Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, F. Abu Zaid, W. Pakusa

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Algorithmic Model Theory — Assignment 10

Due: Monday, 9 January, 12:00

Exercise 1

Let $\mathcal{G} = (V, (E_i)_{1\leq i\leq m}, (P_i)_{1\leq i\leq m})$ be a transition system. In the lecture the following deflationary operator $F\colon \mathcal{P}(V^2) \to \mathcal{P}(V^2)$ (for defining a linear ordering on the bisimulation quotient of $\mathcal{G}$) was considered.

$$\begin{aligned}
\preceq^1 = F(V,V) &= \{(u,v) : \mathcal{G} \models \bigwedge_{i\leq m} P_i u \to (P_i v \vee \bigvee_{j<i} (\neg P_j u \wedge P_j v))\} \\
\preceq^i = F(\preceq^{i-1}) &= \{(u,v) \in \preceq^{i-1} : (v,u) \notin \preceq^{i-1}\} \cup \\
&\quad \{(u,v) \in \preceq^{i-1} : \text{if there is a } \sim^{i-1} \text{-class separating } u \text{ and } v \text{ then} \\
&\qquad \text{for the minimal such class } C \text{ it holds that } uE \cap C = \emptyset \text{ and } vE \cap C \neq \emptyset\}.
\end{aligned}$$

Recall that $v \sim^i w :\Leftrightarrow (v \preceq^i w \text{ and } w \preceq^i v)$.

(a) Show explicitly that $\preceq^i$ defines a total preorder on $V \times V$ for all $i \geq 1$, i.e. show that $\preceq^i$ is transitive and total. Conclude that the deflationary fixed point $\preceq^\infty$ is a linear ordering on the bisimulation quotient of $\mathcal{G}$.

(b) Give an analogous definition of $\preceq^\infty$ in terms of an inflationary rather than a deflationary fixed point induction and formulate your definition in IFP.

Hint: Define for the inflationary induction strict (and not necessarily total) preorders $\prec^i$ such that $\prec^\infty$ is a (strict) linear order on the bisimulation quotient of $\mathcal{G}$.

Exercise 2

Let Σ be an alphabet. A language $L \subseteq \Sigma^*$ can be identified with a class of word structures $\mathcal{K}_L = \{\mathcal{W}_x : x \in L\}$ where

$$\mathcal{W}_x = (\{0, \dots, |x|-1\}, (P_a = \{i : x(i) = a\})_{a\in\Sigma}, <).$$

It is a well-known fact that L is regular, i.e. recognisable by a finite automaton $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$, if and only if $\mathcal{K}_L$ is definable in MSO. Let LFP^M denote the fragment of LFP in which only monadic fixed point variables (but simultaneous fixed point definitions) are allowed.

(a) Prove, using the equivalence of MSO-definability and regularity of languages, that LFP^M and MSO have the same expressive power on the class of word structures.

(b) Prove or disprove, again using the equivalence of MSO-definability and regularity of languages, that the same holds for MSO and full LFP on the class of word structures.

Hint: For the following exercises you can use the inflationary stage comparison theorem for LFP which was presented in the lecture.

Exercise 3

Construct LFP-formulas which define in a rooted tree $\mathcal{T} = (V, E, r)$, where r denotes its root, the following relations.

(a) $R_1 = \{(x, y) :$ the subtrees rooted in x and y have the same height$\}$

(b) $R_2 = \{(x, y) :$ the nodes x and y are on the same level of the tree$\}$

(c) $R_3 = \{x :$ the subtree rooted in x possesses a perfect matching$\}$.

Exercise 4

A k edge coloured connected graph $\mathcal{G} = (V, E, C_1, \ldots, C_k)$ is a connected graph (V, E) which is extended by k binary predicates $C_1, \ldots, C_k \subseteq E$ that encode a valid k edge colouring of (V, E), i.e. the sets $C_1, \ldots, C_k$ are pairwise disjoint, $\bigcup_{i=1}^{k} C_i = E$ and $|(\{v\} \times V) \cap C_i| \leq 1$ for all $v \in V$ and $1 \leq i \leq k$.

(a) Construct an LFP-formula $\varphi(p, x, y)$ such that for all $a \in V$ the relation defined by φ in $\mathcal{G}$ with parameter a, i.e. the relation $\{(v, w) : \mathcal{G} \models \varphi(a, v, w)\}$, is a linear order on V.

Hint: Use the colouring of the vertices to identify a spanning tree of $\mathcal{G}$.

(b) Conclude, using the Immerman-Vardi Theorem, that LFP captures PTIME on the class of connected k edge coloured graphs.