Lehr- und Forschungsgebiet
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SS 2008

Algorithmic Model Theory — Assignment 9

Due: Tuesday, 24 June, 12:00

Exercise 1

Show that $\mathrm{Th}_{\mathrm{mon}}(\mathbb{Q}, <)$ is decidable.

Hint: Any countable dense linear order without endpoints is isomorphic to $(\mathbb{Q}, <)$. Exploit the decidability of the monadic theory of the infinite binary tree.

Exercise 2

It is sometimes useful to consider a modified semantics when evaluating SnS formulae—the so-called *weak semantics* where monadic second-order quantifiers range over finite sets only. To indicate which semantics we want to apply, we use $\models$ and $\models_w$ to denote the usual and the weak semantics, respectively, and we denote the logical system using the weak semantics WS1S.

(a) Show that WS1S and S1S are equally expressive on infinite words.
Hint: Show that, for each Büchi automaton $\mathcal{A}$ on ω-words, one can construct a WS1S-formula ψ such that $\alpha \in L(\mathcal{A})$ if, and only if, $\alpha \models \psi$.

(b) Show that this is not the case for WS2S and S2S, i.e., construct formulae φ and φ_w (do not choose the negation of φ for φ_w!) such that

- $T \models \varphi$ but $T \not\models_w \varphi$, and
- $T \not\models \varphi_w$ but $T \models_w \varphi_w$

where T is the infinite binary tree.

Exercise 3

(a) The *unravelling* of a graph $\mathcal{G} = (V, E)$ from a node $v \in V$ is defined as the structure $\mathcal{T}(\mathcal{G}, v) := (V^*, E, v)$ with $E^{\mathcal{T}} := \{(wa, wab) : wa \in V^* \text{ and } (a, b) \in E^{\mathcal{G}}\}$.

Show that $\mathcal{T}(\mathcal{G}, v)$ is MSO-interpretable in $(\mathcal{G}^*, v)$.

(b) Consider the linearly ordered natural numbers $\mathfrak{N} = (\mathbb{N}, <)$. Show that the expansion $\mathfrak{M} = (\mathfrak{N}, P)$ where $P^{\mathfrak{M}} = \{\frac{1}{2}k(k+1) : k \in \mathbb{N}\}$ is MSO-interpretable in the iteration $\mathfrak{N}^*$.

Hint: a) $\frac{1}{2}k(k+1) = \sum_{i=0}^{k} i$. b) Represent the set of natural numbers by a suitable path through the iteration on which the positions representing numbers in P are definable.