coordinate map h.

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Repair the damage done by $\mathcal{I}$ to the myopic shape and extract values for $\bar{x}$ from their encodings.

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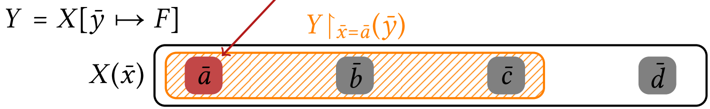

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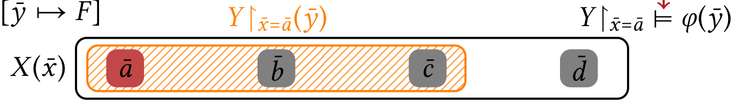

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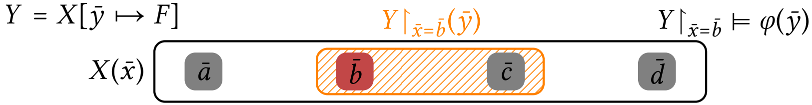

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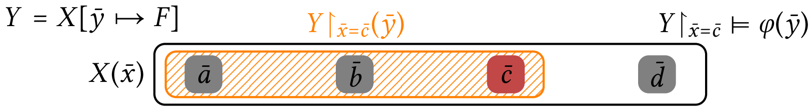

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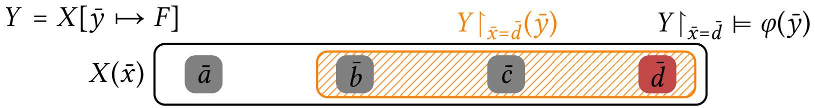

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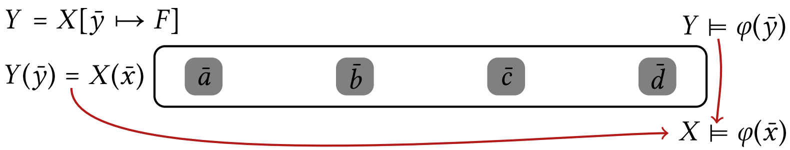

The Game Atom

- Union games are complete for the union-closed fragment

For $X \neq \emptyset$, define

$$\mathfrak{A} \models_X \cup\text{-}\mathsf{game}(\mathcal{V}_k, \bar{x}) :\Longleftrightarrow \begin{array}{l} X \text{ is complete and} \\ \text{if } X \text{ encodes a union game } \mathcal{G}_X^A, \\ \text{then } X(\bar{x}) \in \mathcal{T}(\mathcal{G}_X^A). \end{array}$$

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- Answer: Make sure that unions of satisfying teams cannot encode a *different* game.

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Rönnholm's Question

$\mathrm{FO}(\subseteq_k)$: FO + inclusion atoms $\bar{u} \subseteq \bar{v}$ with $|\bar{u}| = |\bar{v}| \leq k$.

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GFP^+_k: fragment of greatest fixed-point logic where fixed-point relations are of arity $\leq k$

Answering Rönnholm's Question

1. For every $\mathrm{FO}(\subseteq_k)$-formula $\varphi(\bar{x})$ there exists an *equivalent myopic* GFP^+_k-sentence $\psi(X)$.
2. For every *myopic* GFP^+_k-sentence $\psi(X)$ there exists an *equivalent* $\mathrm{FO}(\subseteq_{k'})$-formula $\varphi(\bar{x})$ where $k' := \max\{k, \mathrm{ar}(X)\}$.
3. For every GFP^+_k-formula $\psi(\bar{x})$ there exists a (downwards closed) $\mathrm{FO}(\subseteq_k)$-formula $\gamma(\bar{x})$ s.t. for every $\mathfrak{A}, X$,

$$\mathfrak{A} \models_X \gamma(\bar{x}) \iff \mathfrak{A} \models_s \psi(\bar{x}) \text{ for every } s \in X.$$

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Dependencies Concepts up to Equivalences

$$\mathfrak{A} \models_X \mathrm{dep}(\bar{x}, y) \iff \text{for every } s, s' \in X\text{: } s(\bar{x}) = s'(\bar{x}) \text{ implies } s(y) = s'(y)$$

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Let

- $\approx$ be an equivalence relation of the structure $\mathfrak{A}$
- $\bar{a} \approx \bar{b} :\iff a_i \approx b_i$ for all i

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Expressive Powers in Comparison

$$
\begin{tikzcd}
 & & \Sigma^1_1 & \equiv & \mathrm{FO}(\subseteq, |) \arrow[dll, dash, sloped, ">" description] \arrow[drr, dash, sloped, ">" description] & \equiv & \mathrm{FO}(\perp) & & \\
\mathrm{GFP}^+ & \equiv & \mathrm{FO}(\subseteq) \arrow[drr, dash, sloped, ">" description] & & & & \mathrm{FO}(|) \arrow[dll, dash, sloped, ">" description] & \equiv & \mathrm{FO}(\mathrm{dep}) \\
 & & & & \mathrm{FO} & & & &
\end{tikzcd}
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$$\begin{tikzcd}[column sep=small]
 & & \Sigma^1_1(\approx) \equiv \mathrm{FO}(\subseteq_\approx, |_\approx) \equiv \mathrm{FO}(\perp_\approx) \arrow[dl] \arrow[dr] & & \\
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 & & \mathrm{FO} & &
\end{tikzcd}$$

Expressive Powers in Comparison

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\begin{tikzcd}
 & & \Sigma^1_1 \arrow[d, dash, "\vee" description] & & \\
 & \Sigma^1_1(\approx) \arrow[r, phantom, "\equiv"] & \mathrm{FO}(\subseteq_{\approx}, |_{\approx}) \arrow[r, phantom, "\equiv"] \arrow[dl, dash, "<" description] \arrow[dr, dash, ">" description] & \mathrm{FO}(\perp_{\approx}) & \\
\mathrm{GFP}^{+}(\approx) \arrow[r, phantom, "\equiv"] & \mathrm{FO}(\subseteq_{\approx}) \arrow[dr, dash, ">" description] & & \mathrm{FO}(|_{\approx}) \arrow[r, phantom, "\equiv"] \arrow[dl, dash, "<" description] & \mathrm{FO}(\mathrm{dep}_{\approx}) \\
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Results for these Logics

Joint work with Erich Grädel

1. $\Sigma_1^1(\approx) \equiv \mathrm{FO}(\subseteq_\approx, |_\approx)$ (on the level of sentences).
2. $\Sigma_1^1(\approx) < \Sigma_1^1$ (on the level of sentences).
3. For every $\varphi(X) \in \Sigma_1^1(\approx)$ where X occurs only ≈-guarded there exists an equivalent $\psi(\bar{x}) \in \mathrm{FO}(\subseteq_\approx, |_\approx)$ that cannot distinguish between ≈-equivalent teams and vice versa.
4. $\mathrm{FO}(\subseteq_\approx) \equiv \mathrm{GFP}^+_\approx$ (on the level of sentences).

≈-guarded occurrence of X: $X_\approx \bar{v} := \exists \bar{w}(\bar{v} \approx \bar{w} \wedge X\bar{w})$.