Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
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WS 2019/20

Algorithmic Model Theory — Assignment 8

Due: Tuesday, 03 December, 10:30

Exercise 1 5 Points

Prove that for every $h \geq 1$ there is a formula $\varphi_h \in \mathrm{FO}(E)$ with $\|\varphi_h\| \in \mathcal{O}(h^4)$ such that every local sentence ψ which is equivalent to φ_h on the class $\mathfrak{T}$ of finite **trees** has size at least $Tower(h)$.

Hint: Adapt the proof of Lemma 4.21, which makes a similar statement about forests, to a tree F, instead of a forest. Construct this tree F by introducing new nodes connecting the roots of the trees from 4.21 such that the distance between them is large enough (with respect to h and the maximal locality radius of the χ_i in φ_h). Then show that $F^{-j} \vDash \chi_l$, where $l \leq m$, and $F^{-j} \nvDash \chi_l$, where $m < l \leq L$, still hold (for l again as in 4.21).

Exercise 2 8 Points

Determine the asymptotic probabilities of the following graph properties.

(a) $\mathcal{K}_1 = \{G : G \text{ has no isolated node}\}$

(b) $\mathcal{K}_2 = \{G : G \text{ is bipartite}\}$

(c) $\mathcal{K}_3 = \{G : G \text{ is a tree}\}$

(d) $\mathcal{K}_4 = \{G : G = (V, E) \text{ contains a clique of size } \geq \log(|V|)\}$

Exercise 3 8 Points

Let $\mathcal{K}$ be a class of graphs and let $\psi \in \mathrm{FO}(E)$ be such that $\mu(\psi) = 1$. We say that $\mathcal{K}$ follows from ψ if for every graph $G \models \psi$ it holds that $G \in \mathcal{K}$. For instance, the class of connected graphs follows from the sentencce $\forall x \forall y(\neg Exy \rightarrow \exists z(Exz \wedge Eyz))$ with asympotic propability 1. Of course, each such class $\mathcal{K}$ itself has asympotic propability 1.

We want to show that the class $\mathcal{R}$ of all rigid graphs does not follow from any $\psi \in \mathrm{FO}(E)$ with $\mu(\psi) = 1$. Recall that a graph G is rigid if it has no non-trivial automorphisms. However, it is known, but not so easy to prove, that almost all graphs are rigid. This shows that there are interesting properties of graphs which hold for almost all graphs, but which do not follow from any first-order definable property of almost all graphs (another example is hamiltonicity).

(a) Explain why it suffices to show that for every finite set $T_0 \subseteq T$ there exists a non-rigid graph G, i.e. a graph with non-trivial automorphisms, such that $G \models T_0$.

(b) Consider the class $\mathcal{K}$ of graphs with vertex set $V_\ell = \{-\ell, \ldots, -1, 1, \ldots, \ell\}$, for all $\ell \geq 1$, and with the property that there is an edge between i and j if, and only if, there is an edge between $-i$ and $-j$ for all $i, j \in V_\ell$. All graphs in $\mathcal{K}$ are non-rigid (why?).

(c) Show that every extension axiom $\sigma \in T$ has asympotic propability 1 on the class $\mathcal{K}$ (in particular, each extension axiom has a model in $\mathcal{K}$). To prove this, it can be helpful to observe that a random graph in $\mathcal{K}$ results by tossing a fair coin for every possible edge pair $\{i, j\}, \{-i, -j\}$. Put everything together to prove the claim.

Exercise 4 5 Points

Show that it is decidable, given a sentence FO(E), whether $\mu(\psi) = 0$ or $\mu(\psi) = 1$.