(1) *Find the Nash equilibria of the following strategic game*

		Player 2	
		L	R
Player 1	T	$(2,2)$	$(0,0)$
	B	$(0,0)$	$(1,1)$

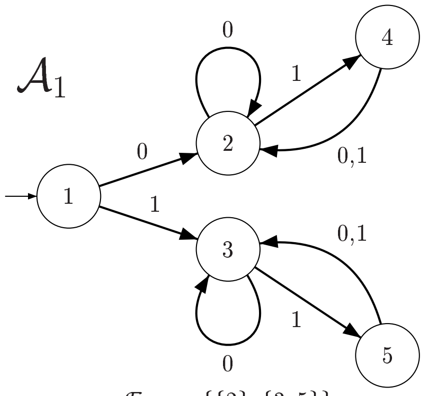
(2) We consider the game of geography played on a directed game graph $G = (V_0, V_1, E)$, from an initial node v.

- *Does the following hold: whoever has a winning strategy in* $\mathbb{G}\mathrm{eo}\,(G\ ,\ i)$, *also has a memoryless winning strategy ?*

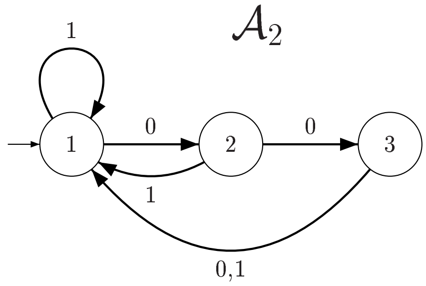
(3) Consider the following four deterministic Muller automata $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and $L(\mathcal{A}_1), L(\mathcal{A}_2), L(\mathcal{A}_3), L(\mathcal{A}_4)$, the languages they recognize. Find the right integers i_1, i_2, i_3, i_4 such that:

$$L(\mathcal{A}_{i_1}) <_W L(\mathcal{A}_{i_2}) <_W L(\mathcal{A}_{i_3}) <_W L(\mathcal{A}_{i_4})$$

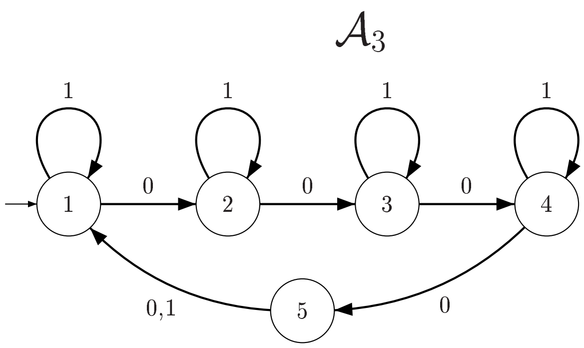
where $<_W$ is the strict Wadge ordering: $A <_W B \iff_{def} A \leq_W B$ but $B \not\leq_W A$.



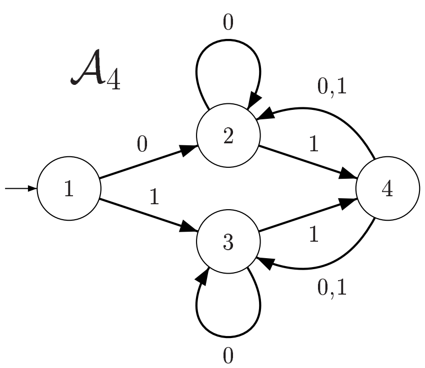
$\mathcal{F}_{A_1} = \{\{2\}, \{3,5\}\}$



$\mathcal{F}_{A_2} = \{\{1,2\}\}$



$\mathcal{F}_{A_3} = \{\{1,2,3,4,5\}\}$

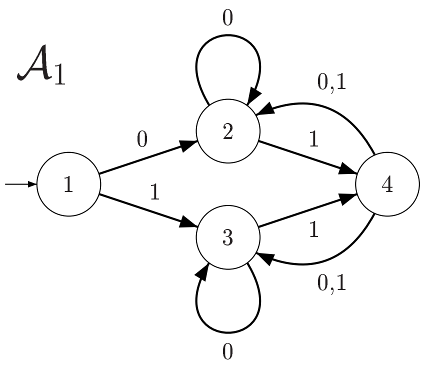


$\mathcal{F}_{A_4} = \{\{4\}\}$

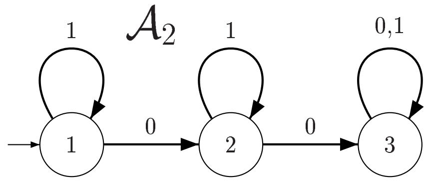
(4) Let $G = (V_0, V_1, E, C, \phi)$ be an arena for a parity games G is called 0-*solitaire* (resp. 1-*solitaire*) if each vertex from V_1 (resp. V_0) has only one successor.
1-solitaire-PARITY-GAMES $= \{(G, v) :$ G is 1-solitaire, player $\mathbf{0}$ has a w.s. from vertex $v\}$.

- *Show that 1-solitaire-PARITY-GAMES* $\in$ **PTIME**.
(*hint*: recall the proof that PARITY-GAMES $\in$ **NP**).

(5) *For the following 2 deterministic Büchi automata* $\mathcal{A}_i$*, answer the questions* $L(\mathcal{A}_1) \in \mathbf{\Sigma}^\circ_1$ *?*, $L(\mathcal{A}_1) \in \mathbf{\Pi}^\circ_1$ *?*, $L(\mathcal{A}_2) \in \mathbf{\Sigma}^\circ_1$ *?*, $L(\mathcal{A}_2) \in \mathbf{\Pi}^\circ_1$ *?* where $L(\mathcal{A}_i)$ stands for the language accepted by $\mathcal{A}_i$.



$\mathcal{F}_{A_1} = \{2, 3, 4\}$



$\mathcal{F}_{A_2} = \{1, 2\}$

(6) We recall that WF is not Borel, where $WF \subseteq \{0,1\}^\omega$ id defined by

$$x = x_1 x_2 \ldots \in WF \iff T_x \begin{cases} \text{is a tree over } \{0,1\} \ (a_1 a_2 \ldots a_k \in T_x \longrightarrow \forall q < k \ \ a_1 a_2 \ldots a_q \in T_x) \\ \text{is well founded (there is no infinite branch)} \end{cases}$$

where given any $x \in \{0,1\}^\omega$, T_x is the subset of $\{0,1\}^*$ defined by:

$$a_1 a_2 \ldots a_k \in T_x \iff x_{2^{(a_1+1)} . 3^{(a_2+1)} \ldots P_k^{(a_k+1)}} = 1$$

where P_i is the i^{th} prime number ($P_1 = 2, P_2 = 3, P_3 = 5, P_4 = 7, P_5 = 11, \ldots$).

(a) *Describe an algorithm for a nondeterministic Turing machine reading infinite words with Büchi acceptance conditions that recognizes* $IF = \{0,1\}^\omega \smallsetminus WF$.

(b) *Is the set* IF *Borel?*

(7) Given $\mathcal{U}$ any free ultrafilter over $\mathbb{N}$, let $A_\mathcal{U} \subseteq \mathbb{N}^\omega$ be the following set:

$$x = x_0 x_1 \ldots \in A_\mathcal{U} \iff \begin{cases} x \text{ is strictly increasing} & \bigwedge & \left([0, x_0] \cup \bigcup_{i \in \mathbb{N}}]x_{2i+1}, x_{2i+2}]\right) \in \mathcal{U} \\ & \text{or} & \\ \exists i \in \mathbb{N} \ \ (x_{2i+1} \geq x_{2i+2} & \wedge & (\forall j < 2i+1 \ \ x_j < x_{j+1})). \end{cases}$$

- *Is the set* $A_\mathcal{U}$ *Borel?*

(8) Let $G = (V_0, V_1, E, C, \phi)$ be an arena for a parity games, we define the following relation between vertices:

$$u \leq_r v \iff_{def} v \text{ is reachable from u}$$

$$u <_r v \iff_{def} u \leq_r v \ \wedge v \not\leq_r u$$

$$u \equiv_r v \iff_{def} u \leq_r v \ \wedge v \leq_r u$$

Given a vertex u, the Strongly Connected Component of u is

$$SCC(u) = \{v \in V_0 \cup V_1 : \ u \equiv_r v\}.$$

- *We assume that for any vertex $u \in V_0 \cup V_1$ the parity game on the arena restricted to $SCC(u)$ can be solved in polynomial time. Show that the whole parity game on G can also be solved in polynomial time.*

  Hint: *Notice that given two vertices u and v, either $SCC(u) = SCC(v)$ or $SCC(u) \cap SCC(v) = \emptyset$. Use an induction on the number of different Stongly Connected Components ($card(\{SCC(u) : \ u \in V_0 \cup V_1\})$) and consider $SCC(u)$ for some u that satisfies $u <_r v$ fails for all vertices v, as for example $SCC(u_4)$ in the picture below.*

Figure 1: The SCC in G