Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
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Algorithmic Model Theory — Assignment 10

Due: Tuesday, 17 December, 10:30

Exercise 1 7 Points

A k edge coloured connected graph $\mathcal{G} = (V, E, C_1, \dots, C_k)$ is a connected graph (V, E) which is extended by k binary predicates $C_1, \dots, C_k \subseteq E$ that encode a valid k edge colouring of (V, E), i.e. the sets $C_1, \dots, C_k$ are pairwise disjoint, $\bigcup_{i=1}^k C_i = E$ and $|(\{v\} \times V) \cap C_i| \leq 1$ for all $v \in V$ and $1 \leq i \leq k$.

(a) Construct an LFP-formula $\varphi(p, x, y)$ such that for all $a \in V$ the relation defined by φ in $\mathcal{G}$ with parameter a, i.e. the relation $\{(v, w) : \mathcal{G} \models \varphi(a, v, w)\}$, is a linear order on V.

Hint: Use the colouring of the vertices to identify a spanning tree of $\mathcal{G}$.

(b) Conclude, using the Immerman-Vardi Theorem, that LFP captures PTIME on the class of connected k edge coloured graphs.

Exercise 2 12 Points

(a) Construct formulae of the multidimensional μ-calculus that define the following classes $\mathcal{C}_i$ of rooted transition systems.

$\mathcal{C}_1 = \{(\mathcal{G}, v) :$ from v a terminal vertex is reachable that satisfies precisely the same predicates$\}$

$\mathcal{C}_2 = \{(\mathcal{G}, v) :$ there are two infinite paths π, σ starting from v such that for all positions $i > 0$ and all predicates P it holds $(\mathcal{G}, \pi[i]) \models P$ if, and only if, $(\mathcal{G}, \sigma[i]) \not\models P\}$

(b) Show that for $\mathcal{K}_1, \bar{v}$ and $\mathcal{K}_2, \bar{w}$ with $\mathcal{K}_1, v_i \sim \mathcal{K}_2, w_i$ for $1 \leq i \leq k$ it holds that $\mathcal{K}_1^k, \bar{v} \sim \mathcal{K}_2^k, \bar{w}$. Conclude, using the bisimulation invariance of L_μ, that the multidimensional μ-calculus is bisimulation invariant as well.

Exercise 3 5 Points

Conway's LIFE is a game played on an undirected graph. At the beginning some vertices are marked with a pebble. In every turn the following rules are applied simultaneously to all vertices:

- marked vertices remain marked if and only if they have 2 or 3 marked neighbours;
- unmarked vertices become marked if and only if they have exactly 3 marked neighbours.

Write a PFP formula over the signature $\{E, P\}$ (E is the edge relation and P the set of vertices marked at the beginning) which holds in an arena $\mathcal{G} = (V, E^{\mathcal{G}}, P^{\mathcal{G}})$ if and only if the game becomes eventually stationary.

Exercise 4 6 Points

Let $\mathcal{K}$ be a class of (finite) τ-structures with the following property. For every $m \in \mathbb{N}$, there exists a structure $\mathfrak{A} \in \mathcal{K}$ such that for all m-tuples $\overline{a}$ in $\mathfrak{A}$ there exists a non-trivial automorphism of $(\mathfrak{A}, \overline{a})$. Show that $\mathcal{K}$ does not admit definable orders (even with parameters) in any logic which is isomorphism-invariant.