Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, K. Dannert

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Algorithmic Model Theory — Assignment 9

Due: Tuesday, 10 December, 10:30

Exercise 1 — 6 Points

In this exercise, graphs are undirected.

A finite graph contains a *Eularian cycle*, if there is a cycle which visits every edge exactly once. This is known to be equivalent to the graph being connected (except for nodes with degree 0) with every node having even degree.

We define two extensions of first-order logic by the following operators:

- FO[Eul] is FO closed under expressions of the form Eul $x, y\ \varphi(x, y)$, $\varphi \in$ FO. The semantic is defined by $\mathfrak{A} \models$ Eul $x, y\ \varphi(x, y)$ if and only if the graph $(A, E_G = \{(a, b) \in A^2 : \mathfrak{A} \models \varphi(a, b)\})$ contains a Eularian cycle.
- FO[Ham] is FO closed under expressions of the form Ham $x, y\ \varphi(x, y)$, $\varphi \in$ FO. The semantic is defined by $\mathfrak{A} \models$ Ham $x, y\ \varphi(x, y)$ if and only if the graph $(A, E_G = \{(a, b) \in A^2 : \mathfrak{A} \models \varphi(a, b)\})$ contains a Hamiltonian cycle.

(a) Prove that FO[Eul] does not have a 0-1-law.

(b) Prove that FO[Ham] does not have a 0-1-law by considering the sentence

$$\exists z \text{ Ham } x, y\ (Exz \land \neg Eyz).$$

Hint: You may use without proof that as n goes to infinity, $G \in \mathcal{G}_{2n}$ almost surely contains a node of degree n.

Exercise 2 — 4 Points

Give examples for operators $F : \mathcal{P}(A) \to \mathcal{P}(A)$ with the following properties:

(a) F has a fixed point but no least fixed point.

(b) F has a least fixed point but F is not monotone.

(c) F is monotone but not inflationary.

(d) F is inflationary but not monotone.

Exercise 3 — 8 Points

Consider the signature $\tau = \{P, Q\}$. Give an L_μ-formula $\varphi \in \mathrm{L}_\mu(\tau)$ such that for each transition system $\mathcal{K} = (V, E, P, Q)$ and each node $v \in V$ we have $\mathcal{K}, v \models \varphi$ if and only if

(a) at each node reachable from v where Q holds, P holds as well.

(b) from each node reachable from v where P holds, there is a reachable node where Q holds.

(c) there is an infinite path from v such that $P \wedge Q$ holds only finitely many times.

Exercise 4 12 Points

Construct IFP-formulae which define in a rooted tree $\mathcal{T} = (V, E, r)$, where r denotes its root, the following relations.

(a) $R_1 = \{(x, y) : \text{the subtrees rooted in } x \text{ and } y \text{ have the same height}\}$

(b) $R_2 = \{(x, y) : \text{the nodes } x \text{ and } y \text{ are on the same level of the tree}\}$

(c) $R_3 = \{x : \text{the subtree rooted in } x \text{ possesses a perfect matching}\}$.