Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, W. Pakusa, F. Reinhardt,
M. Voit

SS 2016

Algorithmic Model Theory — Assignment 11

Due: Friday, 9 July, 13:00

Remark: Structures are finite and graphs are undirected in this exercise.

Exercise 1

Prove or disprove that the following classes are $L^{\omega}_{\infty\omega}$-definable (over the appropriate signatures).

(a) the class of all connected graphs,

(b) the class of all graphs with an odd number of edges,

(c) the class of all cyclic groups,

(d) for every $P \subseteq \mathbb{N}$, the class of all linear orderings $(A, <)$ such that $|A| \in P$,

(e) the class of all structures (A, P) (with a monadic predicate P) such that $|P| \geq |A|/2$,

(f) the class consisting of all pairs $((V, E, <_G), (W, F, <_H))$ of ordered graphs $G = (V, E, <_G)$ and $H = (W, F, <_H)$ such that G is isomorphic to H (as ordered graphs). Before you solve this part of the exercise, explain how you would encode such pairs of ordered graphs as usual relational structures.

Exercise 2

Construct IFP-formulas which define in a rooted tree $\mathcal{T} = (V, E, r)$, where r denotes its root, the following relations.

(a) $R_1 = \{(x, y) : \text{the subtrees rooted in } x \text{ and } y \text{ have the same height}\}$

(b) $R_2 = \{(x, y) : \text{the nodes } x \text{ and } y \text{ are on the same level of the tree}\}$

(c) $R_3 = \{x : \text{the subtree rooted in } x \text{ has a perfect matching}\}$.

Exercise 3

(a) Let $\mathcal{K}$ be a class of finite structures. We say that $\mathcal{K}$ is *fixed-point bounded* if for every first-order formula $\varphi(X, \bar{x})$ (positive in X and the arity of X coincides with the length of $\bar{x}$) there is a constant m_φ such that for all structures $\mathfrak{A} \in \mathcal{K}$ the inductive construction for the least fixed point of the monotone operator $F^{\mathfrak{A}}_{\varphi}$ defined by φ on $\mathfrak{A}$ reaches the least fixed point after at most m_φ many steps (the closure ordinal is $\leq m_\varphi$). Show that LFP $\equiv$ FO over every fixed-point bounded class of structures $\mathcal{K}$.

(b) Show that LFP $\equiv$ FO over the class of all complete graphs.

Hint: Make use of (a) and the fact that complete graphs have the symmetric group as automorphism group.