Automatische Strukturen

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What are automatic structures?

Structures that admit a finite presentation by automata.

The universe, and all relations of an automatic structure $\mathfrak{A} = (A, R_1, \ldots, R_m)$ are recognisable by (synchronous multi-head) finite automata.

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In principle we can use any kind of automata (over finite or infinite words or trees, ...) as long as they are effectively closed under all first-order operations and their emptiness problem is decidable.

- Given an automatic presentation of $\mathfrak{A}$ and a first-order formula $\varphi(\overline{x})$, one can effectively construct an automaton that represents the relation $\varphi^{\mathfrak{A}} := \{\overline{a} : \mathfrak{A} \models \varphi(\overline{a})\}$.
- Every automatic structure has a decidable first-order theory.

Word-automatic structures

$\mathfrak{A} = (A, R_1, \dots, R_m)$ is (word-)automatic if there exist a regular language $L_\delta \subseteq \Sigma^*$ and a surjective function $h : L_\delta \to A$ such that the relations

$$L_= := \{(u,v) : h(u) = h(v)\} \subseteq L_\delta \times L_\delta$$

$$L_{R_i} := \{(u_1, \dots, u_r) : \mathfrak{A} \models R_i h(u_1) \dots h(u_r)\} \subseteq L_\delta \times \dots \times L_\delta$$

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Automatic presentation of $\mathfrak{A}$: list of automata

$$\langle M_\delta, M_=, M_{R_1}, \dots, M_{R_m} \rangle$$

recognising $L_\delta, L_=, L_{R_1}, \dots, L_{R_m}$.

Synchronous automata

Automaton M, recognising a relation $R \subseteq \underbrace{\Sigma^* \times \cdots \times \Sigma^*}_{r}$ works on the alphabet

$\Gamma := (\Sigma \cup \{\square\})^r - \{\square\}^r$

$$(u_1, \ldots, u_r) \in R \iff M \text{ accepts } \underbrace{\begin{matrix} u_{11}u_{12} & \ldots & u_{1j}\square\square\square\ldots\square \\ u_{21}u_{22}\ldots & \ldots & u_{2k}\square\ldots\square \\ \vdots & \vdots & \vdots \\ u_{i1}u_{i_2} & \ldots & u_{i\ell} \\ \vdots & \vdots & \vdots \\ u_{r1}u_{r2} & \ldots & u_{rj}\square\square\square\ldots\square \end{matrix}}_{\ell=\max\{|u_i|:i=1,\ldots,r\}} \in \Gamma^*$$

Examples of automatic structures

- $(\mathbb{N}, +)$ is automatic

  – $L_\delta = \{0,1\}^* 1 \cup \{0\}$

  – $h(w_0 \dots w_{n-1}) = \sum_{i<n} w_i 2^i$ (h injective)

  – L_+ recognised by automaton M_+

  scans
  $$\begin{array}{ll} u_0 u_1 \dots & \dots u_m \Box \dots\dots \Box \\ v_0 v_1 \dots & \dots v_{n-1} \Box \\ w_0 w_1 \dots & \dots w_n \end{array}$$

  remembering carry bit c_i for $u_0 \dots u_{i-1} + v_0 \dots v_{i-1}$

  checks whether $w_i = u_i + v_i + c_i \pmod 2$

- every finite structure is automatic

- the configuration graph of any Turing machine is automatic

Universal automatic structures

- $(\mathbb{N}, +, |_p)$ is automatic

$$x \,|_p\, y \quad :\Longleftrightarrow \quad x \text{ is a power of } p \text{ dividing } y$$

use p-ary representation of numbers

$$L_{|_p} = \left\{ \begin{matrix} u \\ v \end{matrix} : \begin{matrix} u \\ v \end{matrix} = \begin{matrix} 0\dots & 0\,1\,\Box\dots & \dots\Box \\ 0\dots & 0 v_r v_{r+1} & \dots v_n \end{matrix} \right\}$$

- $\mathrm{Tree}(m) = (\{0, \dots, m-1\}^*, \sigma_0, \dots, \sigma_{m-1}, \leq, \mathrm{el})$ is automatic

  - $\sigma_i : \quad u \mapsto ui$
  - $u \leq v : \quad \exists w \quad uw = v$
  - $\mathrm{el}(u, v) : \quad |u| = |v|$

ω-automatic structures

$\mathfrak{A} = (A, R_1, \ldots, R_s)$ is ω-automatic if there exist a ω-regular language $L_\delta \subseteq \Sigma^\omega$ and a surjective function $h : L_\delta \to A$ such that the relations

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- every automatic structure is ω-automatic
- $(\mathbb{R}, +)$ and $(\mathbb{R}, +, \leq, |_m, 1)$ are ω-automatic

$$x \,|_m\, y \quad :\Longleftrightarrow \quad \exists k, r \in \mathbb{Z} : x = m^k, \quad y = r \cdot x$$

- ω-Tree$(m) = (\{0,\dots,m-1\}^{\leq\omega}, \sigma_0, \dots, \sigma_{m-1}, \leq, \mathrm{el})$ is ω-automatic

First-order logic on (ω)-automatic structures

Standard models of automata are effectively closed under first-order operations (union, intersection, complementation, projection) and have a decidable emptiness problem. Hence, every automatic structure has a decidable first-order theory.

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Question: Can we extend such results beyond first-order logic?

Positive results for counting quantifiers: Let FOC be the extension of FO by
- "there exist infinitely many x such that …"
- "there exist k mod m many x such that …"
- "there exist uncountably many x such that …"

Theorem. Given $\varphi(\bar{x}) \in$ FOC and an ω-automatic presentation of $\mathfrak{A}$, one can effectively extend the presentation to one of $(\mathfrak{A}, \varphi^{\mathfrak{A}})$.

Corollary. The FOC-theory of every ω-automatic structure is decidable.

Undecidability

Important Fact: The configuration graph of every Turing machine is automatic.

Hence, any logic that is strong enough for expressing reachability can encode the halting problem, and does therefore not admit effective evaluation on certain automatic structures.

Examples:

- monadic second-order logic
- transitive closure logics
- fixed-point logics

Automatic structures and interpretations

$\mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$: $\mathfrak{A}$ is first-order interpretable in $\mathfrak{B}$

Automatic structures and ω-automatic structures are closed under FO-interpretations:

$$\mathfrak{B} \text{ is } (\omega)\text{-automatic}, \quad \mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B} \quad \Longrightarrow \quad \mathfrak{A} \text{ is } (\omega)\text{-automatic}$$

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In particular, the (ω)-automatic structures are closed under

- expansion by definable relations
- factorisation by definable congruences
- substructures with definable universe

Note: They are not closed under taking arbitrary substructures

The model-theoretic characterization of automatic structures

Theorem. The following are equivalent:

(1) $\mathfrak{A}$ is automatic

(2) $\mathfrak{A} \leq_{\text{FO}} (\mathbb{N}, +, |_p)$ for some (and hence all) $p \geq 2$

(3) $\mathfrak{A} \leq_{\text{FO}} \text{Tree}(p)$ for some (and hence all) $p \geq 2$.

There are analogous results for ω-automatic structures.

ω-automatic structures and injective presentations

For word-automatic structures, we always have injective automatic presentations. This is not so for ω-automatic structures.

End-equivalence of infinite words: $x \sim_e y$ if x and y are equal from some position onwards. Refined equivalences $\sim_e^m$ by making this position explicit.

$\sim_e$ is ω-regular, but does not permit an ω-regular set of representatives. Hence injectivity cannot always be achieved by selecting a regular set of representatives from a given presentation.

Theorem (Hjorth, Khoussainov, Montalban, Nies
There exist ω-automatic structures that do not even permit injective Borel presentations.

Countable structures

Every ω-regular equivalence relation, that has only countably many classes does permit an ω-regular set of unique representatives. Further, an injective ω-automatic presentation of a countable structure can be packed into one over finite words.

Theorem. (Kaiser, Rubin, Bárány) For countable structures are equivalent:

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- $\mathfrak{A}$ admits an injective ω-automatic presentation
- $\mathfrak{A}$ is word-automatic.

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- $\mathfrak{A}$ is word-automatic.

It follows that the structures $(\mathbb{N}, \cdot)$, $(\mathbb{Q}, +)$, and the random graph, which are known not to be word-automatic, are not ω-automatic either.

A fundamental open problem

Open problem: Is the field of reals $(\mathbb{R}, +, \cdot)$ automatic in some sense?

Being uncountable, it cannot be presentable by automata over finite words or finite trees. But it could, a priori, be ω-automatic, or ω-tree-automatic.

Both reducts $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$ admit automatic presentations, even over infinite words. Can we combine these to a presentation of the field of reals?

It is still open, whether $(\mathbb{R}, +, \cdot)$ admits a presentation by automata over infinite trees (or even more general objects).

But we can show that $(\mathbb{R}, +, \cdot)$ is not ω-automatic !

Theorem (Abu Zaid, EG, Kaiser, Pakusa) An integral domain is ω-automatic if, and only if, it is finite. In particular, the field of reals is not ω-automatic.

Vortragsthemen für dieses Seminar

1. Characterising automatic structures via interpretations
2. Set interpretations
3. Structures that are not automatic
4. The isomorphism problem
5. $(\mathbb{Q}, +)$ is not automatic
6. ω-automatic structures are not always injectively presentable
7. Countable ω-automatic structures are word-automatic
8. Model-theoretic properties of ω-automatic structures
9. Automatic groups and the k-fellow traveller property
10. Advice automatic structures