Lehr- und Forschungsgebiet
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Algorithmic Model Theory — Assignment 12

Due: Monday, 23 January, 12:00

Exercise 1

Show that the following classes of (undirected) graphs are definable in FPC.

(a) Graphs having an odd number of edges. *Hint: Handshaking lemma*

(b) Graphs having an even number of connected components.

Exercise 2

Let $\mathfrak{A}$ be a finite τ-structure. We make the following convention: we interpret numerical tuples $\bar{\nu} = (\nu_{k-1}, \dots, \nu_1, \nu_0) \in \{0, \dots, |A| - 1\}^k$ as numbers in $|A|$-adic representation, i.e. we associate the value $\sum_{i=0}^{k-1} \nu_i |A|^i$ to each tuple $\bar{\nu} \in \{0, \dots, |A| - 1\}^k$.

Show that the expressive power of FPC does not increase if we allow counting quantifiers of higher arity, i.e. formulas $\#_{x_0 x_1 \cdots x_{k-1}} \varphi(x_0, \dots, x_{k-1}) \leq (\nu_{k-1}, \dots, \nu_0)$ where in a structure $\mathfrak{A}$ the value of $\#_{x_0 x_1 \cdots x_{k-1}} \varphi(x_0, \dots, x_{k-1})$ is the number of tuples $\bar{a}$ such that $\mathfrak{A} \models \varphi(\bar{a})$ (with respect to the encoding introduced above). For simplicity, you may only consider the case $k = 2$.

Exercise 3

We denote by $\mathrm{C}^k_{\infty\omega}$ the k-variable infinitary logic with counting which is the extension of $\mathrm{L}^k_{\infty\omega}$ by all quantifiers of the form $\exists^{\geq n} x$ with the intended semantics "there are at least n elements x such that". Similarly to the definition of $\mathrm{L}^\omega_{\infty\omega}$ we set $\mathrm{C}^\omega_{\infty\omega} := \bigcup_{k \geq 1} \mathrm{C}^k_{\infty\omega}$.

For illustration, consider the following sentence of $\mathrm{C}^1_{\infty\omega}$ defnining the class of graphs with an even number of vertices

$$\bigvee_{n \in 2\omega} (\exists^{\geq n} x (x = x) \wedge \neg \exists^{\geq n+1} x (x = x)).$$

Show that FPC $\leq \mathrm{C}^\omega_{\infty\omega}$ (for formulas without free numerical variables).

Hints:

- Construct for every formula $\varphi(\bar{x}, \bar{\nu}) \in$ FPC and $n \in \omega, \bar{\nu} \in \{0, \dots, n-1\}^k$ a formula $\varphi^\star_{n,\bar{\nu}}(\bar{x})$ which is equivalent to φ on structures of size n, i.e. for all $\mathfrak{A}$ of size n we have

$$\mathfrak{A} \models \varphi(\bar{a}, \bar{\nu}) \text{ iff } \mathfrak{A} \models \varphi^\star_{n,\bar{\nu}}(\bar{a}), \quad \text{for all } \bar{a} \in A.$$

- For fixed point operators, adapt the construction from the proof showing FP $\leq \mathrm{L}^\omega_{\infty\omega}$.