Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
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WS 2010/11

Mathematical Logic II — Assignment 8

Due: Monday, December 13, 12:00

Exercise 1 3 Points

Prove that a recursively enumerable theory T is decidable, if it has only finitely many complete extensions $T' \supseteq T$.

Exercise 2 3 + 3 + 6 Points

We define a sequence $(\Phi)_{i\in\omega}$ of extrensions of Peano arithmetic by

(1) $\Phi_0 = \Phi_{PA}$,

(2) $\Phi_{i+1} = \Phi_i \cup \{\mathrm{Cons}_{\Phi_i}\}$,

(3) $\Phi_\omega = \bigcup_{i<\omega} \Phi_i$,

where Φ_{PA} is the axiom system of Peano arithmetic and Cons_{Φ_i} is a formula that expresses that Φ_i is consistent.

(a) Prove that all Φ_i are consistent.

(b) Prove that Φ_ω is consistent.

(c) Resolve the following paradox. We extend the sequence by:

(2′) $\Phi_{\alpha+1} = \Phi_\alpha \cup \{\mathrm{Cons}_{\Phi_\alpha}\}$,

(3′) $\Phi_\lambda = \bigcup \Phi_{\alpha<\lambda}$ for limit ordinals λ.

As there are only countably many formulae, there is a fixed-point Φ_∞ of the sequence $(\Phi_\alpha)_{\alpha\in\mathrm{On}}$, thus $\Phi_\infty = \Phi_\infty \cup \{\mathrm{Cons}_\infty\}$. Then we have $\Phi_\infty \vdash \mathrm{Cons}_\infty$, which contradicts the second Gödel's Theorem.

Exercise 3* 6* Points

Resolve the follownig paradox. Let Φ_{PA} be the axiom system of Peano arithmetic and let $\mathrm{Cons}_{\Phi_{\mathrm{PA}}}$ be the formula that expresses the consistency of Φ_{PA} (as defined in the lecture). Let Φ be the formula $\Phi = \Phi_{\mathrm{PA}} \cup \{\neg\mathrm{Cons}_{\Phi_{\mathrm{PA}}}\}$. As $\Phi_{\mathrm{PA}} \nvdash \mathrm{Cons}_{\Phi_{\mathrm{PA}}}$, it follows that Φ is consistent. On the other hand, Φ proves that Φ_{PA} is not consistent: $\Phi \vdash \neg\mathrm{Cons}_{\Phi_{\mathrm{PA}}}$, as $\neg\mathrm{Cons}_{\Phi_{\mathrm{PA}}} \in \Phi$. But then Φ is all the more inconsistent.

Exercise 4 2 + 4 + 2 + 2 Points

Let $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$ be three τ-structures for a signature τ. Prove or disprove the following statements:

(a) Let $\mathfrak{A} \preceq \mathfrak{B}$ and let A be finite or B be finite. Then $\mathfrak{A} = \mathfrak{B}$.

(b) If $\mathfrak{A} \preceq \mathfrak{C}$ then $\mathfrak{A} \preceq \mathfrak{B}$.

(c) If $\mathfrak{A} \preceq \mathfrak{C}$ and $\mathfrak{B} \preceq \mathfrak{C}$ then $\mathfrak{A} \preceq \mathfrak{B}$.

(d) If $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \preceq \mathfrak{B}$.