Chapter 3

Infinite Two-Player Games with Perfect Information: Theoretical Background

3.1 The Corner Stone: The Gale-Stewart Game

Definition 13 *Let Σ be any alphabet, and $A \subseteq \Sigma^\omega$ be any set of infinite words on Σ, $\mathbb{G}(A)$ is the game where player I and II take turns. Player I starts and picks a letter $x_0 \in \Sigma$, player II picks $x_1 \in \Sigma$ and so on.*

$$\begin{array}{lllllllllllll}
I & : & x_0 & & x_2 & & x_4 & & \text{- - - - - - - -} & x_{2n} & & x_{2n+2} & \text{- - - - - -} \\
II & : & & x_1 & & x_3 & & x_5 & \text{- - - - - - - -} & & x_{2n+1} & & x_{2n+3} \ \text{- - - -}
\end{array}$$

This way they build the infinite word $x = x_0x_1x_2x_3 \ldots\ldots$ Player I wins if x belongs to A, player II wins otherwise.

Many infinite games can be described as a particular instance of the *Gale-Stewart Game*, since nothing special is required from the alphabet nor from the set A. Everything would be extraordinary smooth and delicate if this game would be *determined*. But it is not the case.

3.2 Non-Determinacy

Theorem 14 *(AC) Let $|\Sigma| \geq 2$,*

$$\textit{there exists } A \subseteq \Sigma^\omega \textit{ such that } \mathbb{G}(A) \textit{ is not determined}$$

Proof of theorem 14: We first prove the result for $\Sigma = \mathbb{N}$, the generalization to any other alphabet will be straightforward. For this proof, we need to consider subsets of $\mathcal{P}(N)$ called *filters*. A *filter* over $\mathbb{N}$ is a subset $\mathcal{F} \subseteq \mathcal{P}(N)$ that satisfies:

- $\circ$ $U \in \mathcal{F} \ \wedge \ V \in \mathcal{F} \implies U \cap V \in \mathcal{F}$

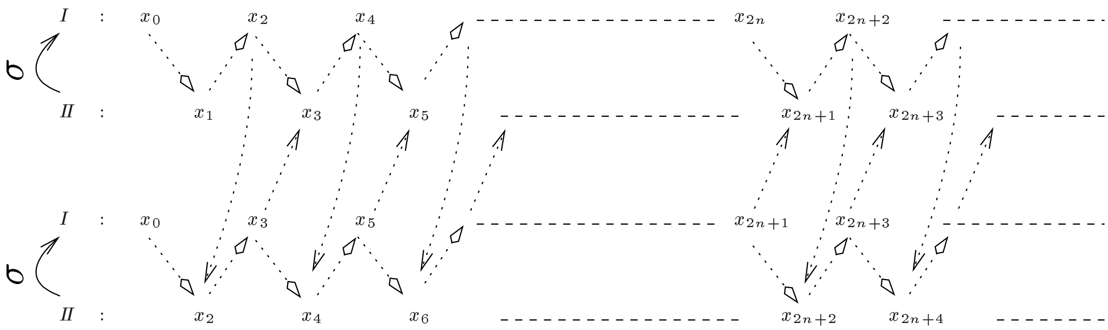
- $U \in \mathcal{F} \ \wedge \ U \subseteq V \implies V \in \mathcal{F}$
- $\emptyset \notin \mathcal{F}$

For example, the set $\{U \subseteq \mathbb{N} : \ \mathbb{N} \smallsetminus U \text{ is finite}\}$ of all co-finite subsets of $\mathbb{N}$ is a filter over $\mathbb{N}$ called *Fréchet filter*.
A *filter* that verifies also the property that *for any* $U \subseteq \mathbb{N}$, $U \in \mathcal{F} \ \vee \ (\mathbb{N} \smallsetminus U) \in \mathcal{F}$ is called a *ultrafilter*. Clearly *Fréchet filter* is not a *ultrafilter* since neither the set of even integers nor the one of odd integers belongs to it. Anyway, using the Axiom of Choice, one can show that there exists *ultrafilters* that contain (extend) *Fréchet filter*. Such *ultrafilters* are called *free* ultrafilters.
Given $\mathcal{U}$ any free ultrafilter over $\mathbb{N}$, let $A_{\mathcal{U}} \subseteq \mathbb{N}^\omega$ be the following set:

$$x = x_0 x_1 \ldots \in A_{\mathcal{U}} \iff \begin{cases} x \text{ is strictly increasing} & \bigwedge \quad \left([0, x_0] \cup \bigcup_{i \in \mathbb{N}}]x_{2i+1}, x_{2i+2}]\right) \in \mathcal{U} \\ & \text{or} \\ \exists i \in \mathbb{N} \ \ (x_{2i+1} \geq x_{2i+2} & \wedge \qquad (\forall j < 2i+1 \ \ x_j < x_{j+1})). \end{cases}$$

In other words, the game $\mathbb{G}(A_{\mathcal{U}})$ is exactly the game where players *I* and *II* take turn playing integers, with the additional rule that the first one to play an integer less or equal than the opponent's previous moves loses. If both *I* and *II* play according to this rule, everything looks like they choose, step by step, an infinite-co-infinite subset of $\mathbb{N}$ as in the picture below, where all integers "picked" by *I* are colored green, and all the ones for *II* are red:

The winning condition then is directly related to the ultrafilter: *I* wins if *Green* $\in \mathcal{U}$. Clearly *Green* $= \mathbb{N} \smallsetminus$ *Red* holds.
Now we show that $\mathbb{G}(A)$ is not determined. Toward a contradiction we first assume that *I* has a w.s. σ. Let *II* play as follow: after *I* plays x_0, *II* plays any $x_1 > x_0$, then *I* answers with x_2, but then *II* considers a second play (let's call it *fake* of the game $\mathbb{G}(A_{\mathcal{U}})$ where *I* still applies σ and *II* in the *fake* game **copies** *I*'s answers in the *original* game, skipping x_0, and also *II* in the *original* game **copies** *I*'s answers in the *fake* game, as shown in the picture below:



Since in both games player I applied a w.s. I wins both plays. Hence, by looking at both games, it follows:

$$[0,x_0]\cup]x_1,x_2]\cup]x_3,x_4]\cup]x_5,x_6]\cup\ldots\ldots\quad\in\mathcal{U}$$

and

$$[0,x_0]\cup]x_2,x_3]\cup]x_4,x_5]\cup]x_6,x_7]\cup\ldots\ldots\quad\in\mathcal{U}.$$

Therefore the intersection of these two sets is still in $\mathcal{U}$:

$$\Big([0,x_0]\cup]x_1,x_2]\cup]x_3,x_4]\cup]x_5,x_6]\cup\ldots\quad\bigcap\quad[0,x_0]\cup]x_2,x_3]\cup]x_4,x_5]\cup]x_6,x_7]\cup\ldots\Big)=[0,x_0]\in\mathcal{U}$$

But $\mathcal{U}$ is a free ultrafilter, thus $(\mathbb{N}\smallsetminus[0,x_0])\in\mathcal{U}$ holds, which leads to $[0,x_0]\bigcap(\mathbb{N}\smallsetminus[0,x_0])=\emptyset\in\mathcal{U}$, a contradiction.
If we assume that $I\!I$ has a w.s. in $\mathbb{G}(A_{\mathcal{U}})$, the same argument leads to the same contradiction.

So far, we just proved the result for $\Sigma=\mathbb{N}$. But it is very easy to simulate the above game with at least two letters. For example, assume $0\in\Sigma$, and imagine that the winning conditions for I are:

(1) either $I\!I$ played finitely many 0,

(2) or both players played infinitely many 0 but the sequence $x_0x_1x_2\ldots\ldots$ belongs to $A_{\mathcal{U}}$, where x_0 is the first position where I plays a 0, x_1 is the first position $>x_1$ where $I\!I$ plays a 0, x_2 is the first position $>x_2$ where I plays a 0, and so on...

$\dashv$ 14

3.3 Borel Determinacy

Assume the payoff set $A\subseteq\Sigma^\omega$ satisfies the following property: for all infinite word $x\in\Sigma^\omega$:

$$x\in A\implies\exists n\in\mathbb{N}\ \ \forall y\in\Sigma^\omega\ \ x_0x_1x_2\ldots x_ny\in A$$
$$x\notin A\implies\exists n\in\mathbb{N}\ \ \forall y\in\Sigma^\omega\ \ x_0x_1x_2\ldots x_ny\notin A$$

We will see that such a set is called *clopen* (a contraction of *closed and open*). To define it differently, let's just say that with this kind of set, the issue to determine whether an infinite word x is in it or not only depends on a prefix of x. In other words, the game $\mathbb{G}(A)$ although it is infinite, really behaves like a finite game. Precisely, for any infinite word x, let u_x be the smallest prefix of x that satisfies one of the two conditions above. And Let T be the tree of all prefixes of all u_x for all infinite word x. By its very definition T is a wellfounded tree that stands as the extended form of a finite two player game where players take turn, player I starting, and the winning conditions being: I wins if the leaf u_x reached verifies $x\in A$ and $I\!I$ wins otherwise.

This shows that finite games and infinite clopen games are essentially the same. Finite games can be easily extended to infinite games by pretending still playing while everything is already settled yet. To go beyond clopen games, we need to consider a classification of the complexity of the payoff sets involved. But the complexity needed here relies on topology.

3.3.1 Topology

Given a set S, a topology on S is a couple $T = (S, \mathcal{O})$ where $\mathcal{O} \subseteq \mathcal{P}(S)$ is the so-called set of *open* sets that must satisfy:

(1) $\emptyset,\ S \in \mathcal{O}$,

(2) $U,\ V \in \mathcal{O} \Longrightarrow U \cap V \in \mathcal{O}$

(3) $\forall i \in I\ \ U_i \in \mathcal{O} \Longrightarrow \bigcup_{i \in I} U_i \in \mathcal{O}$.

Usually a topology is given by its *basis*: a set of *basic* open sets such that any non empty open set can be written as a union of some sets in this basis.
For example, a basis for the usual topology on $\mathbb{R}$ is the set of open intervals of the form $]a, b[$. This means any open set A is of the form $A = \bigcup_{i \in I}]a_i, b_i[$ for reals a_i, b_i.
The topology we are interested in is the usual topology on infinite sequences: the space Σ^ω is equipped with the following basis:

$$\{u \cdot \Sigma^\omega :\ u \in \Sigma^*\}.$$

This means any open set $A \subseteq \Sigma^\omega$ is of the form either $A = \emptyset$, or $A = \bigcup_{u \in W} u \cdot \Sigma^\omega = W \cdot \Sigma^\omega$.
The complement of an open set is called *closed*.

Definition 15 *A Deterministic Büchi Automaton is an automaton* $\mathcal{A} = (\Sigma, Q, q_i, \delta, \mathcal{F})$ *such that:*

(1) Σ *is the alphabet* ($\Sigma \neq \emptyset$),

(2) Q *the set of states* ($Q \neq \emptyset$),

(3) q_i *is the initial state,*

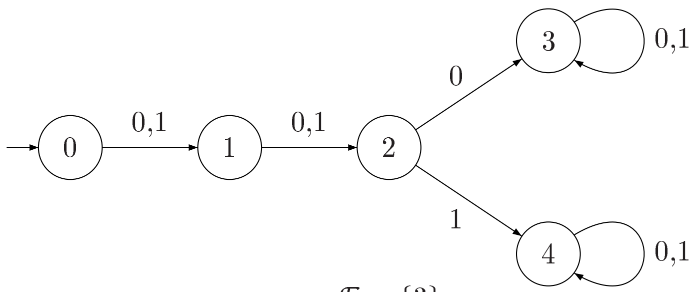
(4) $delta :\ \Sigma \times Q \longmapsto Q$ *is the transition function,*

(5) $\mathcal{F} \subseteq Q$ *are the acceptance conditions.*

Given a word $x \in \Sigma^\omega$, *let* $Inf_{\mathcal{A}}(x)$ *be the set of all states that the machine* $\mathcal{A}$ *runs through infinitely many times while reading* x. *This word* x *is accepted by* $\mathcal{A}$ *iff* $Inf_{\mathcal{A}}(x) \cap \mathcal{F} \neq \emptyset$.

Examples 16

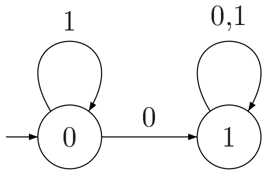
(1)



$\mathcal{F} = \{3\}$

The set A recognized by this Büchi Automaton is $\{0,1\}^2 \cdot 0 \cdot \Sigma^\omega$. It is an open set since $A = W \cdot \Sigma^\omega$ with $W = \{0,1\}^2 \cdot 0$. But one should remark that $\Sigma^\omega \setminus A = \{0,1\}^2 \cdot 1 \cdot \Sigma^\omega$ which is also an open set. So A is both closed and open.

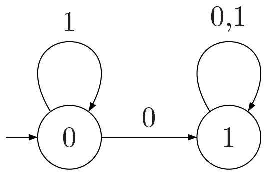
(2)



$$\mathcal{F} = \{1\}$$

The set B recognized by this Büchi Automaton is $\{1\}^* \cdot 0 \cdot \Sigma^\omega$. It is clearly open since $B = W' \cdot \Sigma^\omega$ with $W' = \{1\}^* \cdot 0$. But on the contrary, its complement: $\{1^\omega\}$ is not open. So B is open but not closed.

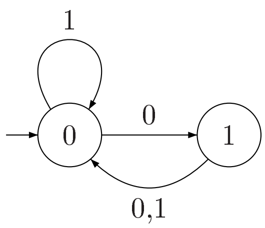
(3)



$$\mathcal{F} = \{0\}$$

The set C recognized by this Büchi Automaton is $\{1^\omega\}$ which is closed but not open.

(4)



$$\mathcal{F} = \{1\}$$

The set D recognized by this Büchi Automaton is the set of words with infinitely many 0. Unfortunately this set is not open nor closed. However, one can see that

$$D = \bigcap_{n \in \mathbb{N}} \left((\Sigma^* 0)^n \cdot \Sigma^\omega \right).$$

Clearly each set $(\Sigma^*0)^n \cdot \Sigma^\omega$ is open, so D is a *countable* intersection of open sets.

3.3.2 Borel Sets

There is a very important class of subsets of topological spaces, called the *Borel* sets. Such sets where introduced, in the case of the reals, at the very end of the nineteenth century, as a foundation of modern analysis.

Definition 17 *Given a topology $T = (S, \mathcal{O})$, the Borel subsets of S is the smallest class $\mathcal{B}$ that:*

- *contains the open sets ($\mathcal{O} \subseteq \mathcal{B}$),*
- *is closed under complementation ($U \in \mathcal{B} \Longrightarrow S \setminus U \in \mathcal{B}$)*
- *is closed under countable union ($(\forall n \in \mathbb{N}\ \ U_n \in \mathcal{B}) \Longrightarrow \bigcup_{n\in\mathbb{N}} U_n \ \in \mathcal{B}$)*

An immediate consequence is that $\mathcal{B}$ contains all the closed sets, and is closed under countable intersection. In particular the set presented in example (4) $D = \bigcap_{n\in\mathbb{N}} ((\Sigma^*0)^n \cdot \Sigma^\omega)$ is Borel. Most of the sets of infinite words we encounter in computer science are Borel. But not all of them. To give an example of a non-Borel set, consider first the following coding of trees:
Given $x \in \{0,1\}^\omega$, let T_x be the subset of $\{0,1\}^*$ defined by:

$$u = (a_1, a_2, \ldots, a_k) \in T_x \quad \Longleftrightarrow \quad x_{2^{a_1+1}.3^{a_2+1}\ldots P_k^{a_k+1}} = 1$$

where P_i is the i^{th} prime number.
Finally define $WF \subseteq \{0,1\}^\omega$ as the set of infinite words over $\{0,1\}$ that satisfies

$$x \in WF \quad \Longleftrightarrow \quad T_x \text{ is a well founded tree over } \{0,1\}.$$

We recall that T_x is a tree if it is closed under subsequence, and it is wellfounded if it has no infinite branch.
The set WF is not Borel, nor its complement.

3.3.3 The Borel Hierarchy

As soon as they were introduced, Baire did set up the Borel sets in a hierarchy: the Borel hierarchy, which relies on counting how many operations "complementation" and "countable union" are necessary to create a set.

Definition 18

(1) $\mathbf{\Sigma}^\circ_1 = \{\textit{open sets}\}$

(2) $\mathbf{\Pi}^\circ_\alpha = \{\Sigma \setminus A : \quad A \in \mathbf{\Sigma}^\circ_\alpha\}$

(3) $\mathbf{\Sigma}^\circ_\alpha = \{A = \bigcup_{n\in\mathbb{N}} A_n : \quad A_n \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^\circ_\beta\}$

In particular, $\mathbf{\Pi}^\circ_1$ is the class of all closed sets, $\mathbf{\Sigma}^\circ_2$ the class of all countable union of closed sets, etc...

Is there a "good" characterization of these classes ? Yes, at least for $\mathbf{\Sigma}^\circ_1$ and $\mathbf{\Pi}^\circ_1$.

Proposition 19 *Let* $A \subseteq \Sigma^\omega$

$$A \text{ is closed} \iff \text{there is a tree } T_A \subseteq \Sigma^* \text{ s.t. } A = \{infinite\ branches\ of\ T_A\}$$

Example 20 $\{1^\omega\}$ is closed and its underlying tree is $T = 1^*$.

Proof of proposition 19:

($\Longrightarrow$) Since A is closed, there is some $W \in \Sigma^*$ s.t. $\Sigma^\omega \smallsetminus A = W \cdot \Sigma^\omega$. Let T_A be the set of all finite words u from Σ^* s.t. u has no prefix in W. Clearly, T_A is a tree (it is closed under prefix), and $x \in A$ iff x has no prefix in W iff x is an infinite branch of T_A.

($\Longleftarrow$) Let A be the set of infinite branches of the tree T_A. For all x that is not an infinite branch of T_A there is a smallest prefix u_x that "exits" this tree. Set $W = \{u_x : x \notin A\}$. Clearly, $x \notin A \iff x \in W \cdot \Sigma^\omega$.

$\dashv$ 19

Theorem 21 (Martin) *Let* $A \subseteq \Sigma^\omega$,

$$A \text{ Borel} \Longrightarrow \mathbb{G}(A) \text{ is determined.}$$

Proof of theorem 21: We only prove the case A is open. The rest is way too difficult for this course. First we remark that if $\mathbb{G}(A)$ is determined for all open set A, then $\mathbb{G}(B)$ is determined for all closed set B. Indeed, let $A = \Sigma^\omega \smallsetminus \Sigma \cdot B$. Clearly, if B is closed, then B is the set of infinite branches of a tree T_B, and $\Sigma \cdot B$ is also the set of infinite branches of a tree (namely $\Sigma \cdot T_B$), hence $\Sigma \cdot B$ is closed. Thus $A = \Sigma^\omega \smallsetminus \Sigma \cdot B$ is open. Now if I has a w.s. in $\mathbb{G}(A)$, this very same strategy, forgetting the first move, is a w.s. for II in $\mathbb{G}(B)$. And if II has a w.s. in $\mathbb{G}(A)$, this strategy leads to a w.s. for I in $\mathbb{G}(B)$. For same reasons, if $\mathbb{G}(B)$ is determined for all closed set B, then $\mathbb{G}(A)$ is determined for all open set A.

Let A be closed. Assume that II has no winning strategy in the game $\mathbb{G}(A)$. Given a legal position for II: $u = a_0 \ldots a_{2n-1}$, we say that it is not losing for I if II has no winning strategy in the subgame starting at position u: $\mathbb{G}(A_u)$ (where $A_u = \{x \in \Sigma^\omega : \quad u \cdot x \in A\}$. Now note that if $u = a_0 \ldots a_{2n-1}$ is not losing for I, then there exists an extending position $u \cdot a_{2n}$ such that no position $u \cdot a_{2n}a_{2n+1}$ is losing for I. As $\emptyset$ is not losing for I, this gives a way to produce a strategy σ that never allows II to reach a position which is losing for I:
I starts by choosing a_0 such that for all a_1, a_0, a_1 is not losing for I. When II then plays some a_1, I responds by choosing some a_2 with a_0, a_1, a_2, such that for all a_3, a_0, a_1, a_2, a_3 is not losing for I, etc...
This strategy is winning for I. Indeed, given $a_0a_1a_2\ldots$ a run where I applied σ, suppose $a_0, a_1, \ldots$ is not in A; as $\Sigma^\omega \smallsetminus A$ is open, there exists $k \in \omega$ such that for all sequence $a_0 \ldots a_{2k-1} \cdot \alpha$ which is in Σ^ω is also in $\Sigma^\omega \smallsetminus A$. But then, any strategy for II is winning in the game $\mathbb{G}(A_{a_0 \ldots a_{2k-1}})$, a contradiction.

$\dashv$ 21

3.3.4 Wadge Reducibility

In computational complexity given a problem and a class, the most general questions are:

(1) does this problem belong to this class ?

(2) If yes, is this problem complete for this class ?

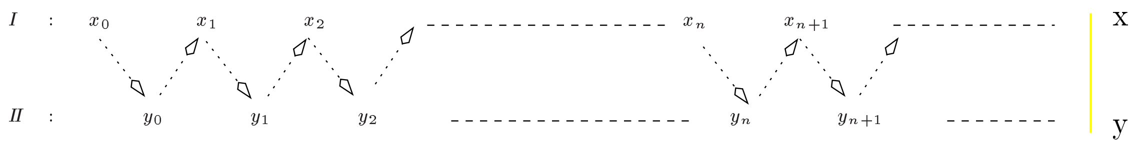
The second question tends to capture the true complexity of the problem. Intuitively, a problem is complete for some class if it is not in some "bigger" class nor in some "smaller" one. Hence this class reveals its complexity.
Same kind of thing takes place here.

(1) does this set belong to $\mathbf{\Sigma}^{\circ}_{\alpha}$, (respectively $\mathbf{\Pi}^{\circ}_{\alpha}$) ?

(2) If yes, is this problem complete for$\mathbf{\Sigma}^{\circ}_{\alpha}$ (respectively $\mathbf{\Pi}^{\circ}_{\alpha}$) ?

But for this notion of completeness we need a natural notion of reduction. Here comes Wadge reduction. We present it in a game theoretic way:

Definition 22 *Given $A \subseteq \Sigma_A^{\omega}$ and $B \subseteq \Sigma_B^{\omega}$, we say A* "Wadge reduces" *to B (denoted $A \leq_W B$) if II has a w.s. in the Wadge game $\mathbf{W}(A\ ,\ B)$ where players take turn, one playing letters among Σ_A, II playing letters among Σ_B. I begins, and this way in infinitely many moves they produce $x \in \Sigma_A^{\omega}$ for I and $y \in \Sigma_B^{\omega}$ for II. Player II wins the game if $x \in A \longleftrightarrow y \in B$.*



Examples 23

(1) Take $\Sigma_A = \Sigma_B = \Sigma = \{0,1\}$, $A = 0 \cdot \Sigma^{\omega}$; $B = \{0^{\omega}\}$; player II has a w.s. in $\mathbf{W}(A\ ,\ B)$: If

I plays $x_0 = 0$, then for sure x will belong to A, hence *II* plays only 0 until the game is over.

If *I* plays a $x_0 = 1$, then for sure x will not end up inside A, hence *II* wins if he plays a 1.

(2) But player *II* does not have a w.s. in $\mathbf{W}(B\ ,\ A)$. On the contrary *I* has. It consists in playing 0 as first move and then play $x_1 = 1 - y_0$ and then play $x_2 = x_3 = \cdots = 0$.

(3) Set $A = \Sigma^\omega \setminus \{0^\omega\}$ and $B = \{0^\omega\}$. So that $B = A^{\complement}$. Player *I* has a w.s. in both games $\mathbf{W}(A\ ,\ B)$ and $\mathbf{W}(B\ ,\ A)$. Namely, play 0 and then copy *II*'s run.

Remark 24

$$A \leq_W B \iff A^{\complement} \leq_W B^{\complement}$$

Proof of remark 24: This is immediate from the very definition of $\leq_W$.

$\dashv$ 24

Determinacy of Wadge games fails in general. There are such games that are not determined. But for sets A and B not too complicated, the game $\mathbf{W}(A\ ,\ B)$ is determined:

Proposition 25 *Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$, with A and B both Borel,*

$$\mathbf{W}(A\ ,\ B) \text{ is determined}$$

Proof of proposition 25: It is tedious but straightforward to see that that the set $C \subseteq (\Sigma_A \cup \Sigma_B)^\omega$ defined by

$$x_0x_1x_2\ldots \in C \iff \begin{cases} \forall n \in \mathbb{N}\ \ (x_{2n} \in \Sigma_A \wedge x_{2n+1} \in \Sigma_B)\ \ \bigwedge\ \ (x_0x_2x_4\ldots \in A \longleftrightarrow x_1x_3x_5\ldots \in B) \\ \bigvee \\ \exists n \in \mathbb{N}\ \ x_{2n+1} \notin \Sigma_B\ \ \bigwedge\ \ \forall m \leq n\ \ x_{2m} \in \Sigma_A \end{cases}$$

is Borel. Hence $\mathbb{G}(C)$ is determined, and a w.s. for a player in $\mathbb{G}(C)$ induces a w.s. for the *same player* in $\mathbf{W}(A\ ,\ B)$.

$\dashv$ 25

Definition 26 *Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$,*

- $A <_W B \iff_{def} (A \leq_W B\ \wedge\ B \not\leq_W A)$
- $A \equiv_W B \iff_{def} (A \leq_W B\ \wedge\ B \leq_W A)$

As a consequence of Borel Wadge Determinacy we obtain

Corollary 27 Let $A \subseteq \Sigma_A^\omega$, $B \subseteq \Sigma_B^\omega$, with A and B both Borel,

(1) $A \not\leq_W B \Longrightarrow B \leq_W A^{\complement}$

(2) A and B are incomparable $\Longrightarrow\ \ B \equiv_W A^{\complement}$

(3) There are no three Borel sets that are incomparable with each other.

Proof of corollary 27:

(1) If $A \not\leq_W B$, then *II* does not have a w.s. in $\mathbf{W}(A\ ,\ B)$, hence, by Determinacy, *I* has a w.s. But clearly, a w.s. for *I* in $\mathbf{W}(A\ ,\ B)$ leads to a w.s. for *II* in $\mathbf{W}\left(B\ ,\ A^{\complement}\right)$. Therefore $B \leq_W A^{\complement}$ holds.

(2) If A and B are incomparable, this means $A \not\leq_W B$ and $B \not\leq_W A$ both hold. By previous result, this leads to both $B \leq_W A^{\complement}$ and $A^{\complement} \leq_W B$. Thus $A^{\complement} \equiv_W B$.

(3) Toward a contradiction assume there exists A, B, C that are incomparable with each other. Since $A \not\leq_W B$ and $B \not\leq_W A$, we get $A^{\complement} \equiv_W B$. Since $C \not\leq_W A$ and $A \not\leq_W C$, we get $A^{\complement} \equiv_W C$. Hence $B \equiv_W C$, a contradiction.

$\dashv$ 27

Theorem 28

$$<_W \quad \textit{is wellfounded on Borel sets}$$

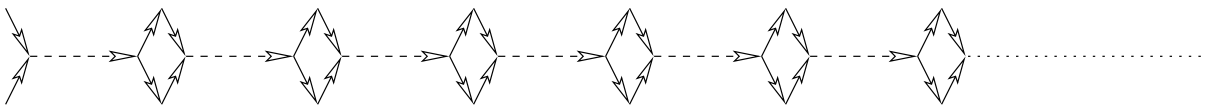
which means that there is no infinite sequence of Borel sets that satisfies:

$$A_1 \;{}_W\!\!> A_2 \;{}_W\!\!> A_3 \;{}_W\!\!> \dots \;{}_W\!\!> A_n \;{}_W\!\!> A_{n+1} \;{}_W\!\!> \dots$$

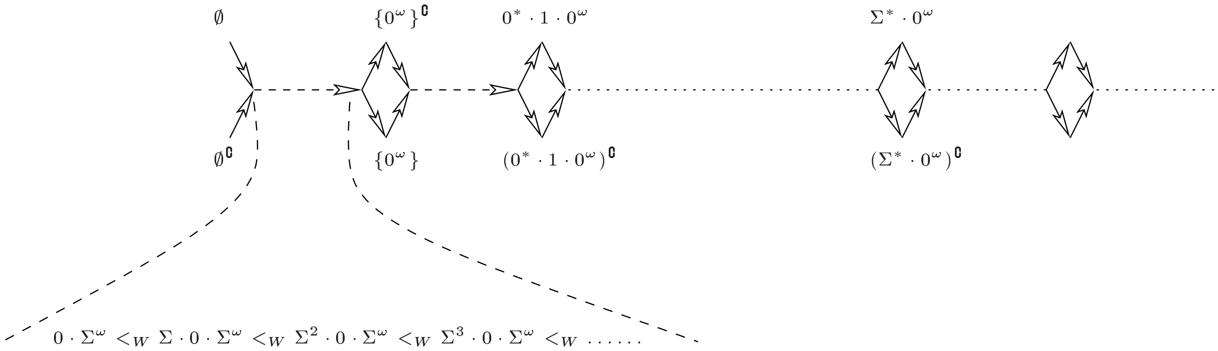
Proof of theorem 28: This is a difficult technical result which also relies on Borel Determinacy.

$\dashv$ 28

This ordering looks like:



Examples 29 Set $\Sigma = \{0, 1\}$,



3.3.5 Canonical Σ^0_n-Complete Sets

Definition 30 *Let $Z \subseteq (\Sigma^\omega \cup \Sigma^*)$, $n \in \mathbb{N}$, given a letter s (called* skip*) and letters (called* erasers*) $\{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(n)\}$ such that $|\Sigma \cup \{s\} \cup \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(n)\}| = |\Sigma| + n + 1$, we define the set $Z^{\sim n} \subseteq (\Sigma \cup \{s\} \cup \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(n)\})^\omega$ by induction on n:*

- $Z^{\sim 0} = \{x : \ x(\ /s) \in Z\}$, *where $x(\ /s)$ stands for x after every occurrence of s has been removed.*
- $Z^{\sim n+1} = \{x : \ \left(\ldots\left((x(\ /s))^{\twoheadleftarrow(n+1)}\right)^{\twoheadleftarrow(n)}\ldots\right)^{\twoheadleftarrow(1)} \in Z\}$. *Where, given, $y \in (\Sigma \cup \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(h)\})^\omega \cup (\Sigma \cup \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(h)\})^*$, $(y)^{\twoheadleftarrow(h)}$ is the sequence obtained from y by interpreting $\twoheadleftarrow(h)$ as an eraser, and reading y from left to right. The letter $\twoheadleftarrow(h)$ works just like* backspace *on your keyboard. So the rules of the erasing process are:*
  - **ERASE**$(u \cdot a \cdot \twoheadleftarrow(h) \cdot y')$ =**ERASE**$(u \cdot y')$ *(if $u \in (\Sigma \cdot \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(h-1)\})^*$, $a \in \Sigma \cup \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(h-1)\}$)*
  - **ERASE**$(\twoheadleftarrow(h) \cdot y')$ =**ERASE**(y')
  - **ERASE**$(y) = y$ *(if $y \in (\Sigma \cup \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(h-1)\}^\omega \cup (\Sigma \cup \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(h-1)\})^*)$).*

  So for any sequence $y \in (\Sigma \cup \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(h)\})^\omega \cup (\Sigma \cup \{\twoheadleftarrow(1), \ldots, \twoheadleftarrow(h)\})^*$, this erasing process induces a sequence of words $(y_k)_{k\in\mathbb{N}}$ such that at any step the right rule is applied. set $\lim\limits_{k\in\mathbb{N}} y_k = z$ if $\forall i < lh(z) \ \ \exists j \ \ \forall k > j \ \ z_0 z_1 z_2 \ldots z_i = y_{k0} y_{k1} y_{k2} \ldots y_{ki}$.

  We set $y^{\twoheadleftarrow(h)} = \lim\limits_{k\in\mathbb{N}} y_k$

Example 31 Let $\ominus\!\xrightarrow{0}\!\oplus$ be the set of words on the alphabet $\{0\}$ that contains a letter (i.e. it is every possible word except the empty one).

- $\ominus\!\xrightarrow{0}\!\oplus^{\sim 0}$ is the subset of $\{0, s\}^\omega$ of all words with at least a 0.
- $\ominus\!\xrightarrow{0}\!\oplus^{\sim 1}$ is the subset of $\{0, \twoheadleftarrow(1), s\}^\omega$ of all words with at least a 0 **that is never erased** when $\twoheadleftarrow(1)$ is interpreted as *backspace*.

By convention we write $\oplus\!\xrightarrow{0}\!\ominus$ for the complement of $\ominus\!\xrightarrow{0}\!\oplus$. So that $\oplus\!\xrightarrow{0}\!\ominus$ is the subset of $0^* \cup \{0^\omega\}$ that only contains the empty word.

Proposition 32 *Let $n \in \mathbb{N}$,*

$$(\ominus\!\xrightarrow{0}\!\oplus)^{\sim n} \ \textit{is} \ \boldsymbol{\Sigma}^\circ_{n+1}\textit{-complete}$$

Proof of proposition 32: We only prove this result for $n = 1, 2$; the other cases being much more difficult.

n=0: By definition, $(\ominus\!\xrightarrow{0}\!\oplus)^{\sim 0}$ is the set of infinite words $x \in \{0, s\}^\omega$ such that x contains the letter 0.

(1) $(\ominus\!\xrightarrow{0}\!\oplus)^{\sim 0} = s^* \cdot 0 \cdot \{0, s\}^\omega$, which proves that $(\ominus\!\xrightarrow{0}\!\oplus)^{\sim 0}$ belongs to $\boldsymbol{\Sigma}^\circ_1$.

(2) Given any open set $A \subseteq \Sigma^\omega$, we show that $A \leq_W (\ominus\!\xrightarrow{0}\!\oplus)^{\sim 0}$ holds: by definition, there exists $W \subseteq \Sigma^*$ such that $A = W \cdot \Sigma^\omega$. The following strategy is winning for *II* in $\mathbf{W}\left(A\ ,\ (\ominus\!\xrightarrow{0}\!\oplus)^{\sim 0}\right)$: assume *I* has played $x_0 x_1 \ldots x_k$, *II* has already played $y_0 y_1 \ldots y_{k-1}$ and must now play y_k:

- $y_k = s$ if $x_0 x_1 \ldots x_k \notin W$
- $y_k = 0$ if $x_0 x_1 \ldots x_k \in W$

n=1: By definition, $(\ominus\xrightarrow{0}\oplus)^{\sim 1}$ is the set of infinite words $x \in \{0, s, \twoheadleftarrow(1)\}^\omega$ such that x, after removing all skips and having $\twoheadleftarrow(1)$ act as backspace, leads to a sequence that contains the letter 0.

(1) $(\ominus\xrightarrow{0}\oplus)^{\sim 1} = \bigcup_{k\in\mathbb{N}} [T_k]$, where T_k is the tree of all finite words $u \in \{0, s, \twoheadleftarrow(1)\}^*$ such that either $lh(u) < k+1$ or $lh(u) \geq k+1$ and in this case $u_k = 0$ and once the skips are thrown away and $\twoheadleftarrow(1)$ acts as backspace, this precise letter 0 *is not erased* (e.g. the word "$00\twoheadleftarrow(1)s0s0ss\twoheadleftarrow(1)s$" belongs to T_i only for $i = 0, 5$ and $i > 10$). We recall that $[T_k]$ is the set of infinite branches of T_k. The idea is that a word x belongs to $(\ominus\xrightarrow{0}\oplus)^{\sim 1}$ if there is somewhere an occurrence of the letter 0 that will not be removed in the erasing process. Since $[T_k]$ is closed, this proves that $(\ominus\xrightarrow{0}\oplus)^{\sim 1}$ belongs to $\mathbf{\Sigma}^\circ_2$.

(2) Given any set $A \subseteq \Sigma^\omega$ that belongs to $\mathbf{\Sigma}^\circ_2$, we show that $A \leq_W (\ominus\xrightarrow{0}\oplus)^{\sim 1}$ holds. By definition, since A is a countable union of closed sets, there exists $(T_n)_{n\in\mathbb{N}}$ a sequence of trees over Σ such that $A = \bigcup_{n\in\mathbb{N}} [T_n]$. The following strategy is winning for *II* in $\mathbf{W}\left(A\ ,\ (\ominus\xrightarrow{0}\oplus)^{\sim 1}\right)$: first, *II* has T_0 in mind, plays 0 and, at each next move, checks if *I*'s position is still in T_0 or not. If this position $x_0 x_1 \ldots x_i$ belongs to T_0, then *II* skips (i.e. plays s), if eventually $x_0 x_1 \ldots x_i \notin T_0$, then *II* plays $\twoheadleftarrow(1)$ (in order to erase the letter 0 played first) and starts all again having T_1 in mind instead of T_0; and so on... This is clearly winning since, if x played by *I* belongs to A, then there is a minimal k such that x is an infinite branch of T_k, and *II*, after changing k times, will spend the rest of the game having T_k in mind, so that he will have played a 0 directly followed by s^ω (a word that also belongs to $(\ominus\xrightarrow{0}\oplus)^{\sim 1}$).
On the contrary, if $x \notin A$, then x is not an infinite branch of any tree T_i, so *II* will have switched from T_j to $T_j + 1$ infinitely often, which means he will have played a sequence in $(0 \cdot s^* \cdot \twoheadleftarrow(1))^\omega$ which, once the skips are thrown away and $\twoheadleftarrow(1)$ acts as backspace, leads to the empty sequence (which is not in $\ominus\xrightarrow{0}\oplus$); showing that *II*'s answer does not belong to $(\ominus\xrightarrow{0}\oplus)^{\sim 1}$.

$\dashv$ 32

Remark 33 Let $Z \subseteq (\Sigma^\omega \cup \Sigma^*)$, $n \in \mathbb{N}$,

$$(Z^{\sim n})^{\complement} = (Z^{\complement})^{\sim n}$$

Proof of remark 33: immediate from the very definition. $\dashv$ 33

Corollary 34 Let $n \in \mathbb{N}$,

$$(\oplus\xrightarrow{0}\ominus)^{\sim n} \text{ is } \mathbf{\Pi}^\circ_{n+1}\text{-complete}$$

Proof of corollary 34: immediate from remark 33 and proposition 32.

$\dashv$ 34

Proposition 35 *Let $A \subseteq \Sigma_A^\omega$, A Borel, $n \in \mathbb{N}$,*

(1) the following are equivalent:

(a) $A \in \mathbf{\Sigma}^\circ_{n+1} \smallsetminus \mathbf{\Pi}^\circ_{n+1}$

(b) A is $\mathbf{\Sigma}^\circ_{n+1}$-complete

(c) $A \equiv_W (\ominus \xrightarrow{0} \oplus)^{\sim n}$

(2) the following are equivalent too:

(a) $A \in \mathbf{\Pi}^\circ_{n+1} \smallsetminus \mathbf{\Sigma}^\circ_{n+1}$

(b) A is $\mathbf{\Pi}^\circ_{n+1}$-complete

(c) $A \equiv_W (\oplus \xrightarrow{0} \ominus)^{\sim n}$

Proof of proposition 35:

(1)(b)⟶(1)(c) : if A is $\mathbf{\Sigma}^\circ_{n+1}$-complete then $A \in \mathbf{\Sigma}^\circ_{n+1}$, therefore $A \leq_W (\ominus \xrightarrow{0} \oplus)^{\sim n}$. On the other side, $(\ominus \xrightarrow{0} \oplus)^{\sim n} \in \mathbf{\Sigma}^\circ_{n+1}$, hence$(\ominus \xrightarrow{0} \oplus)^{\sim n} \leq_W A$.

(1)(c)⟶(1)(a) : this relies on the fact $(\ominus \xrightarrow{0} \oplus)^{\sim n}$ and $(\oplus \xrightarrow{0} \ominus)^{\sim n}$ are incomparable. To see this consider the game $\mathbf{W}\left((\oplus \xrightarrow{0} \ominus)^{\sim n}, (\ominus \xrightarrow{0} \oplus)^{\sim n}\right)$. In this game, *I* has a w.s. that consists in skipping at first move and then copying *II*'s run. Same remark holds for the game $\mathbf{W}\left((\ominus \xrightarrow{0} \oplus)^{\sim n}, (\oplus \xrightarrow{0} \ominus)^{\sim n}\right)$. Now, toward a contradiction, assume that $A \in \mathbf{\Pi}^\circ_{n+1}$ holds, this leads to both $A \leq_W (\oplus \xrightarrow{0} \ominus)^{\sim n}$ (since $(\oplus \xrightarrow{0} \ominus)^{\sim n}$ is $\mathbf{\Pi}^\circ_{n+1}$-complete) and $A \equiv_W (\ominus \xrightarrow{0} \oplus)^{\sim n}$, which yields $(\ominus \xrightarrow{0} \oplus)^{\sim n} \leq_W (\oplus \xrightarrow{0} \ominus)^{\sim n}$, a contradiction.

(1)(a)⟶(1)(b) : toward a contradiction, assume that there exists some $B \in \mathbf{\Sigma}^\circ_{n+1}$ such that $B \not\leq_W A$. By Determinacy, one gets $A^\complement \leq_W B \leq_W (\ominus \xrightarrow{0} \oplus)^{\sim n}$, hence $A^\complement \leq_W (\ominus \xrightarrow{0} \oplus)^{\sim n}$ showing that $A^\complement \in \mathbf{\Sigma}^\circ_{n+1}$, a contradiction.

Case (2) is symmetric.

$\dashv$ 35