Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
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Algorithmic Model Theory — Assignment 8

Due: Monday, 12 December, 12:00

Exercise 1

Show that the syntax of LFP ensures operators used in fixed point definitions to be monotone. Let $\varphi(S, R, \bar{x})$ be a first-order formula over the signature $\tau \cup \{R, S\}$ such that R and S occur only positively (i.e. in the scope of an even number of negations) in φ and such that the length of $\bar{x}$ matches the arity k of R. Show that for every τ-structure $\mathfrak{A}$ the following holds.

(a) For all $S \subseteq A^l$ the operator $F_\varphi^{\mathfrak{A},S} : \mathcal{P}(A^k) \to \mathcal{P}(A^k), R \mapsto \{\bar{a} \in A^k : \mathfrak{A} \models \varphi(S, R, \bar{a})\}$ is monotone.

(b) The operator $G_\varphi^{\mathfrak{A}} : \mathcal{P}(A^l) \to \mathcal{P}(A^l), S \mapsto \{\bar{a} \in A^l : \bar{a} \in \mathbf{lfp}(F_\varphi^{\mathfrak{A},S})\}$ is monotone.

Exercise 2

Let $\mathcal{G} = (V, E, P)$ be a finite directed graph extended by a monadic predicate P. Describe the relations in $\mathcal{G}$ that are defined by the following LFP-formulae.

(a) $[\mathbf{gfp}\, Rx.\, (Px \land \exists y(Exy \land [\mathbf{lfp}\, Sx.\, (Rx \lor \exists y(Exy \land Sy))]\,(y))\,](x)$

(b) $[\mathbf{lfp}\, Uxy.\, (Exy \lor \exists z(Exz \land [\mathbf{lfp}\, Gxy.\, (\exists z(Exz \land Uzy))]\,(z, y))\,](x, y)$

Exercise 3

We consider two monotone operators

$$F : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A)$$
$$G : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(B),$$

and define the simulataneous operator of F and G by

$$H : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A) \times \mathcal{P}(B), H(X, Y) \mapsto (F(X), G(Y)).$$

This operator is monotone as well and we denote its least fixed point (where inclusion is considered component-wise) by $H^\infty = (F^\infty, G^\infty)$. As in the case of a single operator, the fixed point H^∞ can be constructed inductively starting with $H^0 = (\emptyset, \emptyset)$ and iterating H.

For $X \subseteq A$ we define the (monotone) operator $G_X : \mathcal{P}(B) \to \mathcal{P}(B), Y \mapsto G(X, Y)$. Furhermore, we let $E : \mathcal{P}(A) \to \mathcal{P}(A), X \mapsto F(X, \mathbf{lfp}(G_X))$. Prove that E is monotone and $\mathbf{lfp}(E) = F^\infty$.