**Lehr- und Forschungsgebiet**
**Mathematische Grundlagen der Informatik**
RWTH Aachen
Prof. Dr. E. Grädel, F. Abu Zaid, W. Pakusa

WS 2011/12

# Algorithmic Model Theory — Assignment 4

Due: Monday, 14 November, 12:00

## Exercise 1

Let $\tau_n$ be the vocabulary consisting of the unary relation symbols $P_0, \dots, P_{n-1}, Q_0, \dots, Q_{n-1}$.

Construct a formula $\varphi_n \in \mathrm{FO}(\tau_n)$ whose length is bounded by a polynomial in $n$ such that every model of $\varphi_n$ contains at least $2^{2n}$ elements.
*Hint:* Encode an $(2^n \times 2^n)$-grid in models of $\varphi_n$.

## Exercise 2

$\varepsilon$-FO is the extension of FO by Hilbert's *choice operator* (also known as $\varepsilon$-operator). The syntax of $\varepsilon$-FO is given by the usual rules together with an additional $\varepsilon$-rule: If $\psi$ is a formula, and $x$ is a variable, then $\varepsilon x \psi$ is a term (read "an $x$ such that $\psi$").

An interpretation for an $\varepsilon$-FO formula is given by an FO-interpretation $(\mathfrak{A}, \mathcal{I})$ together with a choice function on the universe of $\mathfrak{A}$, i.e. a mapping $F : \mathcal{P}(A) \to A$ such that $F(X) \in X$ for all $X \neq \emptyset$. The value of a term $\varepsilon x \psi$ is defined as $F(\{a \in A : (\mathfrak{A}, \mathcal{I}) \models \psi(a)\})$.

(a) Show that the quantifiers $\exists$ and $\forall$ can be expressed with the $\varepsilon$-operator.

(b) Construct an infinity axiom $\varphi$ in $\varepsilon$-$\mathrm{FO}^2$ over the empty vocabulary, i.e., $\varphi$ contains only the $\varepsilon$-operator, two variables $x$ and $y$, and equality, but neither relation nor function symbols.

## Exercise 3

Specify a conservative reduction from $[\forall^2, (0,1), (1)]$ to $[\forall^2, (1), (0,1)]$, i.e., from the class of formulae with quantifier prefix $\forall^2$ over a vocabulary containing only one binary relation and a unary function symbol to the class of formulae with the same quantifier prefix but over a vocabulary containing a unary relation and a binary function symbol. Note that equality may not be used in the formulae.

*Hint:* A binary relation $P$ and a unary function $f$ can be encoded using a unary relation $Q$ and a binary function $g$ in the following way: $P = \{(a,b) : g(a,b) \in Q\}$, and $f(a) = g(a,a)$.