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Algorithmic Model Theory — Assignment 12

Due: Friday, 15 July, 13:00

Remark: Structures are finite and relational and graphs are undirected in this exercise.

Exercise 1

A lazy engineer has designed a system $\mathfrak{A}$ (a τ-structure) which should satisfy a first-order specification $\forall x \varphi(x) \in \mathrm{FO}(\tau)$. Unfortunately, $\mathfrak{A} \not\models \forall x \varphi(x)$.

The lazy engineer doesn't want to start from the beginning and thus he defines the substructure $\mathfrak{A}'$ of $\mathfrak{A}$ induced on all elements $A' = \{a \in A : \mathfrak{A} \models \varphi(a)\}$ in $\mathfrak{A}$ which satisfy $\varphi(x)$. If he is very lucky, then $A' \neq \emptyset$ and $\mathfrak{A}' \models \forall x \varphi(x)$. In this case he found a system which meets the specification. Otherwise he continues in this way until he finds an appropriate system (or until he obtains the "empty structure").

Construct an $\mathrm{IFP}(\tau)$-sentence which holds in a τ-structure $\mathfrak{A}$ if the lazy engineer finds, starting with $\mathfrak{A}$, a (non-empty) system $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B} \models \forall x \varphi(x)$ by following his strategy.

Exercise 2

A graph $\mathcal{G} = (V, E^{\mathfrak{A}})$ encodes an $(V \times V)$-matrix $M^{\mathcal{G}}$ over $\mathbb{F}_2$ which is

$$M^{\mathcal{G}}(a,b) = \begin{cases} 0, & \text{if } (a,b) \notin E \\ 1, & \text{if } (a,b) \in E. \end{cases}$$

In other words, $M^{\mathcal{G}}$ is the adjacency matrix of the graph $\mathcal{G}$. In the same way, every FPC-formula $\varphi(x,y)$ defines an $(V \times V)$-matrix $\varphi^{\mathcal{G}}$ over $\mathbb{F}_2$ in the graph $\mathcal{G}$. We want to show that matrix multiplication (over $\mathbb{F}_2$) is definable in FPC.

(a) Construct a formula $\varphi(x,y) \in \mathrm{FPC}(E)$ such that for all graphs $\mathcal{G}$ it holds $\varphi^{\mathcal{G}} = (M^{\mathcal{G}})^2$.

(b) Construct a formula $\psi(x,y) \in \mathrm{FPC}(E)$ such that for all graphs $\mathcal{G}$ it holds $\psi^{\mathcal{G}} = (M^{\mathcal{G}})^{2^{|V|}}$.

Exercise 3

Let $\mathfrak{A}$ be a τ-structure. We make the following convention: we interpret numerical tuples $\bar{\nu} = (\nu_{k-1}, \dots, \nu_1, \nu_0) \in \{0, \dots, |A|-1\}^k$ as numbers in the $|A|$-adic representation, i.e. we associate the number $\sum_{i=0}^{k-1} \nu_i |A|^i$ to each tuple $\bar{\nu} \in \{0, \dots, |A|-1\}^k$.

Show that the expressive power of FPC does not increase if we allow counting quantifiers of higher arity, i.e. formulas $\#_{x_0 x_1 \cdots x_{k-1}} \varphi(x_0, \dots, x_{k-1}) \leq (\nu_{k-1}, \dots, \nu_0)$ where in a structure $\mathfrak{A}$ the value of $\#_{x_0 x_1 \cdots x_{k-1}} \varphi(x_0, \dots, x_{k-1})$ is the number of tuples $\bar{a}$ such that $\mathfrak{A} \models \varphi(\bar{a})$ (with respect to the encoding introduced above). For simplicity, it suffices to consider the case $k = 2$.