Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
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WS 2010/11

Mathematical Logic II — Assignment 9

Due: Monday, December 20, 12:00

Exercise 1 2 + 2 Points

Let $T \subseteq FO(\tau)$ be a theory for a signature τ and let $\mathcal{K}$ be the class of τ-structures closed under isomorphisms, i.e. if $\mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{A} \in \mathcal{K}$ then $\mathfrak{B} \in \mathcal{K}$. Let $\text{Mod}(T)$ be the class of models of T and let $\text{Th}(\mathcal{K})$ be the theory of $\mathcal{K}$, i.e. $\text{Th}(\mathcal{K}) = \bigcap_{\mathfrak{A} \in \mathcal{K}} \text{Th}(\mathfrak{A})$. Prove or disprove:

(a) $\text{Mod}(\text{Th}(\mathcal{K})) = \mathcal{K}$,

(b) $\text{Th}(\text{Mod}(T)) = T$.

Exercise 2 2 + 3 + 3 Points

A theory $T \subseteq \text{FO}(\tau)$ is model complete if for arbitrary τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A}, \mathfrak{B} \models T$, $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{A} \preceq \mathfrak{B}$.

(a) Let $T \subseteq \text{FO}(\tau)$ be a model complete theory and let $\mathfrak{A}$ be a finite model of T. Prove that there is no proper extension $\mathfrak{B} \supsetneq \mathfrak{A}$ with $\mathfrak{B} \models T$.

(b) Prove or disprove the model completeness of the theories $\text{Th}(\mathbb{N}, S)$ where S is the successor function and $\text{Th}(\mathbb{Z}, <)$.

(c) Prove that all complete theories over the signautre $\sigma = \{P\}$ with one unary relation symbol P are model complete.

Hint: What are the the complete theories over σ?

Exercise 3 3 + 3 + 4 + 6* Points

A theory $T \subseteq \text{FO}(\tau)$ *permits quantifier elimination* if for each formula $\varphi(\overline{x}) \in \text{FO}(\tau)$ there exists a quantifier free formula $\vartheta(\overline{x}) \in \text{FO}(\tau)$ such that $T \models \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \vartheta(\overline{x}))$.

(a) Prove that a theory $T \subseteq \text{FO}(\tau)$ permits quantifier elimination if and only if for every quantifier free formula $\psi(\overline{x}, y) \in \text{FO}(\tau)$ there is a quantifier free formula $\vartheta(\overline{x}) \in \text{FO}(\tau)$ such that $T \models \forall \overline{x}(\exists y \psi(\overline{x}, y) \leftrightarrow \vartheta(\overline{x}))$.

(b) Let $T \subseteq \text{FO}(\tau)$ be a theory that permits quantifier elimination. Prove that T is model complete.

(c) Let $T \subseteq \text{FO}(\tau)$ be a theory that permits quantifier elimination where τ is a signature without constants. Prove that T is complete.

(d*) Prove that the theory of dense linear orders permits quantifier elimination.

Hint: Use the dijunctive normal form to represent the quantifier free formulae over $<$.

Exercise 4 4 Points

An *elementary chain* is a sequence $(\mathfrak{A}_\beta)_{\beta<\alpha}$ of structures with $\mathfrak{A}_\gamma \preceq \mathfrak{A}_\beta$ for all $\gamma < \beta < \alpha$. Let $(\mathfrak{A}_\beta)_{\beta<\alpha}$ an elementary chain. Prove that $\mathfrak{A}_\beta \preceq \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$ holds for $\beta < \alpha$.