Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, W. Pakusa, F. Reinhardt,
M. Voit

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Algorithmic Model Theory — Assignment 7

Due: Friday, 10 June, 13:00

Exercise 1

Show that connectedness of (undirected) graphs cannot be expressed in existential second-order logic (Σ_1^1) if we also allow infinite graphs.

*H*int: Use the compactness theorem for first-order logic (consider the existentially quantified relations as part of models).

Exercise 2

In this exercise we want to show that the classical "Łoś-Tarski Theorem" does not hold if we restrict to finite structures. Recall that this theorem says the following: for a sentence φ of first-order logic the following statements are equivalent:

- φ is preserved under substructures, i.e. for all $\mathfrak{B} \subseteq \mathfrak{A}$ we have $(\mathfrak{A} \models \varphi \Rightarrow \mathfrak{B} \models \varphi)$.
- φ is equivalent to a universal sentence, i.e. a sentence of the form

$$\psi = \forall x_1 \cdots \forall x_k \, \eta(x_1, \ldots, x_k),$$

where η is quantifier-free.

Let $\tau = \{<, R, P, \min, \max\}$ where $<, R$ are two binary relation symbols, where P is a unary relation symbol, and where $\min, \max$ are two constant symbols.

Furthermore, let φ be a (! universal) $\mathrm{FO}(\tau)$-sentence which says that "$<$ is a linear order with minimal element min and maximal element max, and R is a subset of the corresponding successor relation". Finally, let $\psi = \forall x(x = \max \vee \exists y Rxy)$.

(a) Show that for every finite τ-structure $\mathfrak{A}$ with $\mathfrak{A} \models \varphi \wedge \psi$ it holds that for each substructure $\mathfrak{B} \subseteq \mathfrak{A}$ with $\mathfrak{B} \models \psi$ we have $\mathfrak{A} = \mathfrak{B}$.

(b) Consider the sentence $\vartheta = \varphi \wedge (\psi \rightarrow \exists z Pz) \in \mathrm{FO}(\tau)$. Show that ϑ is preserved under finite substructures, i.e. for all $\mathfrak{B} \subseteq \mathfrak{A}$ with finite $\mathfrak{A}$ we have $\mathfrak{A} \models \vartheta \Rightarrow \mathfrak{B} \models \vartheta$.

(c) Show that ϑ is not equivalent to a universal sentence over finite τ-structures.

Exercise 3

We restrict to finite, relational vocabularies.

We define a *counting-variant* of the Ehrenfeucht-Fraïssé game denoted as $G_m^{\#}(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$. One round in the game proceeds as follows. First, Spoiler selects one of the structures $\mathfrak{A}$ and $\mathfrak{B}$ and a *finite subset* $M_A \subseteq A$ (or $M_B \subseteq B$). Duplicator answers with a corresponding subset

$M_B \subseteq B$ (or $M_A \subseteq A$) such that $|M_A| = |M_B|$. Then Spoiler selects an element $d \in M_B$ (or $c \in M_A$) and Duplicator answer by picking $c \in M_A$ (or $d \in M_B$). The new position is $G^{\#}_{m-1}(\mathfrak{A}, \bar{a}, c, \mathfrak{B}, \bar{b}, d)$. The sets M_A, M_B are forgotten after each round. The winning condition is as in the usual game, i.e. after m rounds Duplicator has to guarantee that at the final position $G^{\#}_0(\mathfrak{A}, \bar{a}, c_1, \dots, c_m, \mathfrak{B}, \bar{b}, d_1, \dots, d_m)$, $(\bar{a}, \bar{c}) \mapsto (\bar{b}, \bar{d})$ defines a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Furthermore, we let $\mathrm{FO}^{\#}$ denote the extension of first-order logic by all counting quantifiers $\exists^{\geq k} x$ for all $k \geq 1$ ("there exists at least k elements x such that...").

(a) Prove that $\mathrm{FO}^{\#}$ has the same expressive power as FO, but that for the translation of $\mathrm{FO}^{\#}$-formulas into equivalent FO-formulas one has to increase the quantifier rank of formulas.

(b) Show that if Duplicator wins the game $G^{\#}_m(\mathfrak{A}, \bar{a}, \mathfrak{B}, \bar{b})$, then no formula $\varphi(\bar{x}) \in \mathrm{FO}^{\#}$ of quantifier rank m can distinguish between $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$.

(c) Show that, in contrast to the classical case, the corresponding game equivalence classes cannot be defined by $\mathrm{FO}^{\#}$-sentences. Construct for some $m \geq 1$ an example of a (possibly infinite) structure $\mathfrak{A}$ such that for no sentence $\varphi \in \mathrm{FO}^{\#}$ of quantifier rank m it holds that

$$\text{Duplicator wins } G^{\#}_m(\mathfrak{A}, \mathfrak{B}) \Leftrightarrow \mathfrak{B} \models \varphi.$$