Choiceless Computation and Logic

Svenja Schalthöfer

January 24, 2019

The Most Important Question in Finite Model Theory

Open Question (Chandra, Harel 1982, Gurevich 1988)

Is there a logic capturing PTIME?

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Theorem (Fagin 1974)

$\exists$SO *captures* NP.

Is there a Logic Capturing PTIME?

PTIME

$\subsetneq$

FP

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PTIME

$\cup\!\!\!\!\!\not\;$

FP captures PTIME on ordered structures

Immerman 1986, Vardi 1982

Is there a Logic Capturing PTIME?

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$\supsetneq$

FPC — captures PTIME on many interesting classes of structures

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$\subsetneq$ Cai, Fürer, Immerman 1992

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$\mathrm{FP} + \mathsf{rk}$ — Dawar, Grohe, Holm, Laubner 2009

$\subsetneq$

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Is there a Logic Capturing PTIME?

PTIME

$\subsetneq$ $\subseteq$

FP + rk CPT Blass, Gurevich, Shelah 1999

$\subsetneq$ $\subsetneq$

FPC captures PTIME on many interesting classes of structures

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Is there a Logic Capturing PTIME?

PTIME

$\subseteq$ $\subseteq$

$\mathrm{FP} + \mathsf{rk}$ $\quad$ CPT $\quad$ captures PTIME on even more classes

$\subsetneq$ $\subsetneq$

FPC $\quad$ captures PTIME on many interesting classes of structures

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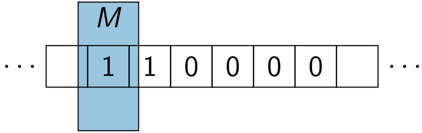

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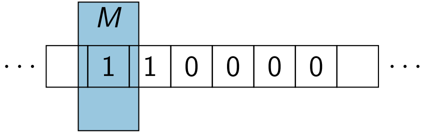

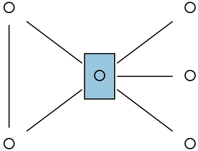

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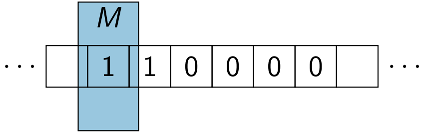

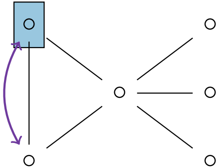

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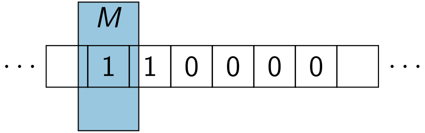

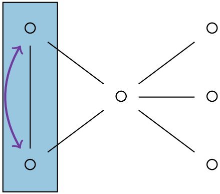

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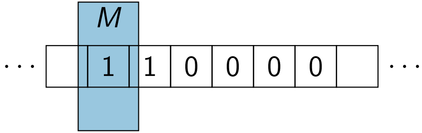

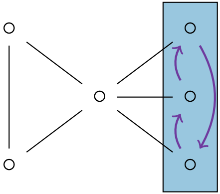

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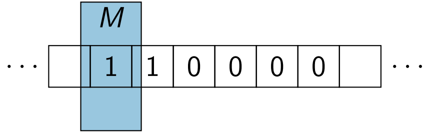

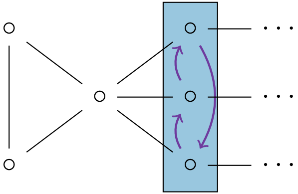

1. Choiceless Polynomial Time

2. Choiceless Logarithmic Space

Computation over Hereditarily Finite Sets

Parallel computations are modelled by sets

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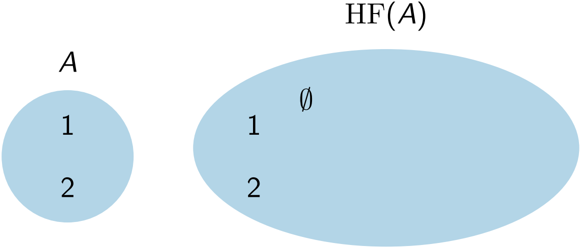

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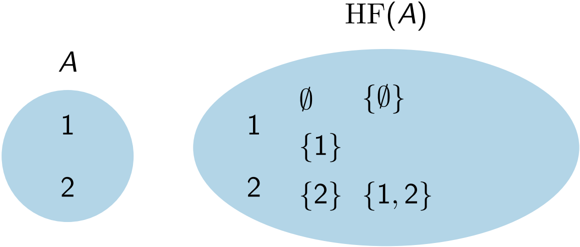

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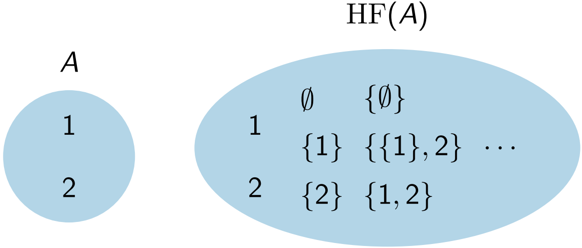

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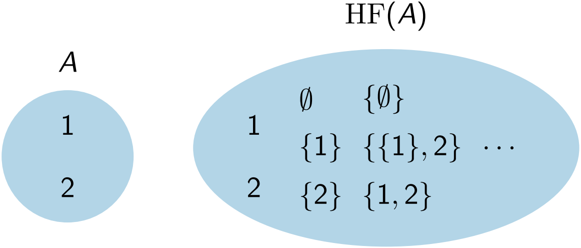
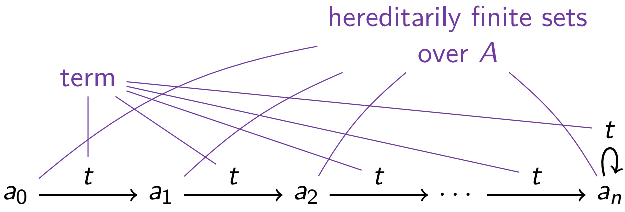


Terms

- Atoms domain of input structure
- $\bigcup\{x, y\}$
- $\{x : x \in \text{Atoms} : \{y : y \in \text{Atoms} : Exy\} \neq \emptyset\}$

Choiceless Polynomial Time

Blass, Gurevich, Shelah 1997



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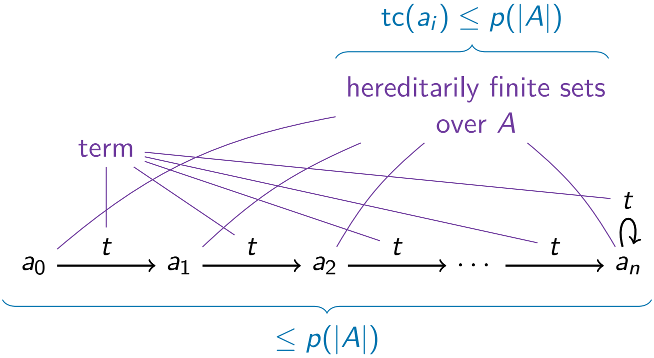
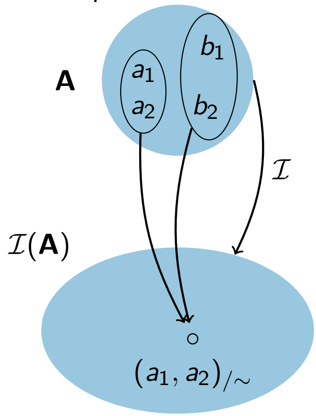


Use: CPT-computations are iterated first-order interpretations

CPT



Fragments and Extensions of CPT

joint work with Erich Grädel, Łukasz Kaiser and Wied Pakusa

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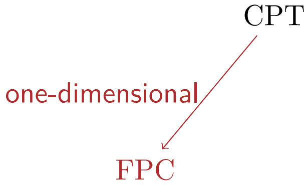

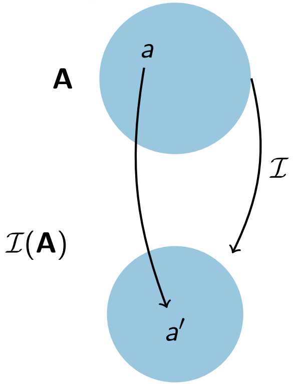

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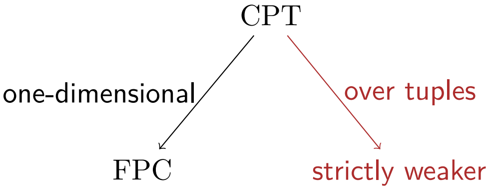

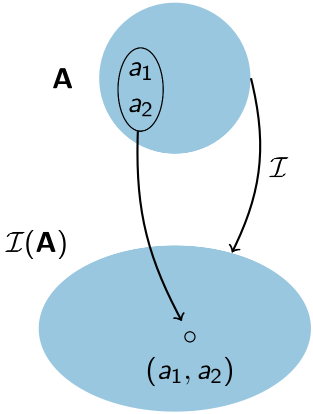

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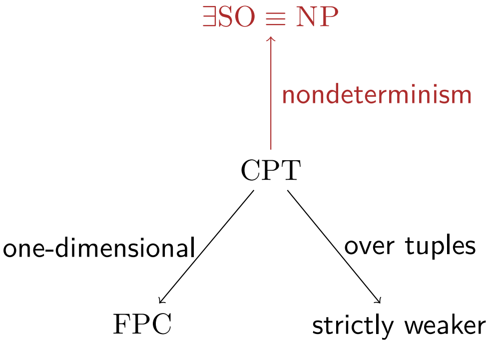

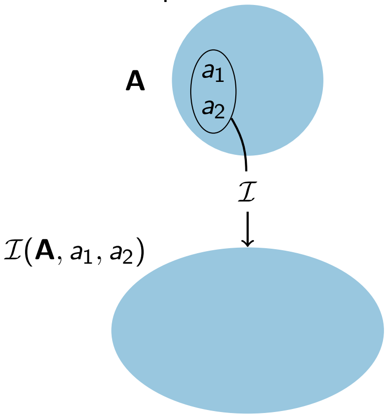

Padded Structures

Blass, Gurevich, Shelah 1999, Laubner 2011

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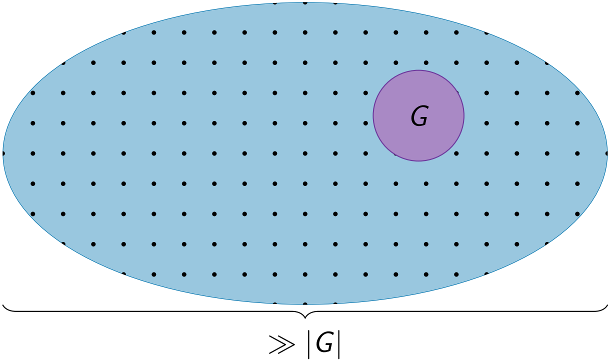

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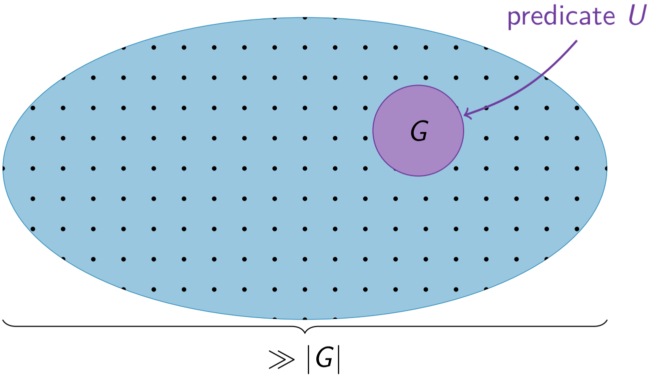
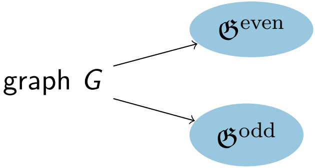


CPT and the Cai-Fürer-Immerman Graphs

joint work with Wied Pakusa and Erkal Selman



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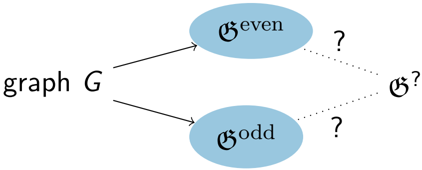

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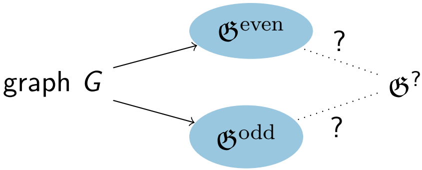

G / sets	ordered
all	✓*
rank $\leq k$	✗*

$*$ Dawar, Richerby, Rossman 2008

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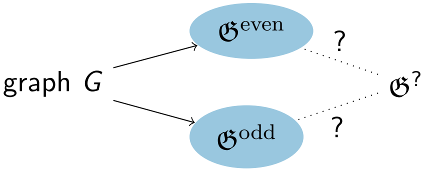

G / sets	ordered	logarithmic colour classes	large degree
all	✓*	✓	✓
rank $\leq k$	✗*	✗	✓

∗ Dawar, Richerby, Rossman 2008

CPT and the Cai-Fürer-Immerman Graphs

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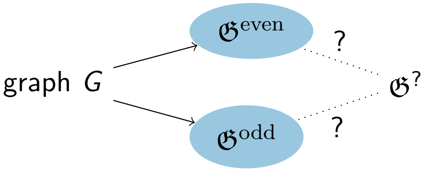

G / sets	ordered	logarithmic colour classes	large degree
all	✓*	✓	✓
rank $\leq k$	✗*	✗	✓
tuple-like	✗	✗	✗

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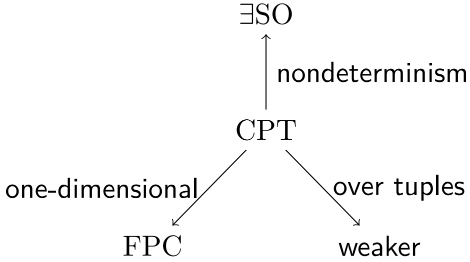
Choiceless Polynomial Time: Conclusion

Structure



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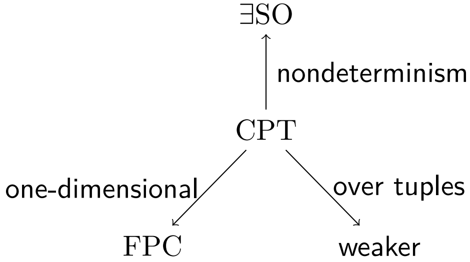

Cai-Fürer-Immerman Query

- logarithmic colours ✓
- large degree
  - bounded rank ✓
  - tuples ✗

1. Choiceless Polynomial Time

2. Choiceless Logarithmic Space

joint work with Erich Grädel

Is there a Logic Capturing Logspace?

Ptime	Logspace
FP	DTC

Captures on ordered structures

Is there a Logic Capturing LOGSPACE?

PTIME

LOGSPACE

FP + rk

CPT

$\subsetneq$ $\subsetneq$

FP

DTC

Captures on ordered structures

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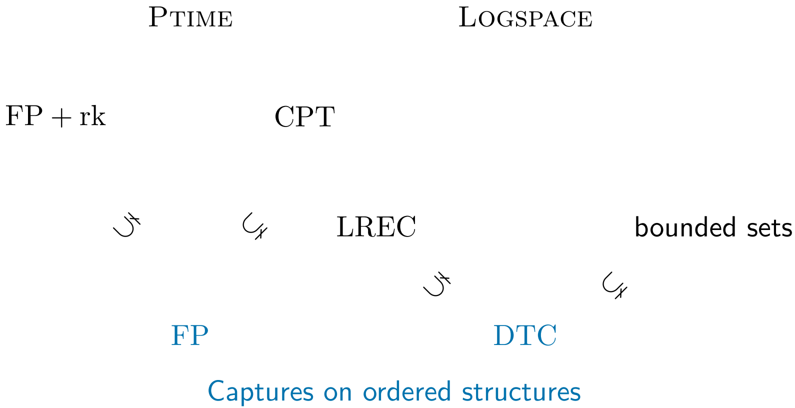

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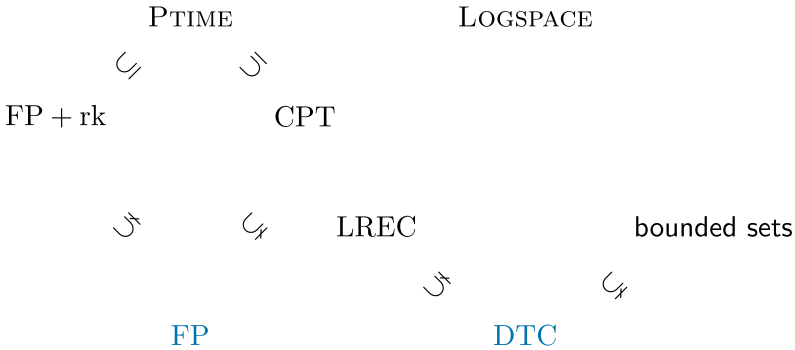

Captures on ordered structures

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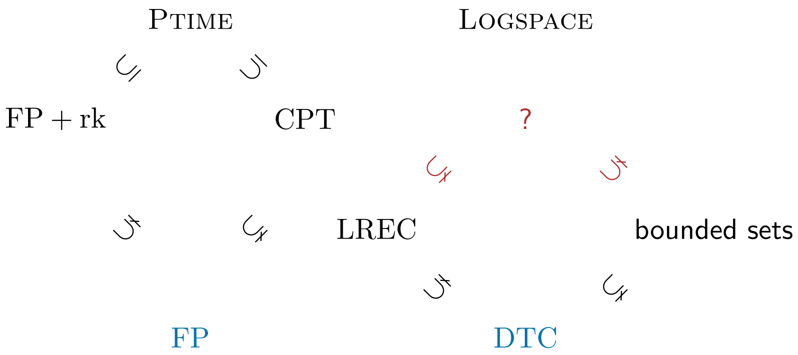
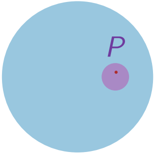


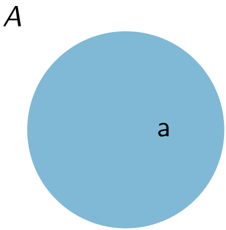
Captures on ordered structures

Why is LOGSPACE complicated?

Size of objects



Sizes of Atoms



"ith element of A"
$\log |A|$ bits

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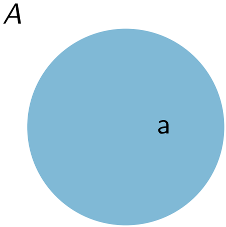

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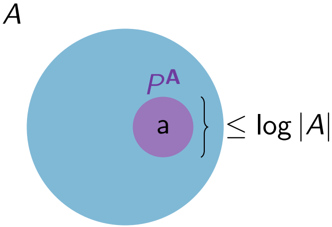

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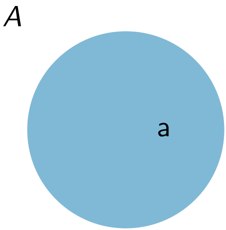

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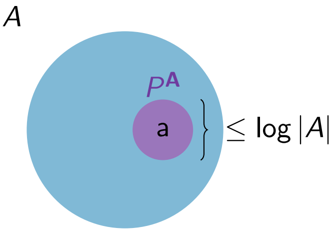
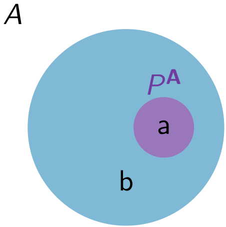


"jth element of P"
$\log \log |A|$ bits

Size-annotated Objects



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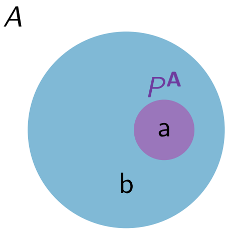

$a \mapsto \log|P| \quad b \mapsto \log|A|$

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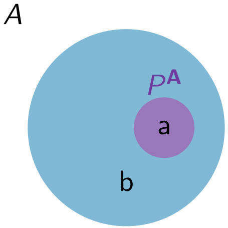

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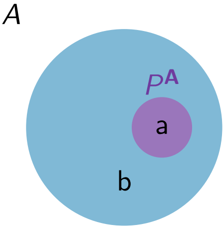
$\{a, b\} \mapsto \log|P| + \log|A| + 1$



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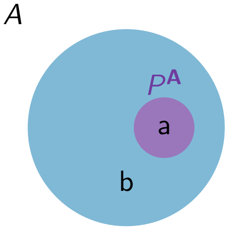

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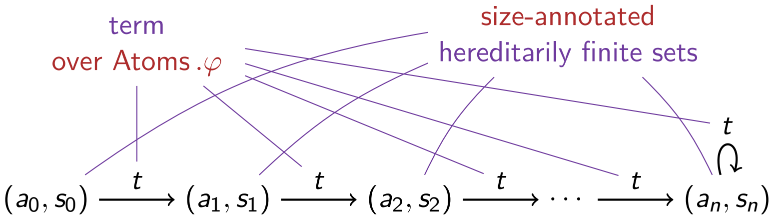
} size annotation of $\{a, b\}$

Induced by terms Atoms $.\varphi$:

$\{\{x, y\} : x \in \text{Atoms}\,.Pu \wedge y \in \text{Atoms}\,.v = v\}$

Choiceless Logarithmic Space



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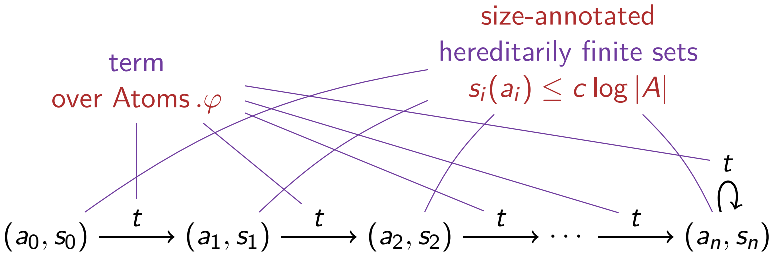

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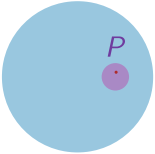
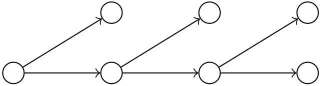


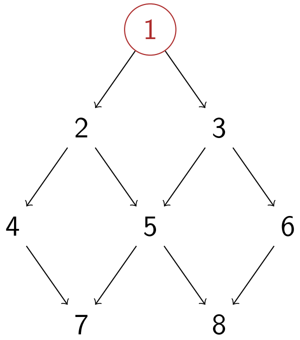
✓

Recursion



L-Recursion

Grohe, Grußien, Hernich, Laubner 2013



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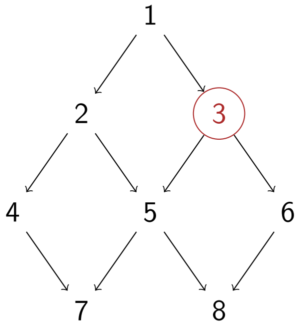
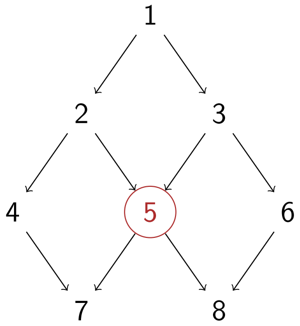


L-Recursion

Grohe, Grußien, Hernich, Laubner 2013



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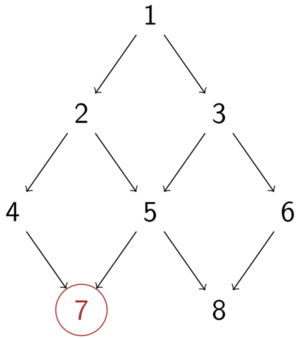

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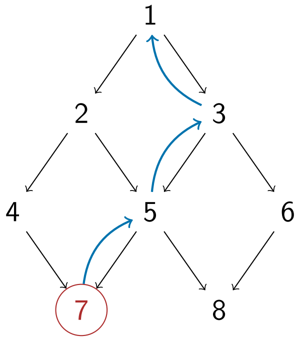
return addresses



unique parent

2nd parent

2nd parent

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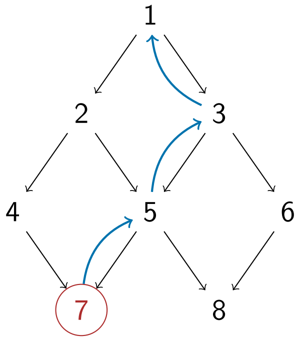

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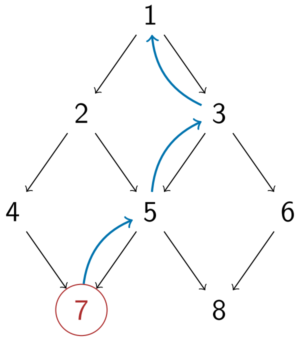

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$\leq c \log n$

input structure → (interpretation) → graph → (unfolding with resources) → DAG

Choiceless Logarithmic Space

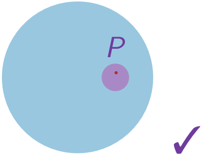
term
over Atoms $.\varphi$
with L-recursion

size-annotated
hereditarily finite sets
$s_i(a_i) \leq c \log |A|$

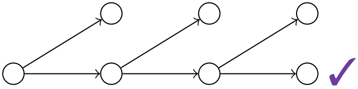
$$(a_0, s_0) \xrightarrow{t} (a_1, s_1) \xrightarrow{t} (a_2, s_2) \xrightarrow{t} \cdots \xrightarrow{t} (a_n, s_n) \circlearrowright t$$

Why is LOGSPACE complicated?

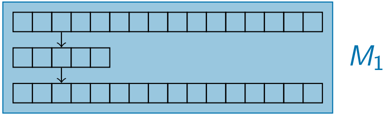
Size of objects



Recursion



Large intermediate results



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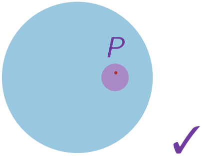

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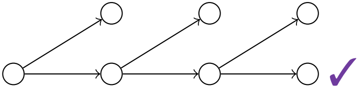

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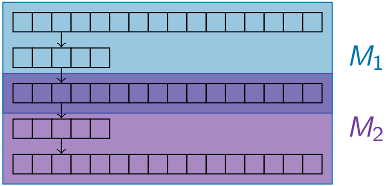
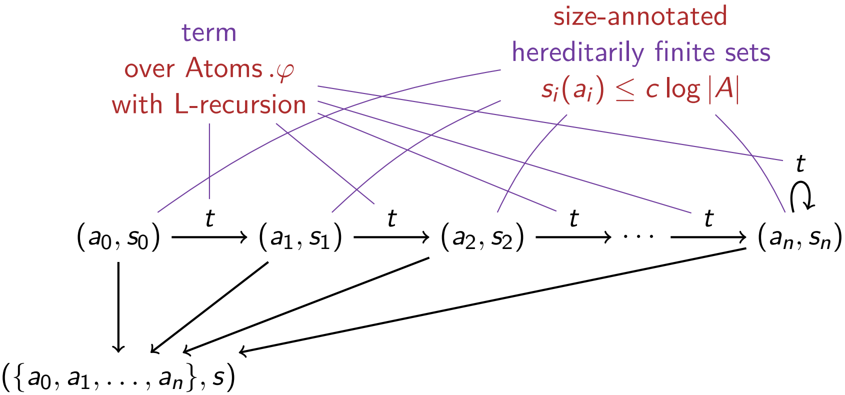


Choiceless Logarithmic Space



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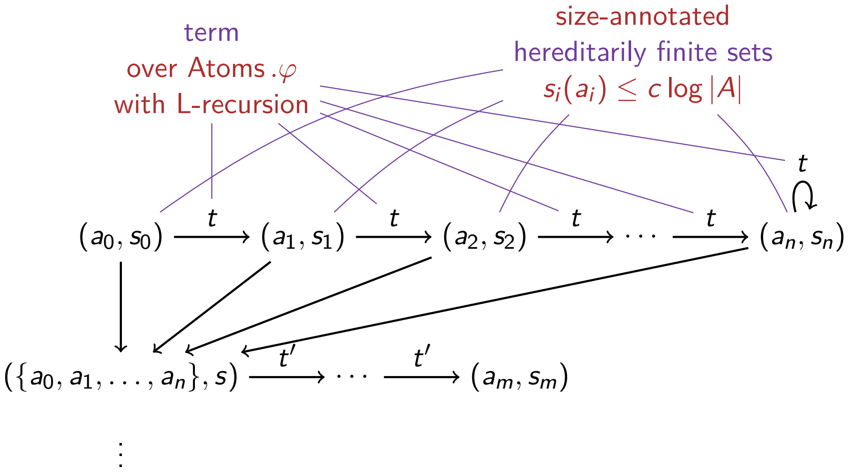


Expressive Power of Choiceless Logarithmic Space

$\text{CPT} \supsetneq \text{CLogspace} \subseteq \textsc{Logspace}$

Expressive Power of Choiceless Logarithmic Space

CPT   LOGSPACE

$\rotatebox{45}{$\subseteq$}$ CLogspace $\subseteq$

CLogspace $\subseteq$ CPT

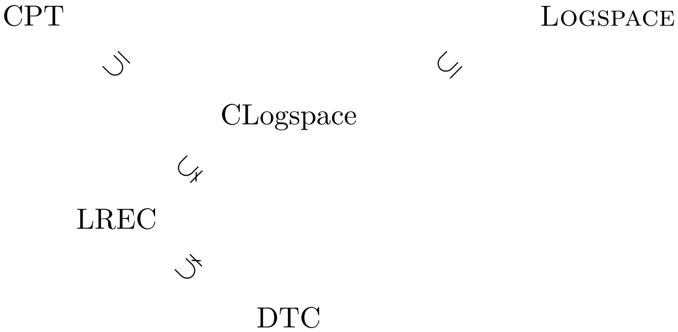
CLogspace $\subseteq$ LOGSPACE

Theorem

CLogspace *captures* LOGSPACE *on very small substructures.*

Structure over A with predicate U such that

$$2^{|U|!(|U|^2(3\log|U|+2)+1)} \leq |A|.$$



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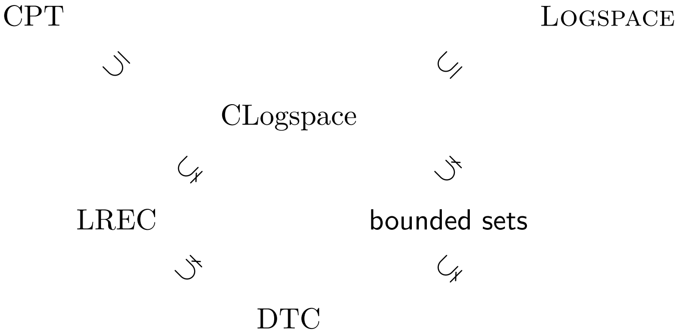

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Choiceless Logarithmic Space: Conclusion

- Size annotations
- L-recursion
- large intermediate results

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