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SS 2016

Algorithmic Model Theory — Assignment 8

Due: Friday, 17 June, 13:00

Exercise 1

Let $r \leq s$. Show that every $\mathrm{FO}(\tau)$-formula $\varphi(\bar{x})$ which is r-local is also s-local.

Exercise 2

In this exercise we want to apply Hanf's Theorem to show that connectivity of graphs cannot be defined in *existential monadic second-order logic (EMSO)*, that is in the logic consisting of all Σ_1^1-sentences of the form

$$\exists P_1 \cdots \exists P_k \vartheta, \vartheta \in \mathrm{FO},$$

where the existential quantifiers over the second-order variables P_i are restricted to *monadic* predicates P_i.

(a) Show that EMSO can define the class of all (finite) graphs which are *not* connected.

(b) Show that EMSO cannot define the class of all (finite) graphs which are connected.

- Assume that some sentence $\psi = \exists P_1 \cdots \exists P_k \vartheta \in$ EMSO defines this class.
- For $n \geq 1$, consider a *connected* graph $\mathcal{G}_n$ consisting of a directed cycle of lenght n, i.e. $\mathcal{G}_n \models \psi$.
- Think of the predicates P_i as colours of the nodes of this cycle and determine the number of isomorpishm types of r-neighbourhoods (to get the appropriate value of r apply Hanf's Theorem to ϑ).
- Choose the parameter n large enough such that at least two nodes on the (coloured) cycle have disjoint and isomorphic r-neighbourhoods. Use these two nodes to construct a new graph consisting of two disjoint cycles which is a model of ψ.

Exercise 3

We consider the class $\mathcal{K}_d$ of (undirected, finite) graphs $G = (V, E)$ with degree $\leq d$ for some constant d. We want to apply Gaifman's Theorem to show that for each fixed first-order sentence $\varphi \in \mathrm{FO}(\{E\})$, the model-checking problem $G \models \varphi$ for graphs $G \in \mathcal{K}_d$ is decidable in linear time.

- Explain why it suffices to solve this problem for basic local sentences

$$\exists x_1 \cdots \exists x_\ell (\bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi^r(x_i)).$$

- Show that one can compute in linear time, given a graph $G = (V, E) \in \mathcal{K}_d$, the set P_ψ of elements $v \in V$ such that $G \models \psi^r(v)$.

  Hint: Use that r and d are constants.

- Show that one can decide in linear time, given a graph $G = (V, E) \in \mathcal{K}_d$, whether the set P_ψ (as defined above) contains an r-scattered tuple of length ℓ.

  Hint: Use the fact that the $2r$-neighbourhood of a maximal r-scattered tuple of P_ψ elements covers P_ψ.