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Algorithmic Model Theory — Assignment 3

Due: Friday, 6 May, 13:00

Exercise 1

We say that a domino system $\mathcal{D} = (D, H, V)$ admits a *horizontially periodic* tiling if we can find a tiling $\tau : \mathbb{N} \times \mathbb{N} \to D$ and an integer $h \geq 1$ such that for all $(x, y) \in \mathbb{N} \times \mathbb{N}$ we have $\tau(x, y) = \tau(x + h, y)$.

Prove or disprove that every domino systems $\mathcal{D}$ which admits a horizontially periodic tiling also admits a periodic tiling.

Exercise 2

In this exercise we want to show that the model construction for FO^2-formulae from the lecture is optimal in the following sense: in general it does not suffice to take, for each $\forall\exists$-subformula, only two copies (instead of three) of the set P which consists of atomic 1-types which are realised at least twice in $\mathfrak{A}$.

Find a satisfiable FO^2-sentence $\varphi = \forall x \forall y \alpha \wedge \forall x \exists y \beta$ where α, β are quantifier-free such that:

- no model of φ contains a king (i.e. $K = \emptyset$) and
- for every model $\mathfrak{A}$ of φ there is no corresponding finite model over the universe $P \times \{0, 1\}$.

Exercise 3

In the lecture we saw that the class $\mathcal{F}$ consisting of all FO-sentences of the form $\forall x\, \eta(x)$ (for quantifier-free η) which only contain unary function symbols is a conservative reduction class.

Prove that the more restricted class $\mathcal{F}_2 \subseteq \mathcal{F}$ consisting of sentences in $\mathcal{F}$ that contain only two unary function symbols is also a conservative reduction class.

Hint: Transform sentences $\forall x \varphi$ with unary function symbols $f_1, \dots, f_m$ into sentences $\forall x \tilde{\varphi} := \forall x \varphi[x/hx, f_i/hg^i]$ where h, g are fresh unary function symbols.

Exercise 4

Show that the following classes of FO-sentences, where R is a binary relational symbol and f is a unary function symbol, contain infinity axioms.

(i) $\exists x \forall y \forall z\, \eta(x, y, z)$, $\eta \in \mathrm{FO}(\{f\})$ quantifier-free.

(ii) $\forall x \exists y \forall z\, \eta(x, y, z)$, $\eta \in \mathrm{FO}(\{R, f\})$ quantifier-free and without equality.

(iii) $\forall x \exists y\, \eta(x, y)$, $\eta \in \mathrm{FO}(\{f\})$ quantifier-free.

(iv) The two variable fragment FO^2 extended by the counting quantifiers $\exists^{\leq n}$ for every $n \in \mathbb{N}$, where $\exists^{\leq n} x \varphi$ expresses that there are no more than n elements x that satisfy φ.