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Algorithmic Model Theory — Assignment 6

Due: Monday, 2 December, 12:00

Exercise 1

To justify the definition of SO-HORN, show that the admission of arbitrary first-order prefixes would make the restriction to Horn clauses pointless. This extension of SO-HORN has the full power of second-order logic.

Exercise 2

weak-SO-HORN is the set of all formulae of the form

$$QR_1 \dots QR_k \forall x_1, \dots \forall x_l \bigwedge_{1 \le i \le r} C_i.$$

The C_i are of the form $B_1 \wedge \dots \wedge B_n \to H$ where the B_i are either atoms or negated atoms with the restriction that the relations $R_1, \dots, R_k$ may only occur positively. That means weak-SO-HORN differs from SO-HORN in that fact that only atomic or negated atomic first order formulae are allowed in the clauses.

(a) Show that on ordered structures weak-SO-HORN is strictly less expressive than SO-HORN.

Hint: Show that for every weak-SO-HORN sentence ψ the class $\{\mathfrak{A} : \mathfrak{A} \models \psi\}$ is closed under substructures.

(b) Show that, however, on ordered structures with the additional successor relation and constants $0, e$ for the first and last element in the order weak-SO-HORN and SO-HORN are equally expressive.

Hint: show that on this domain weak-SO-HORN captures PTIME.

Exercise 3

A finite game $\mathcal{G} = (V, V_0, V_1, E)$ is called *k-bounded* if every $v \in V$ has at most k outgoing edges. Give SO-HORN formulae $\varphi_k(v)$ so that for all k-bounded games $\mathcal{G}$ and all $v \in V$

$$\mathcal{G} \models \varphi_k(v) \Leftrightarrow \text{ player 0 has no winning strategy from } v$$

Remark: For a definition of finite games see chapter 2 of the mathematical logic 1 lecture notes.

Exercise 4

An operator $F : \mathcal{P}(A) \to \mathcal{P}(A)$ is called *inflationary* if $F(X) \supseteq X$ for all $X \subseteq A$.
Give examples for operators $F : \mathcal{P}(A) \to \mathcal{P}(A)$ with the following properties:

(i) F has a fixed point but no least one.

(ii) F has a least fixed point but is not monotone.

(iii) F is monotone but not inflationary.

(iv) F is inflationary but not monotone.