Limitations of Choiceless Definability

Benedikt Pago

Mathematical Foundations of Computer Science - RWTH Aachen University

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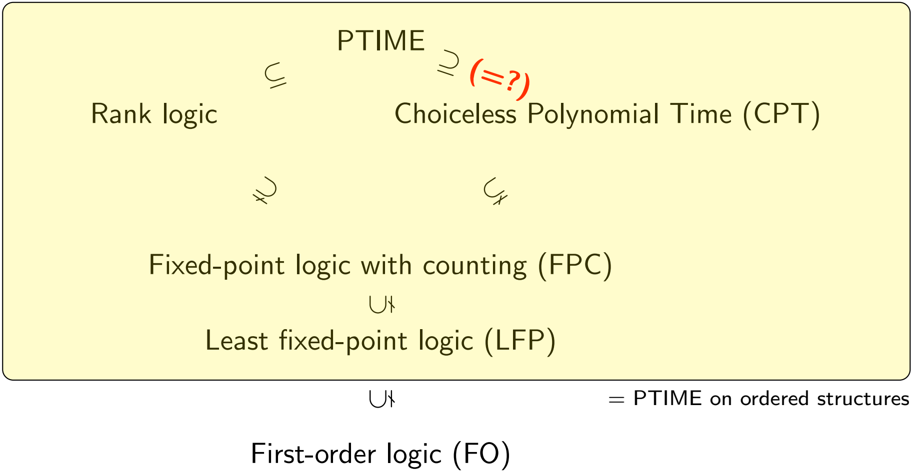
Choiceless Computation

- **"Classical" computation:** Turing Machines manipulating bits on a tape.
- **Choiceless computation:** Abstract computations on finite structures by logic.
- Respects, at any stage, all *symmetries* of the input; choices among indistinguishable objects are forbidden.
- **Question:** (How) does this restrict the computational power?

Is there a logic for PTIME?

Logic for PTIME: Expresses exactly the *PTIME-decidable* properties of *finite structures*.



Previous work on CPT

- CPT was introduced in 1999 (Blass, Gurevich, Shelah).
- **Positive results:**
  - More powerful than fixed-point logic with counting.
  - Captures PTIME on certain classes of structures (padded structures, structures with Abelian/dihedral colours).
  - Definability of certain versions of the CFI-problem.
- **Negative results:** CPT cannot define the dual of a finite vector space (Rossman).
- However, this does not limit the power of CPT on *decision problems*.
- **Main open problem:** Does CPT capture PTIME on **all** finite structures?

Our contribution

Theorem (Main result)

No **Choiceless Polynomial Time** *program can define a* **preorder** *with colour classes of at most* **logarithmic size** *in arbitrary* **hypercubes***.*

Content of this talk:

1. What is Choiceless Polynomial Time?
2. Why can CPT not define certain objects?
3. What are preorders with logarithmic colour classes?
4. Why does this matter?

Choiceless Polynomial Time

Various formalisations:

- Originally: Abstract State Machines (Blass, Gurevich, Shelah, 1999).
- Polynomial Interpretation Logic (Grädel, Pakusa, Schalthöfer, Kaiser).
- BGS-logic (Rossman): Terms for manipulation of *hereditarily finite sets.*

BGS-terms

Let $\mathfrak{A}$ be a structure and a, b *hereditarily finite sets* over the universe of $\mathfrak{A}$.

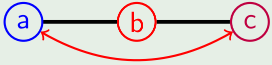
- $\texttt{Pair}(a, b) := \{a, b\}$.
- $\texttt{Union}(a) := \bigcup a$.
- Comprehension: $\{t \ : \ x \in a \ : \ \varphi\} := \{t(x) \mid x \in a, \mathfrak{A} \models \varphi(a)\}$.
- ...

Symmetric computations on hereditarily finite sets

Abstract view on CPT:
Symmetry-invariant computations on *polynomially-sized* hereditarily finite sets over input structure.

Example: Computing a linear order on the input.

Input structure:



Stages of computation:

$$\{a\} \to \{b, \{a\}\} \longrightarrow \{c, \{b, \{a\}\}\}$$

Symmetric result:

$$\{\ \{c, \{b, \{a\}\}\},\ \{a, \{b, \{c\}\}\}\ \}$$

$\Rightarrow$ Any h.f. set x over input $\mathfrak{A}$ is computed together with all its automorphic images w.r.t. $\mathrm{Aut}(\mathfrak{A})$ (= the *orbit* of x).
$\Rightarrow$ Objects with *super-polynomial orbit* can**not** be computed.

Proving the main result

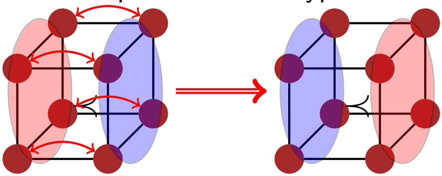
Definition (Preorders and colour classes)

- A preorder $\prec$ on a set A partitions A into *colour classes* $C_1, \dots, C_m$ of indistinguishable elements.
- $\prec$ induces a linear order $C_1 \prec C_2 \prec \dots \prec C_m$ on the colour classes.

Theorem (Main result)

No **Choiceless Polynomial Time** *program can define a* **preorder** *with colour classes of at most* **logarithmic size** *in arbitrary* **hypercubes***.*

Proof: Compute a *super-polynomial* lower bound on the *orbit size* of such preorders, w.r.t. the automorphisms of the hypercube.



A benchmark for logics: The CFI problem

- CFI (due to Cai, Fürer, Immerman) is a version of the Graph Isomorphism problem.
- It separates fixed-point logic with counting from PTIME.
- Potential candidate for separating CPT and PTIME.

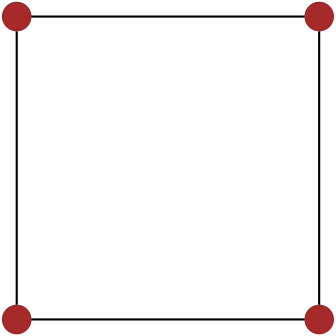


Figure: Underlying graph (Hypercube)

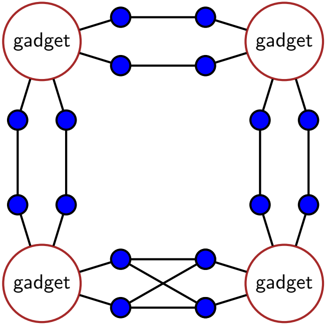


Figure: Even CFI graph

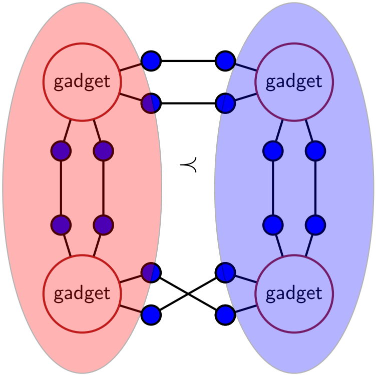
CFI query: Is a given CFI graph odd or even?

Solving the CFI problem in CPT

Currently strongest positive result:

Theorem (Pakusa, Schalthöfer, Selman)

CFI can be solved in CPT if the underlying graph comes with a preorder with colour classes of logarithmic size.



Such preorders are not CPT-definable on hypercubes.

$\Rightarrow$ The current techniques for CFI are *not* applicable to *unordered CFI structures*.

Summary

- Choiceless Polynomial Time: A potential candidate logic for capturing PTIME?
- Potential problem separating CPT from PTIME: CFI on unordered hypercubes.
- Evidence: *No symmetric polynomial time* computation model on h.f. sets can define a *preorder with logarithmic colour classes* in hypercubes.
- **Work in progress:**
  - New choiceless CFI-algorithms using other "auxiliary structures", e.g. tree decompositions instead of preorders.
  - Solving CFI does **not** require logarithmic preorders!