# Definability of Cai-Fürer-Immerman Problems in Choiceless Polynomial Time

Wied Pakusa, *Svenja Schalthöfer*, Erkal Selman

CSL 2016

# Origin: Database theory

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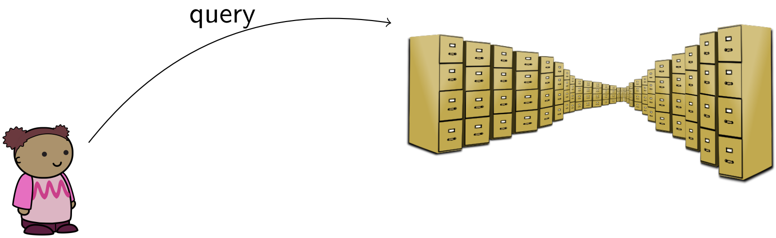

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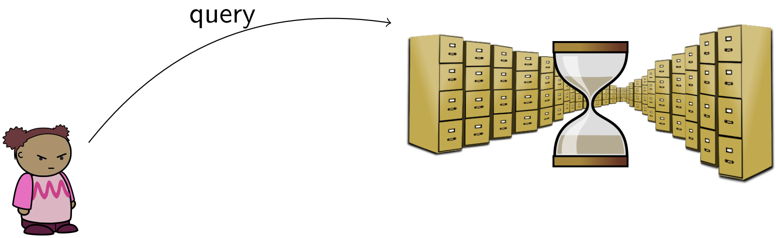

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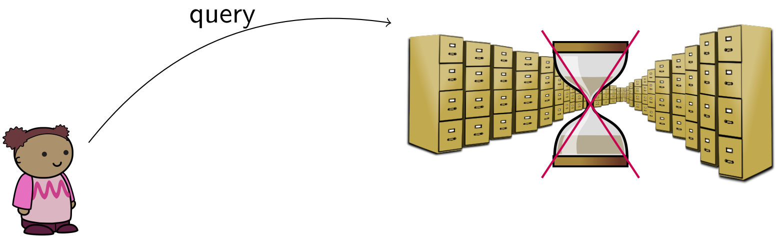

# The most important problem in finite model theory

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### Question (Gurevich 1988)

Is there a logic capturing PTIME?

# What is a logic capturing PTIME?

Informal definition: A logic $L$ captures $\textsc{Ptime}$ if it defines precisely those properties of finite structures that are decidable in polynomial time:

1. For every sentence $\psi \in L$, the set of finite models of $\psi$ is decidable in polynomial time.
2. For every $\textsc{Ptime}$-property $S$ of finite structures, there is a sentence $\psi \in L$ such that $S = \{\mathfrak{A} \in \mathsf{Fin} : \mathfrak{A} \models \psi\}$.

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**Question (Gurevich 1988)**

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**Theorem (Fagin)**

$\exists\mathrm{SO}$ *captures NP.*

# Candidates for a logic capturing PTIME

Ptime

$\subsetneq$

FO cannot define transitive closure

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# Candidates for a logic capturing PTIME

$\textsc{Ptime}$

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$\mathrm{FP} + \mathsf{rk}$ $\mathrm{CPT} + \mathrm{C}$

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**Theorem (Dawar,Richerby,Rossman)**

*The Cai-Fürer-Immerman query over ordered graphs is* $\mathrm{CPT}$*-definable.*

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# The logic: Choiceless Polynomial Time

Iterated creation of hereditarily finite sets,
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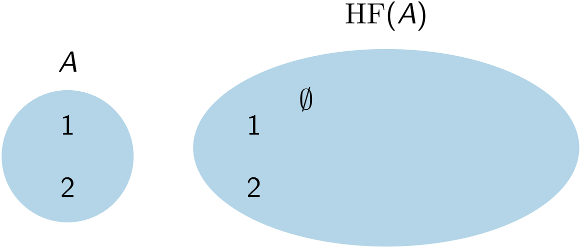
## Hereditarily finite sets



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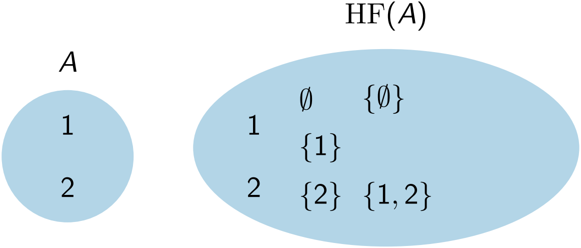

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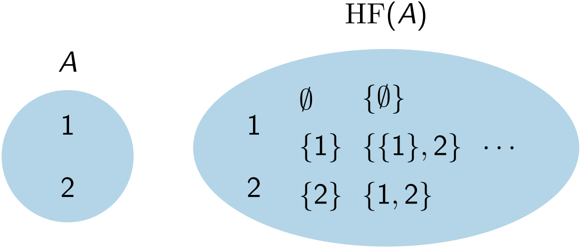

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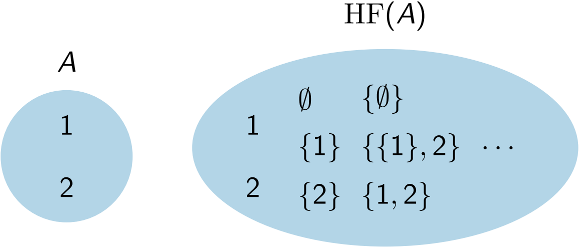


## Operations

- Set-theoretic operations (∅, ∈, ∪,...), boolean connectives
- Comprehension terms: $\{s(x) : x \in t : \varphi(x)\}$

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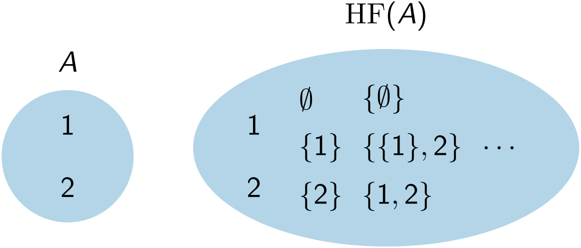


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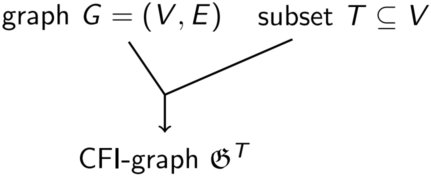


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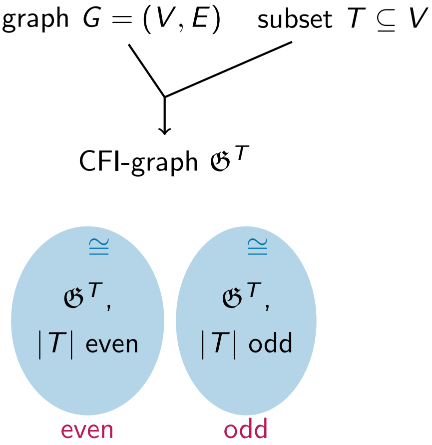


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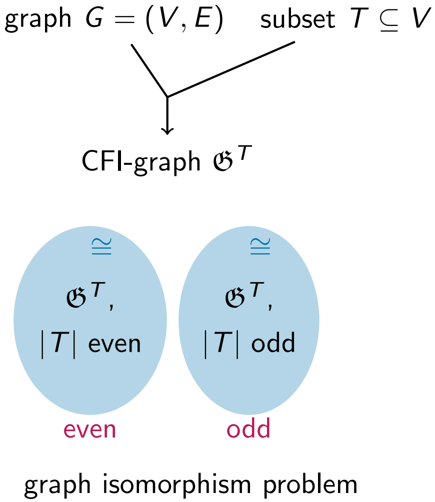
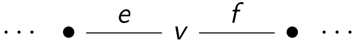


# Cai-Fürer-Immerman graphs: construction

Graph $\mathfrak{G}$

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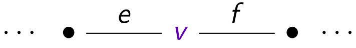


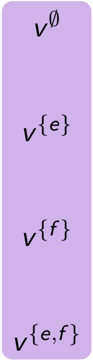

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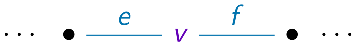


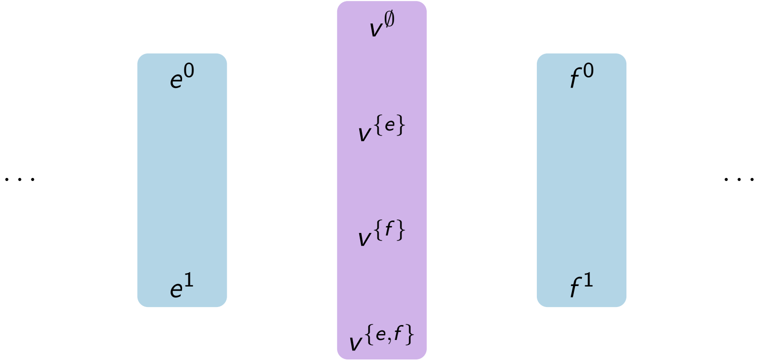

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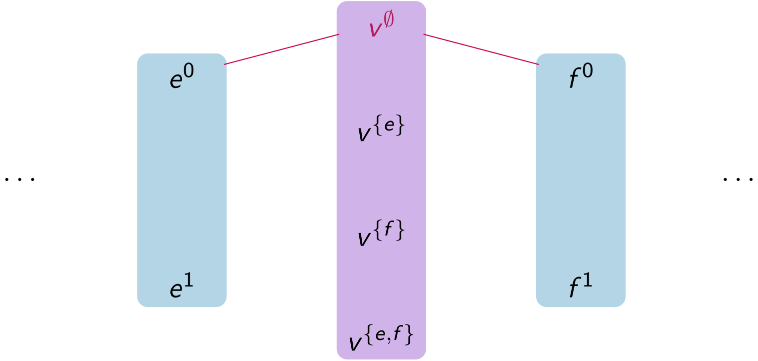

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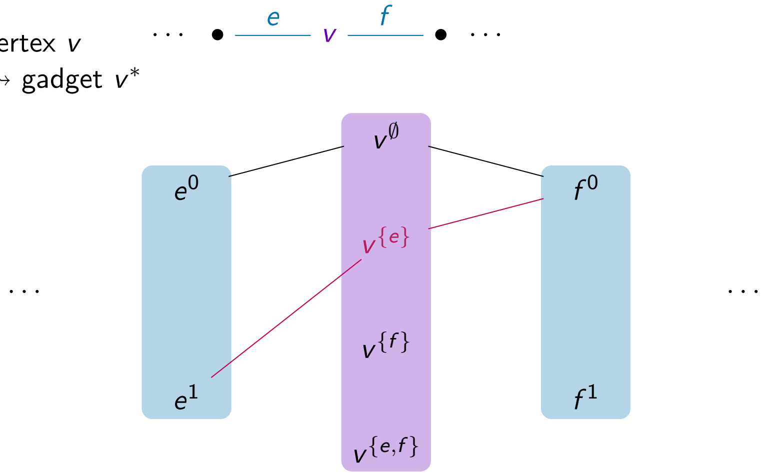

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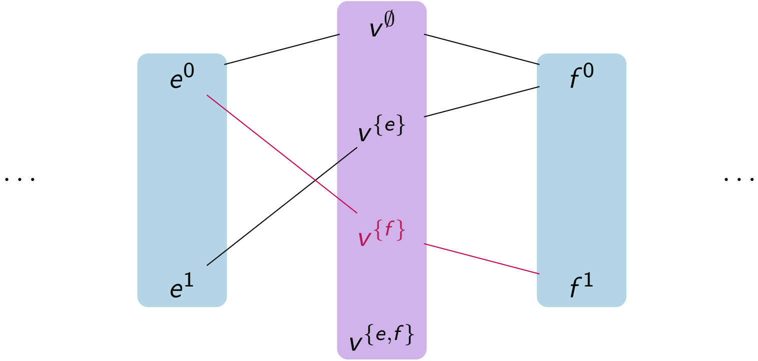

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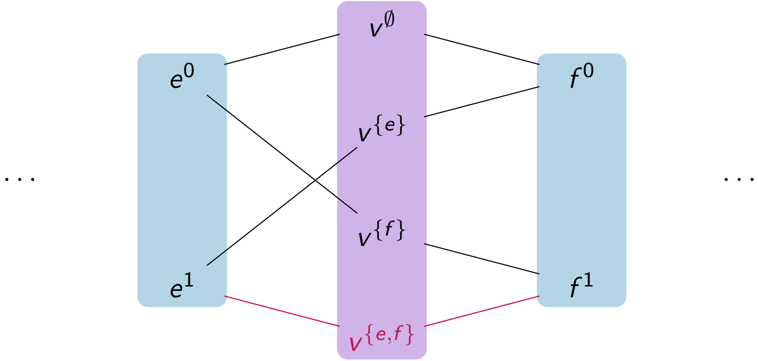

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Graph $\mathfrak{G}^T$

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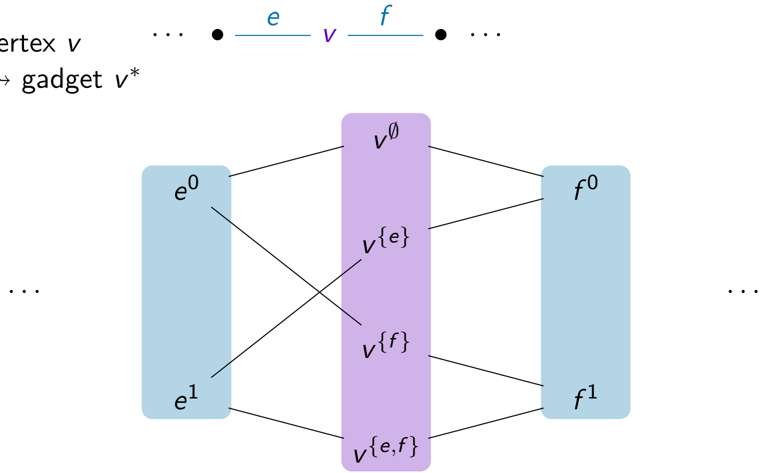

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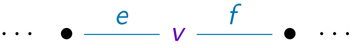


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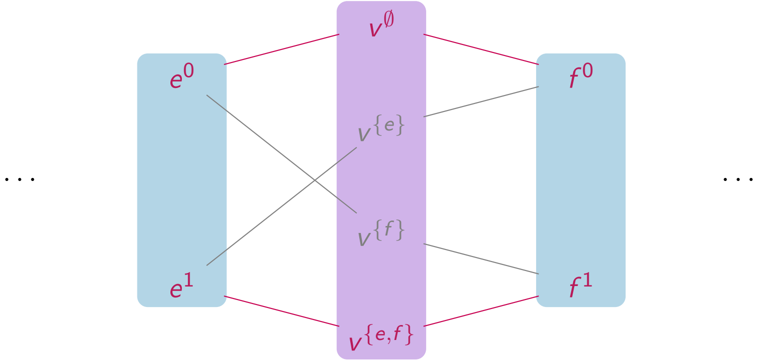


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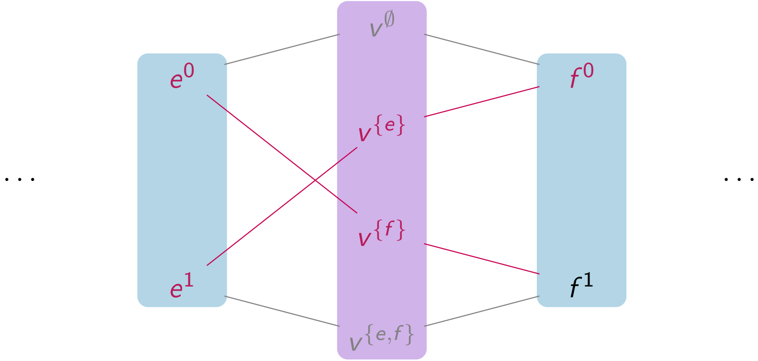
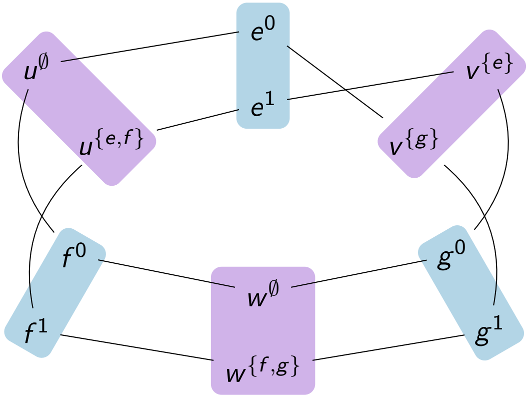



odd gadget, $v \in T$

# Cai-Fürer-Immerman graphs: example



# Results

**Theorem**
*The CFI query over graphs with logarithmic colour classes is* $\mathrm{CPT}$*-definable.*

**Theorem**
*The CFI query over graphs with large degree is* $\mathrm{CPT}$*-definable using only sets of bounded rank.*

**Theorem**
*The CFI query over complete graphs is not* $\mathrm{CPT}$*-definable without using set-like objects.*

**Corollary**
$\approx$ *-free* $\mathrm{PIL} \not\equiv \mathrm{CPT}[\mathrm{rk} \leq k]$

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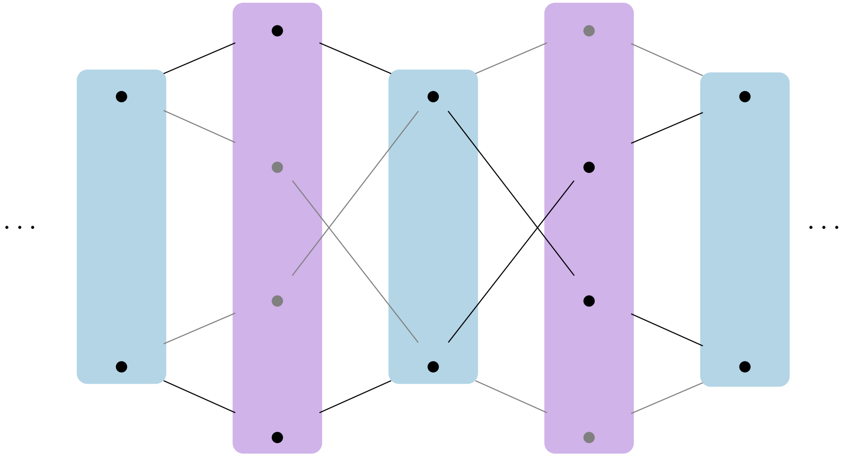
Theorem (Dawar,Richerby,Rossman)
*The CFI query over ordered graphs is* $\mathrm{CPT}$*-definable.*

# Computing the parity of CFI graphs

## Easy PTIME procedure

1. Label edge gadgets
2. Count odd vertex gadgets

# Assigning edge labels



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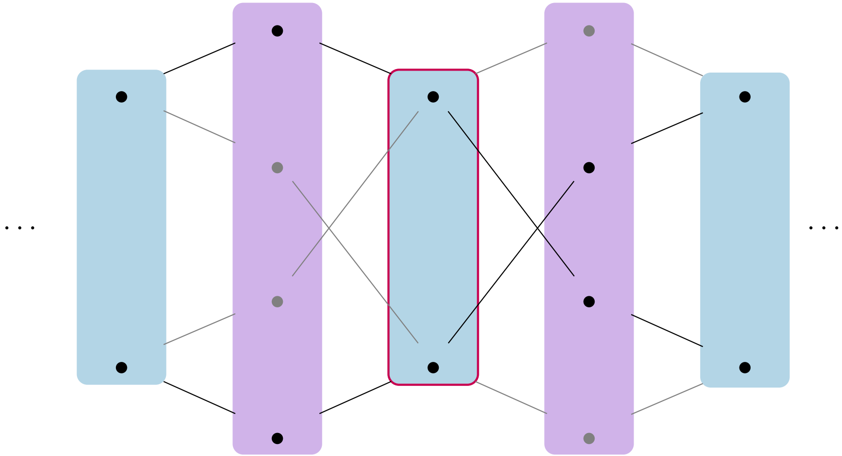

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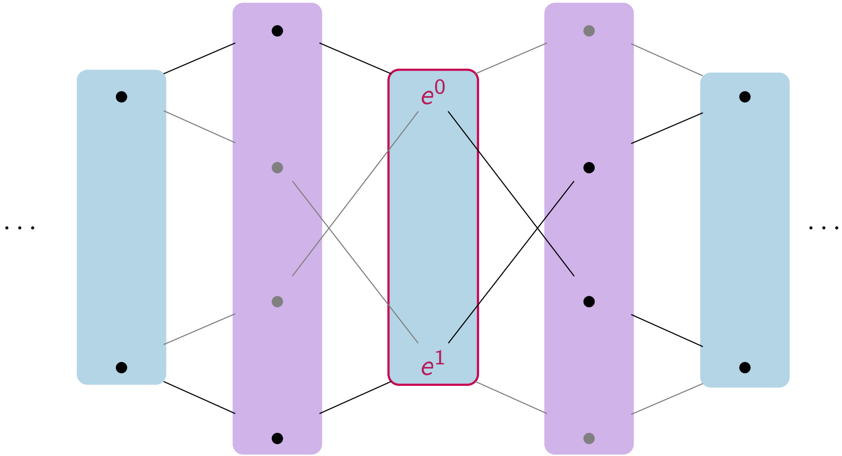

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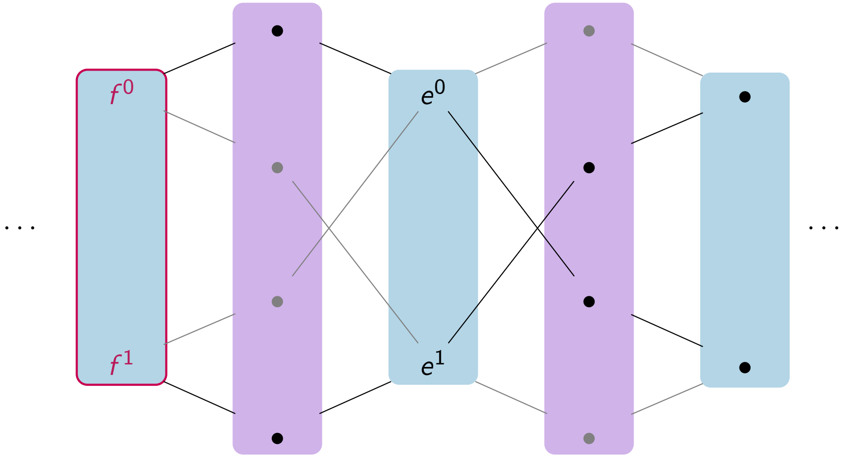

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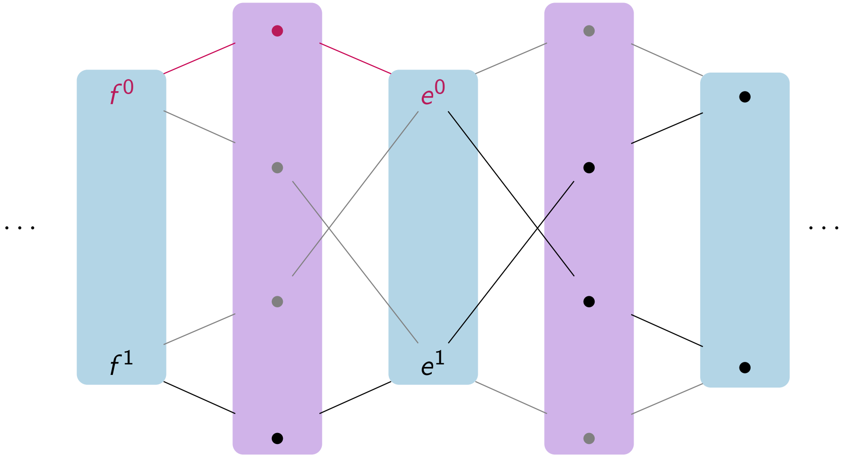

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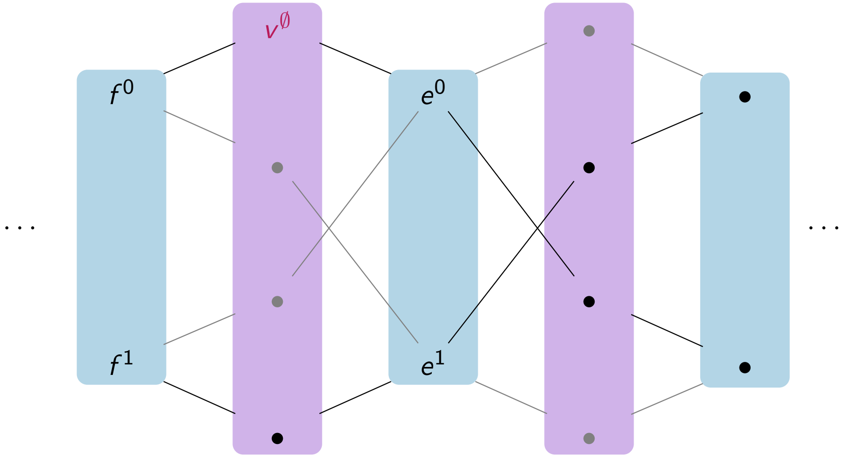

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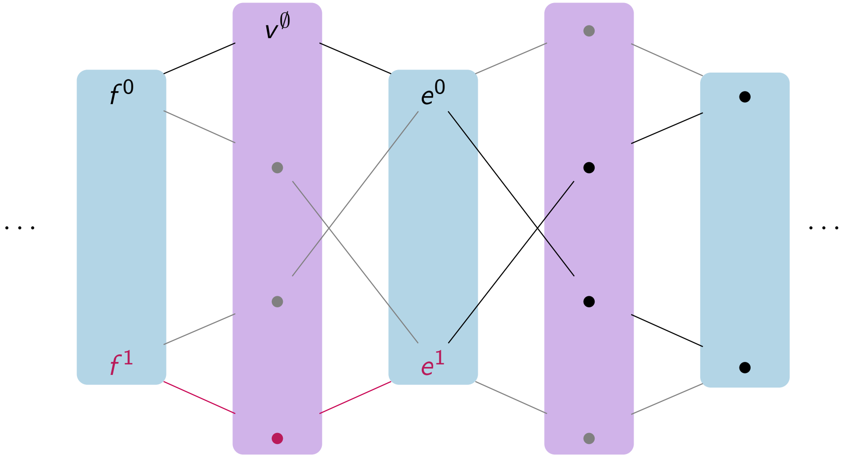

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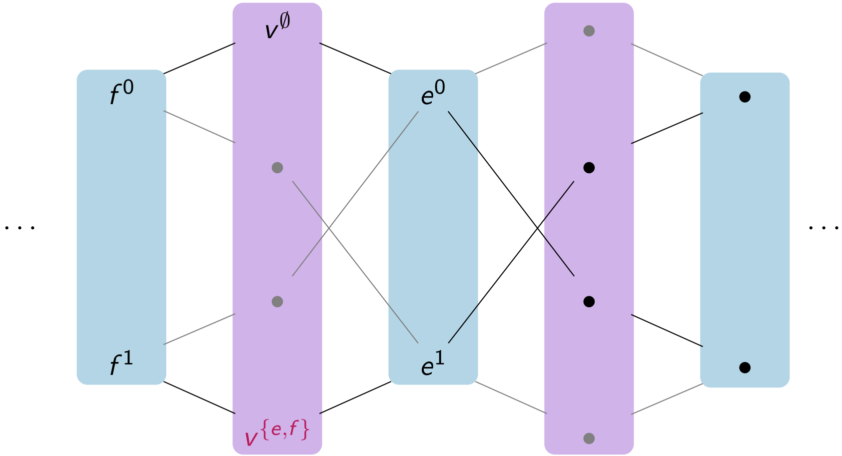

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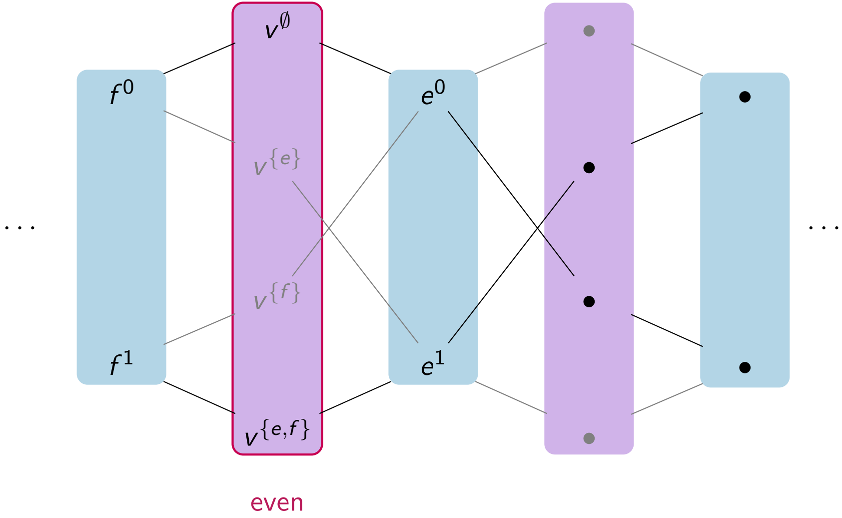

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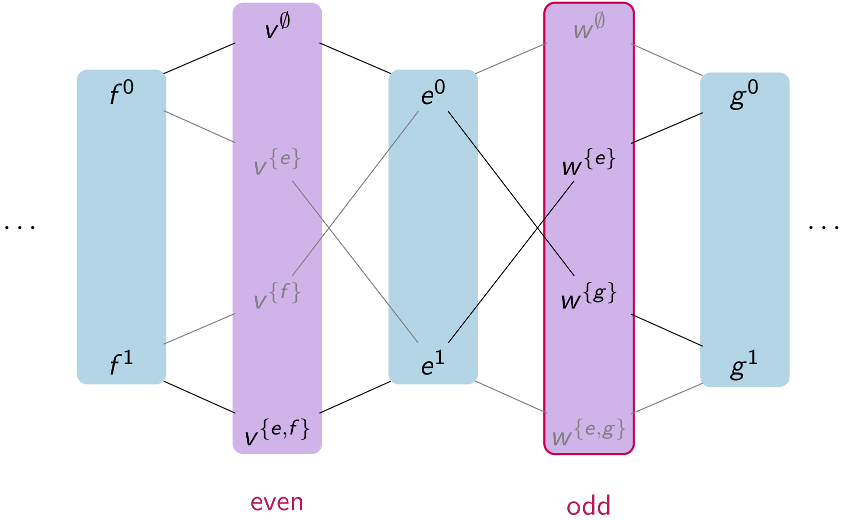

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## CPT procedure (Dawar, Richerby, Rossman)

1. Construct super-symmetric objects
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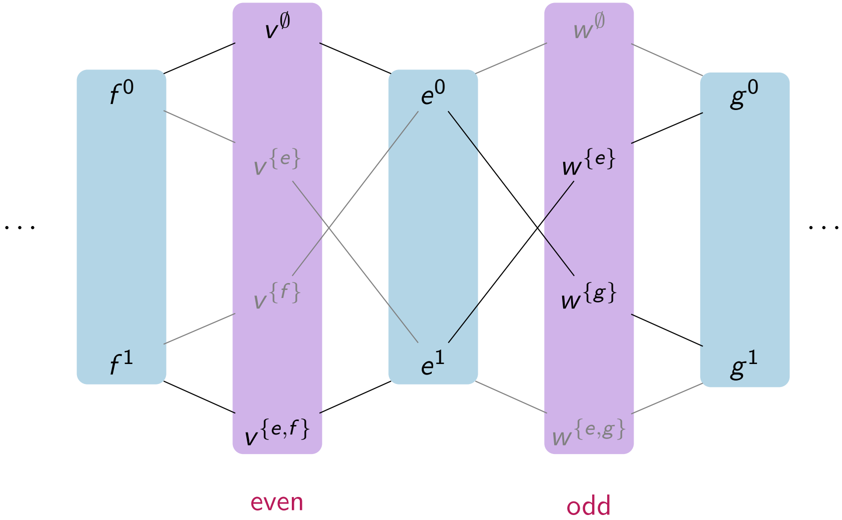
# Why super-symmetry is useful



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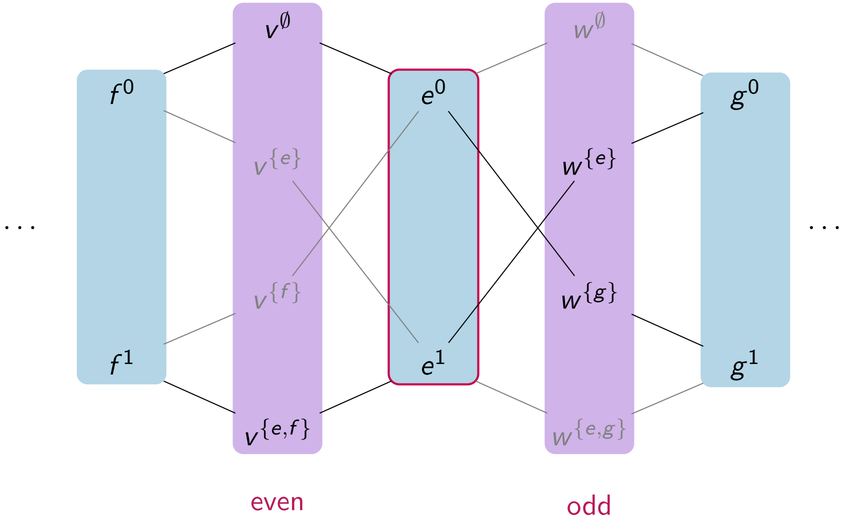

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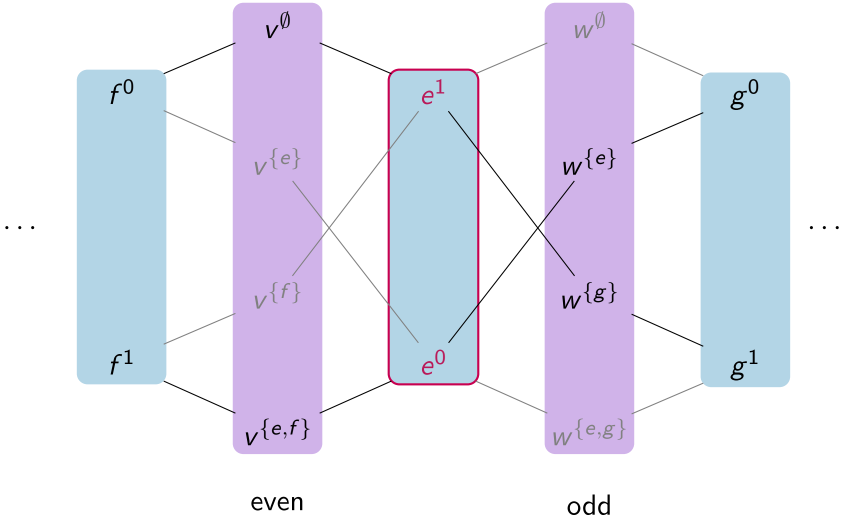

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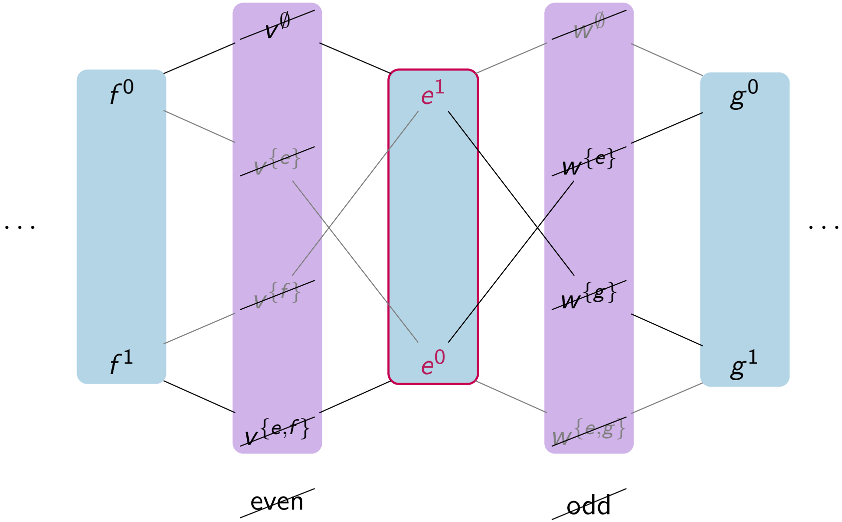

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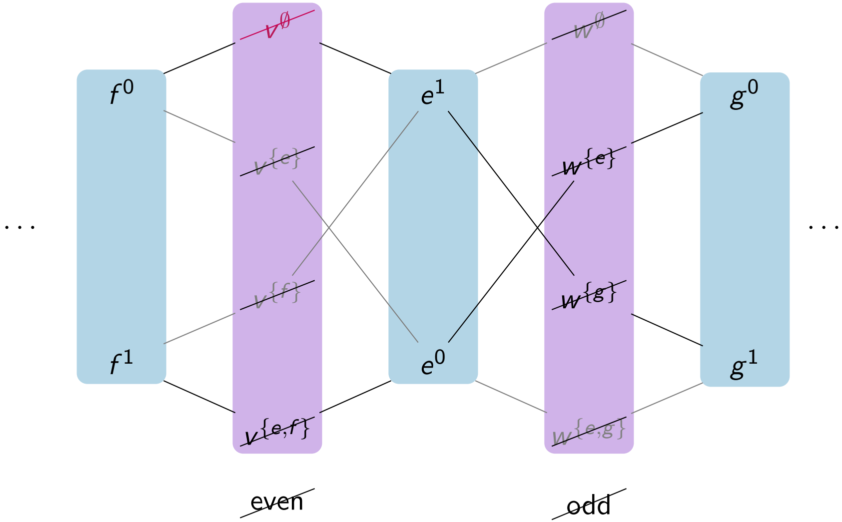

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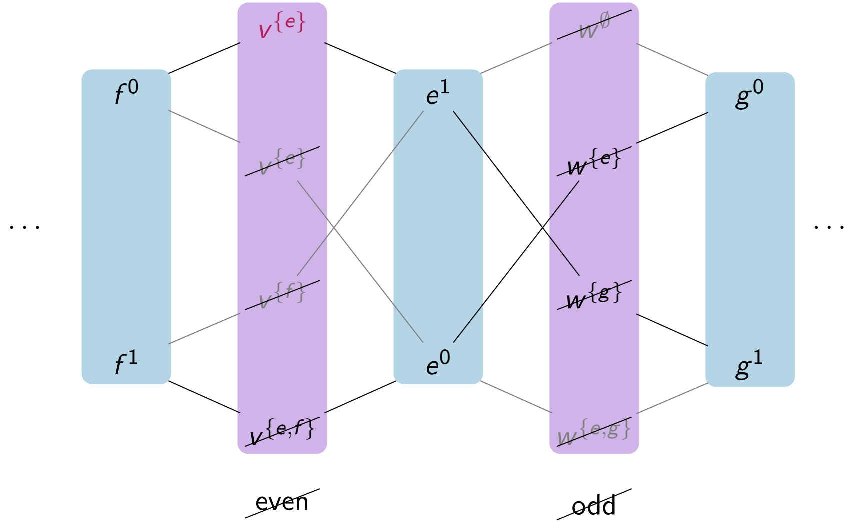

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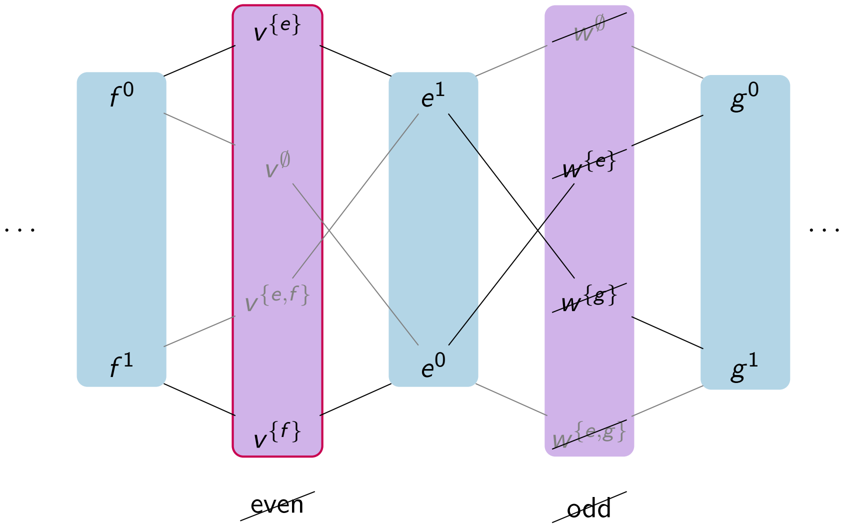

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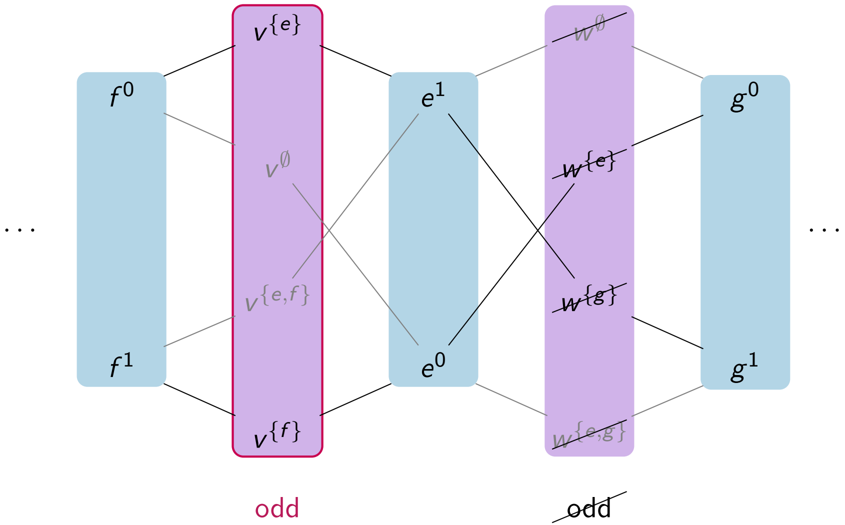

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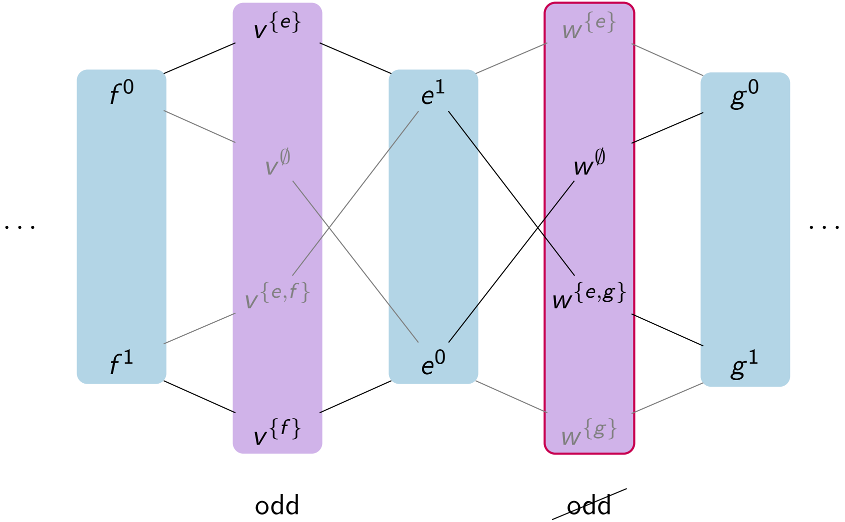

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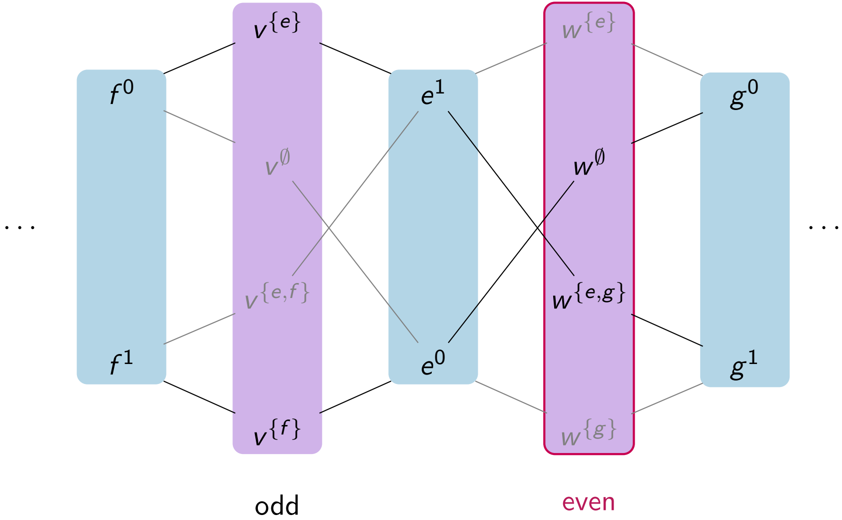

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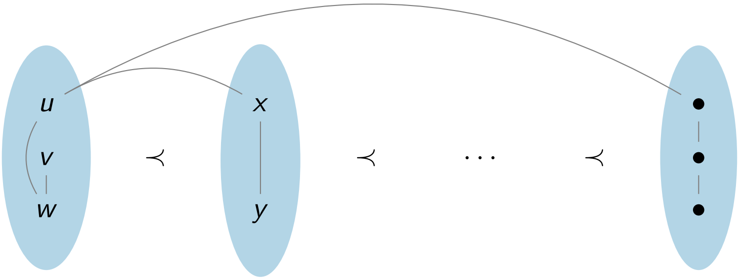

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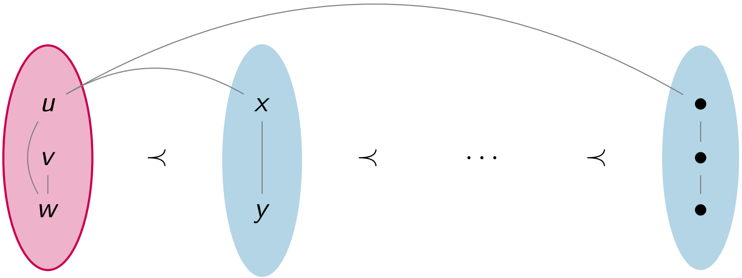


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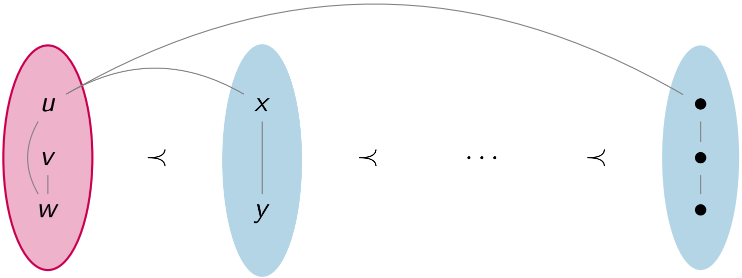


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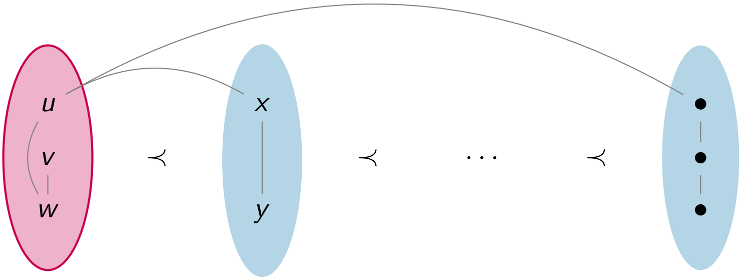


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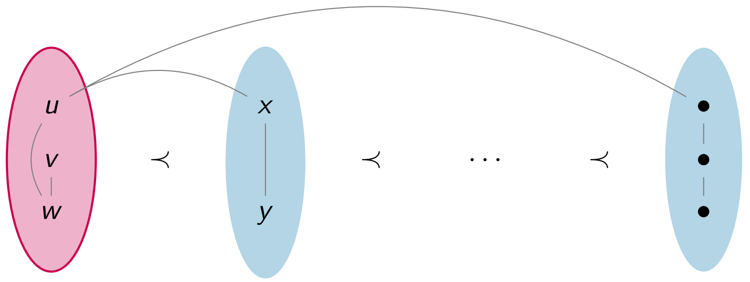


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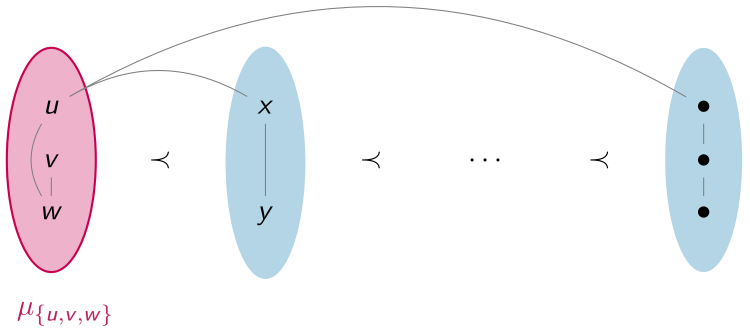
Construct $\mathcal{O}(2^{|C|})$ many sets

# Graphs with colour classes of logarithmic size



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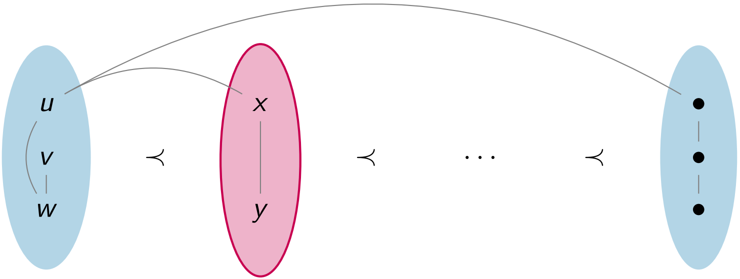


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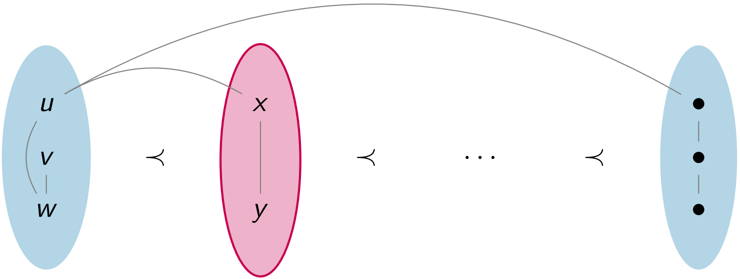


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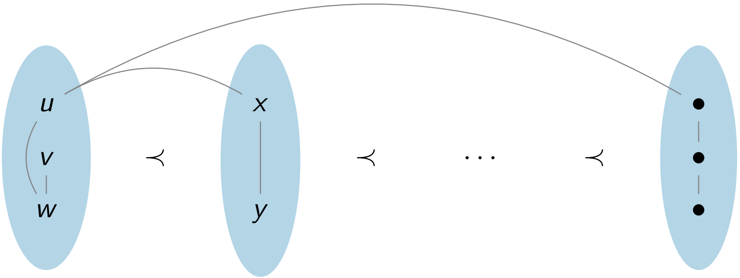


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$\mu_{\{u,v,w,x,y\}}$ $\ddots$

$\mu_V$

# Results

**Theorem**
*The CFI query over graphs with logarithmic colour classes is* $\mathrm{CPT}$*-definable.* ✓

**Theorem**
*The CFI query over graphs with large degree is* $\mathrm{CPT}$*-definable using only sets of bounded rank.*

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*The CFI query over complete graphs is not* $\mathrm{CPT}$*-definable without using set-like objects.*

**Corollary**
$\approx$ *-free* $\mathrm{PIL} \neq \mathrm{CPT}[\mathrm{rk} \leq k]$

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**Theorem (Dawar, Richerby, Rossman)**
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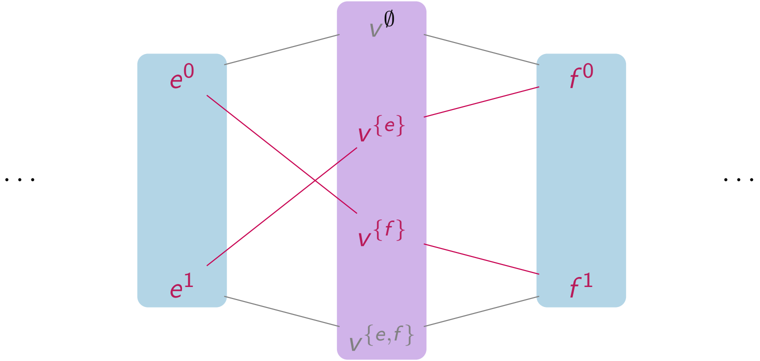
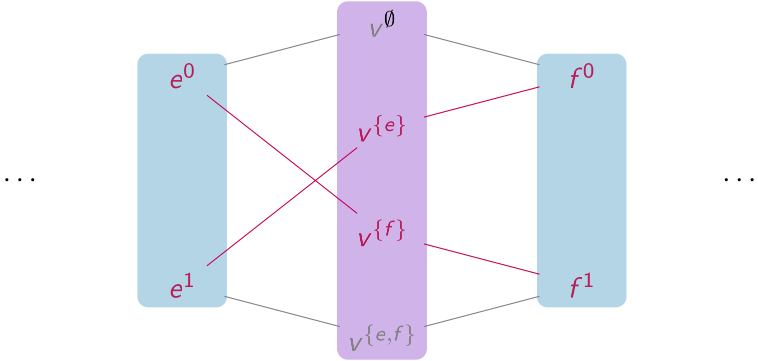


# Graphs with large degree: Keeping the rank small

- Access to all subsets of $V$
- Intuition: “Ordered” objects need nesting



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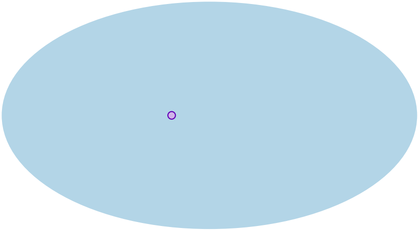
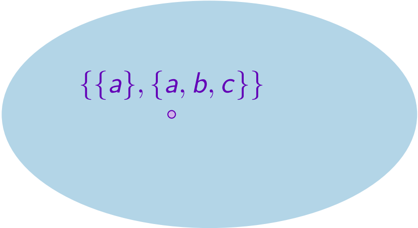
- $\mathfrak{A} \equiv^{\mathcal{C}^{\ell \cdot k}} \mathfrak{B} \Rightarrow \mathsf{HF}(\mathfrak{A})_\ell \equiv^{\mathcal{C}^k} \mathsf{HF}(\mathfrak{B})_\ell$ if $\mathfrak{A}, \mathfrak{B}$ $\mathcal{C}^{\ell \cdot k}$-homogeneous

Move in HF($\mathfrak{A}$):



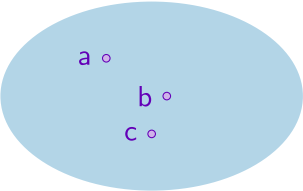
# Supports

Move in HF($\mathfrak{A}$):



# Supports

"Corresponding" move in $\mathfrak{A}$:

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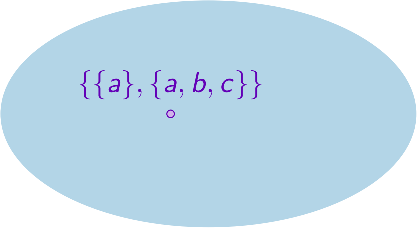

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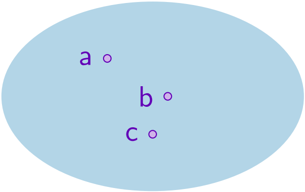


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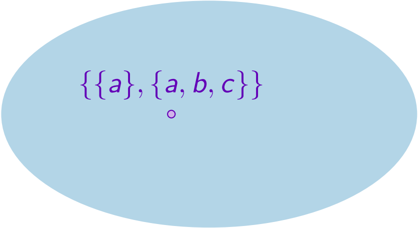


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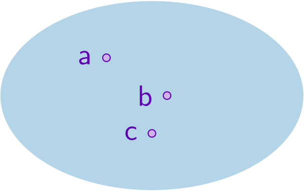


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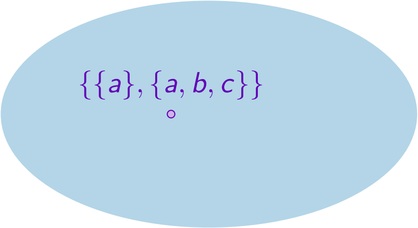


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Example

$A = \{a, b, c, d, e\}$
Supports of $\{a, b, c\}$: $A$, $\{a, b, c\}$, $\{d, e\}$

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What are sequence-like objects?

# Sequence-like objects: strong supports

$v_1, \ldots, v_k$ vertices of $\mathcal{K}_n$.

$x = \{v_1, \ldots, v_k\}$

$y = (v_1, \ldots, v_k)$

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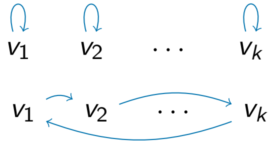


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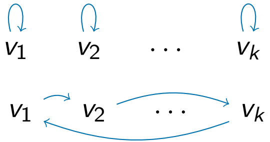


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$\rightsquigarrow$ strong support

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