Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, F. Abu Zaid, W. Pakusa,
F. Reinhardt

WS 2013/14

Algorithmic Model Theory — Assignment 11

Due: Monday, 20 January, 12:00

Exercise 1

A graph $\mathcal{G} = (V, E^{\mathfrak{A}})$ encodes an $(V \times V)$-matrix $M^{\mathcal{G}}$ over $\mathbb{F}_2$ which is

$$M^{\mathfrak{A}}(a,b) = \begin{cases} 0, & \text{if } (a,b) \notin E \\ 1, & \text{if } (a,b) \in E. \end{cases}$$

In other words, $M^{\mathcal{G}}$ is just the adjacency matrix of the graph $\mathcal{G}$. In the same way, every FPC-formula $\varphi(x,y)$ defines an $(V \times V)$-matrix $\varphi^{\mathcal{G}}$ over $\mathbb{F}_2$ in the graph $\mathcal{G}$.

We want to show that matrix multiplication is definable in FPC. Construct an FPC-formula $\varphi(x,y)$ such that for any graph $\mathcal{G} = (V, E)$ it holds $\varphi^{\mathcal{G}} = (M^{\mathcal{G}})^2$.

Exercise 2

We encode linear equation systems over the finite field $\mathbb{F}_2$ as relational structures $\mathfrak{A}$ over the signature $\tau = \{E, V, R_0, R_1\}$ where the intended meaning of the relations is as follows.

- E, V are unary predicates which partition the universe into equations and variables, and
- the equation $e \in E$ corresponds to the linear equation $\sum_{v \in V : R_i(e,v)} v = i$.

(a) Construct an FO(τ)-sentence φ such that $\mathfrak{A} \models \varphi$ if, and only if, $\mathfrak{A}$ encodes a linear equation over $\mathbb{F}_2$ in the described way.

(b) For any fixed finite field $\mathbb{F}$, generalise the above encoding for linear equation systems over $\mathbb{F}$.

Exercise 3

Recall the encoding of linear equation systems over $\mathbb{F}_2$ as relational structures from Exercise 2. Here we want to reduce bipartiteness of undirected graphs to the solvability of linear equation systems over $\mathbb{F}_2$.

Construct FO($\{F\}$)-formulae $\psi_E(x,y), \psi_V(x,y)$ and $\psi_{R_i}(x,y,x',y')$ such that for any (finite, undirected) graph $\mathcal{G} = (W, F)$ the $\{E, V, R\}$-structure $\mathcal{G}^{\psi} = (W^2, E^{\psi}, V^{\psi}, R_0^{\psi}, R_1^{\psi})$ where

- $E^{\psi} = \{(w,w') : \mathcal{G} \models \psi_E(w,w')\}$, $V^{\psi} = \{(w,w') : \mathcal{G} \models \psi_V(w,w')\}$ and
- $R_i^{\psi} = \{((u,u'),(w,w')) : \mathcal{G} \models \psi_{R_i}(u,u',w,w')\}$,

encodes a linear equation system over $\mathbb{F}_2$ which has a solution if, and only if, $\mathcal{G}$ is bipartite.