Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, R. Rabinovich

WS 2010/11

Mathematical Logic II — Assignment 5

Due: Monday, November 22, 12:00

Exercise 1 (2 + 3 + 1 + 4) + 2 Points

(a) Let a be a nonempty set of ordinals.

  (i) What are $\bigcup a$ and $\bigcap a$ for $a = \{\emptyset\}$, $a = \{n \in \omega \mid n \text{ odd}\}$, $a = \omega$ and $a = \omega \cup \{\omega\}$?

  (ii) Prove that $\bigcup a$ is an ordinal and describe it in terms of arithmetical operations and the canonical order on $\mathfrak{On}$.

  (iii) Give a corresponding description for $\bigcap a$.

  (iv) Prove that
$$\alpha = \bigcup \alpha \iff \alpha \text{ is a limit ordinal}$$
  holds for every ordinal α.

(b) Let a be a class of ordinals. Give a sufficient and necessary condition for $\sup a$ to be an ordinal.

Exercise 2 12 Points

Compute the following expressions:

(a) $(((1 + \omega) + 1) + \omega) + 1$,

(b) $(((2 \cdot \omega) \cdot 2) \cdot \omega) \cdot 2$,

(c) $\sup\{n + m \mid m, n \in \omega\}$,

(d) $\sup\{\omega + n \mid n \in \omega\}$,

(e) $\sup\{\omega \cdot n \mid n \in \omega\}$,

(f) $\sup\{\omega \cdot n + 3 \mid n \in \omega\}$,

(a) $(2 \cdot (\omega + 1)) \cdot \omega$,

(b) $2 \cdot (\omega + 1) \cdot 2$,

(c) $\bigcup \omega$,

(d) $\bigcup \{\omega\}$,

(e) $\bigcup \{n \in \omega \mid n \text{ gerade}\}$,

(f) $\sup\{\omega^n + \omega \mid n \in \omega\}$.

Exercise 3 4 Points

We consider the following variants of the Axiom of Choice:

AC*: For every set x there exists a choice function on $\mathcal{P}(x)$.

KP: For every family $(X_i)_{i \in I}$ of nonempty sets, the cartesian product $\Pi_{i \in I} X_i$ is not empty.

ER: Every equivalence relation on a set x has a set of class representatives.

(a) Formalise the notions used in these statements.

(b) Prove that **AC***, **KP**, and **ER** are equivalent to the Axiom of Choice (on the basis of ZF).

Exercise 4* 6^* Points

A (totally) ordered class $\langle A, \leq \rangle$ is *perfectly ordered* if it satisfies the following conditions:

- A has a least element;
- each element of A has an unambigous successor (except the greatest one, if there is any);
- each element of A is a finite successor (via finitely many steps) of either the least element of A or of a limit element of A (an element without any direct ancestor in A).

Prove that each well-ordered class is perfectly ordered, but the converse doesn't hold.