Convergence and Nonconvergence Laws for Random Expansions of Product Structures

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For Yuri Gurevich on the occasion of his 80th birthday

Abstract. We prove (non)convergence laws for random expansions of product structures. More precisely, we ask which structures $\mathfrak{A}$ admit a limit law, saying that the probability that a randomly chosen expansion of $\mathfrak{A}^n$ satisfies a fixed first-order sentence always converges when n approaches infinity. For the groups $\mathbb{Z}_p$, where p is prime, we do indeed have such a limit law, even for the infinitary logic $L^{\omega}_{\infty\omega}$, and these probabilities always converge to dyadic rational numbers, whose denominator only depends on the expansion vocabulary. This can be used to prove that the Abelian group summation problem is not definable in $L^{\omega}_{\infty\omega}$. Further examples for structures with such a limit law are permutation structures and structures whose vocabulary only consists of monadic relations. As a negative example, we prove that the very simple structure $(\{0,1\},\leq)$ does not have a limit law. Furthermore, we develop a method based on positive primitive interpretations that allows transferring (non)convergence results to other structures. Using this method, we are able to prove that structures with binary function symbols or unary functions that are not interpreted by permutations do not have a limit law in general.

1 Introduction

The study of convergence and nonconvergence laws for logical formulae on random finite structure has been an important topic of finite model theory since the discovery of the celebrated 0-1 law for first-order logic, discovered 50 years ago by Glebskiĭ et al. [6] and, independently, by Fagin [5]. Informally, this law says that any property of finite graphs or finite relational structures that is definable by a first-order sentence is either almost surely true or almost surely false on (sufficiently large) randomly chosen finite structures or graphs. More precisely, let ψ be a first-order sentence of vocabulary τ and, consider, for any positive natural number n, the probability $\mu_n(\psi)$, that a random τ-structure with universe $[n] := \{0,\dots,n-1\}$ (chosen with uniform probability from all such structures) is a model of ψ. The 0-1 law says that, for each first-order sentence of any relational vocabulary, the sequence $\mu_n(\psi)$ converges exponentially fast to either 0 or 1, as n goes to infinity.

Since then, there has been an enormous amount of work on variations of such questions, related to many different logical systems as well as to more general probability distributions, focussing not just on 0-1 laws but on more general questions about convergence and nonconvergence of such sequences of probabilities $\mu_n(\psi)$. Yuri Gurevich has made significant contributions to this area. Together with Blass and Kozen, he proved the 0-1 law for the fixed-point logic LFP [2], a result that later motivated the generalization to the 0-1 law for $L^{\omega}_{\infty\omega}$, the infinitary logic with a bounded number of variables [10]. In [7] he presented a lucid survey on 0-1 laws. For further results we refer to [3].

Here we consider a further variation of questions about limit laws, which had originally been motivated by investigations concerning the logical definability of the Abelian group summation problem. Given a finite group or semigroup $(G, +, 0)$ and a subset $X \subseteq G$, we want to determine the sum over all elements of X. Algorithmically this is a very simple problem. If the elements of X come in some order, then we process them along that order and calculate the sum in a trivial way. However, the logical definability of this problem is much more delicate. If we consider G as an abstract structure and X as an abstract set, without a linear order and hence without a canonical way to process elements one by one, then it is unclear how to define the sum in any logic that does not have the power to quantify over a linear order. Indeed it had been conjectured that the Abelian summation problem would not even be expressible in Choiceless Polynomial Time with counting, one of the most powerful known candidates for a logic that might be capable of defining all polynomial-time computable properties of finite structures. Although it has eventually been proved in [1] that this conjecture is false and that, indeed, the summation problem for Abelian semigroups is even definable in fixed-point logic with counting (FPC), it turned out that even for the restricted case of groups $\mathbb{Z}_p^n$, the summation problem cannot be defined in fixed-point logic *without counting*, and in fact not even in the infinitary logic $L^{\omega}_{\infty\omega}$. This last result relied on a new limit kind of limit law for random expansions of $\mathbb{Z}_p^n$ that was also established in [1]. Here we investigate the question what kind of base structures, beyond the groups $\mathbb{Z}_p$, admit a limit law of this kind.

In the next section, we precisely define this problem. We shall then explain the proof from [1] for the groups $\mathbb{Z}_p$. In Sect. 4 we discuss some further cases where a similar limit law can be established, before we turn to nonconvergence results. The simplest base structure $\mathfrak{A}$ for which random expansions of $\mathfrak{A}^n$ do not admit a limit law for first-order logic is $\mathfrak{A} = (\{0, 1\}, <)$. We finally discuss a method based on positive primitive interpretations to transfer such convergence and nonconvergence results among different base structures, and establish a few more cases for nonconvergence laws of this kind.

2 The problem

Let $\mathfrak{A}$ be a structure with a finite universe A of finite (not necessarily relational) vocabulary σ, and let τ be another finite relational vocabulary with $\sigma \cap \tau = \emptyset$. For each n, let $\mathfrak{A}^n$ be the n-fold product of $\mathfrak{A}$, defined in the usual way: The universe of $\mathfrak{A}^n$ is A^n, the set of n-tuples over A, written as functions $a\colon [n] \to A$ (sometimes also denoted as $a = (a(0), \dots, a(n-1))$); for each relation symbol $R \in \sigma$ of arity r and $a_1, \dots a_r \in A^n$, we have that $\mathfrak{A}^n \models R(a_1, \dots, a_r)$ if, and only if, $\mathfrak{A} \models R(a_1(i), \dots, a_r(i))$ for all $i \in [n]$, and for each function symbol $f \in \sigma$ of arity r and $a_1, \dots a_r, b \in A^n$, we have that $\mathfrak{A}^n \models f(a_1, \dots a_r) = b$ if, and only if, $\mathfrak{A} \models f(a_1(i), \dots, a_r(i)) = b(i)$ for all $i \in [n]$.

We consider the probability spaces $S^n_\tau(\mathfrak{A})$ consisting of all $(\sigma \cup \tau)$-expansions of $\mathfrak{A}^n$, with the uniform probability distribution. For every sentence $\psi \in \mathcal{L}(\sigma \cup \tau)$ (in whatever logic $\mathcal{L}$), let $\mu_n(\psi)$ denote the probability that a randomly chosen structure $\mathfrak{B} \in S^n_\tau(\mathfrak{A})$ is a model of ψ.

We are interested to know for which finite structures $\mathfrak{A}$ the following limit law holds: For every finite relational vocabulary τ and for every sentence $\psi \in \mathrm{FO}(\sigma \cup \tau)$ there exists a (dyadic rational) number q such that

$$\mu(\psi) \coloneqq \lim_{n\to\infty} \mu_n(\psi) = q.$$

3 The groups $\mathbb{Z}_p$

It has been shown in [1] that such a limit law holds for $\mathfrak{A} \coloneqq (\mathbb{Z}_p, +, 0)$, for any prime p, not just for FO but also for $L^\omega_{\infty\omega}$. The proof generalizes the classical techniques, based on extension axioms, for proving the 0-1 law for FO and $L^\omega_{\infty\omega}$ on random graphs and random finite relational structures.

Consider the group $(\mathbb{Z}_p, +, 0)$, for some prime p, and an arbitrary finite relational vocabulary $\tau = \{X_1, \dots, X_\ell\}$. For each $n \in \mathbb{N}$, we consider the probability spaces $S_n(\mathbb{Z}_p)$, consisting of all expansions of (the additive group of) the vector space $(\mathbb{Z}_p)^n$ by relations from τ, with the uniform probability distribution. We prove the following limit law.

Theorem 1. *For every relational vocabulary τ and for every sentence $\psi \in L^\omega_{\infty\omega}(\{+, 0\} \cup \tau)$,*

$$\lim_{n\to\infty} \mu_n(\psi) = \frac{r}{2^\ell}, \text{ for } \ell = |\tau| \text{ and some } r \le 2^\ell.$$

Proof. Let $\delta_1, \dots, \delta_m$ be the $m = 2^\ell$ atomic τ-types in the constant 0 (and without variables). For each j, δ_j is a conjunction over ℓ atoms or negated atoms of form $X_i(0, \dots, 0)$, for $X_i \in \tau$. Obviously, for all $j \le m$ and all n, $\mu_n(\delta_j) = 1/m$.

For any collection $a_1, \dots, a_k$ of elements of $(\mathbb{Z}_p)^n$ let $\mathrm{span}(a_1, \dots, a_k)$ be the subspace generated by $a_1, \dots, a_k$. Clearly, the size of $\mathrm{span}(a_1, \dots, a_k)$ in $(\mathbb{Z}_p)^n$ is bounded by p^k, for any n.

Recall that an atomic k-type $t(x_1, \dots, x_k)$ of a vocabulary σ is a maximal consistent set of atoms and negated atoms in the variables $x_1, \dots, x_k$. In our case, a k-type $t(x_1, \dots, x_k)$ specifies the linear dependencies and independencies of $x_1, \dots, x_k$ and the truth values of all atoms $X(y_1, \dots, y_r)$ where $X \in \tau$, and each y_i is a $\mathbb{Z}_p$-linear combination of $x_1, \dots, x_k$.

Definition 2. For each $j \leq m$, we define AT_j to be the set of all atomic types $t(x_1, \dots, x_k)$ of vocabulary $\{+, 0\} \cup \tau$ such that

(1) t is consistent, i.e. realisable in some $\mathfrak{B} \in S^n_\tau(\mathfrak{A})$,
(2) $t \models \delta_j$,
(3) t implies, for each $i \leq k$, that $x_i \notin \mathrm{span}(x_1, \dots, x_{i-1})$.

We then define T_j to be the theory of all extension axioms

$$\mathrm{ext}_{s,t} \coloneqq \forall \bar{x}(s(\bar{x}) \rightarrow \exists x_{k+1} t(\bar{x}, x_{k+1}))$$

where s and t are, respectively, atomic k and $k+1$-types in AT_j with $t \models s$.

Please notice that condition (3) of Definition 2 is equivalent to: t entails the linear independence of $x_1, \dots, x_k$.

Proposition 3. *Every extension axiom* $\mathrm{ext}_{s,t} \in T_j$ *has asymptotic probability one on the sequence of spaces* $S^n_\tau(\mathbb{Z}_p)$.

Proof. Let $(a_1, \dots, a_k)$ be a realisation of the atomic type $s(\bar{x}) \in \mathrm{AT}_j$ in some randomly chosen expansion $\mathfrak{B}$ of $(\mathbb{Z}_p)^n$. The type $s(\bar{x})$ fixes the truth values of all τ-atoms in the variables $x_1, \dots, x_k$ and the constant 0, and $t(\bar{x}, x_{k+1})$ additionally fixes truth-values for the τ-atoms that contain at least one term with the variable x_{k+1}. There is a bounded number q of such atoms. Therefore, if we fix some element $b \in (\mathbb{Z}_p)^n \setminus \mathrm{span}(a_1, \dots, a_k)$, then the probability that $\mathfrak{B} \models t(\bar{a}, b)$ is 2^{-q}.

The elements b that we have to explore are those outside of $\mathrm{span}(a_1, \dots, a_k)$. Each of them fixes $|\mathrm{span}(a_1, \dots, a_k, b) \setminus \mathrm{span}(a_1, \dots, a_k)| \leq (p-1)p^k$ new elements, so there are at least p^{n-k-1} independent choices for b. Since there are fewer than p^{nk} realisations of $s(\bar{x})$ in $\mathfrak{B}$, the probability that one of them cannot be extended to a realisation of $t(\bar{x}, x_{k+1})$ is at most

$$p^{nk}(1 - 2^{-q})^{p^{n-k-1}}$$

which tends to 0 exponentially fast as n goes to infinity.

Thus, the asymptotic probability of every extension axiom $\mathrm{ext}_{s,t} \in T_j$ is one on $S^n_\tau(\mathbb{Z}_p)$. □

For every $j \leq m$, $k < \omega$, let ϑ^k_j be the conjunction of all extension axioms in T_j with at most k variables. Further, let $E(k, j)$ be the class of all expansions $\mathfrak{B}$ of $\mathfrak{A}^n$ (for any finite $n \geq k$) such that $\mathfrak{B} \models \delta_j \wedge \vartheta^k_j$.

Lemma 4. $\mu(\delta_j \wedge \vartheta^k_j) = 1/m$ *for all* $j \leq m, k < \omega$.

Proposition 5. *For every $\psi \in L^k_{\infty\omega}$ and every $j \leq m$, either $\mathfrak{A} \models \psi$ for all $\mathfrak{A} \in E(k,j)$, or $\mathfrak{A} \models \neg\psi$ for all $\mathfrak{A} \in E(k,j)$.*

Proof. Take any two structures $\mathfrak{A}, \mathfrak{B} \in E(k,j)$. From the fact that both structures satisfy $\delta_j \wedge \vartheta^k_j$ we immediately get a winning strategy for the k-pebble game on $\mathfrak{A}$ and $\mathfrak{B}$ (for background on the model comparison games for k-variable logic, see [4]). Hence the two structures are $L^k_{\infty\omega}$-equivalent, so it cannot be the case that ψ is true in one and false in the other. □

Given any formula $\psi \in L^k_{\infty\omega}$, let $r(\psi) = |\{j \leq m : \psi \text{ is true in all } \mathfrak{A} \in E(k,j)\}|$. It follows that

$$\mu(\psi) = \lim_{n\to\infty} \mu_n(\psi) = \frac{r(\psi)}{m}.$$

Hence the limit law holds for $L^\omega_{\infty\omega}$. □

Theorem 6. *The Abelian group summation problem is not definable in $L^\omega_{\infty\omega}$.*

Proof. Suppose that the Abelian group summation problem is definable by a formula $\varphi(x) \in L^k_{\infty\omega}$ such that for every finite Abelian group $(H, +, 0)$, all $X \subseteq H$ and every $h \in H$,

$$(H, +, 0, X) \models \varphi(h) \iff \sum X = h.$$

Consider the sentence $\psi \coloneqq \exists x(\varphi(x) \wedge X(x) \wedge X(0))$, which expresses that both 0 and the sum over all elements of X are contained in X. Let $G = (\mathbb{Z}_2, +, 0)$ and $H = \mathbb{Z}^n_2$. For a randomly chosen $X \subseteq H$ all elements of H have equal probability to be the sum of all elements of X. The probability that this sum is itself an element of X quickly converges to $1/2$. Thus the asymptotic probability of ψ on the spaces $S^n_\tau(\mathbb{Z}_2)$ converges to $1/4$.

However, since we use only one random relation, the denominator of the asymptotic probabilities in the limit law is 2, so $\mu_n(\psi)$ should converge to either 0,1, or $1/2$. Contradiction. □

Categoricity. A classical result about limit laws for finite random structures states that the theory of all extension axioms is ω-categorical, i.e. it has, up to isomorphism, precisely one countable model. We can prove an analogous categoricity result in our setting.

Let $\mathbb{Z}^*_p$ be the weak ω-product of $\mathbb{Z}_p$. Its elements are the functions $\boldsymbol{g} : \omega \to \mathbb{Z}_p$ such that $\boldsymbol{g}(n) = 0$ for all but finitely many n, addition is defined componentwise in the obvious way, and $\mathbf{0}$ is the constant function mapping all $n \in \omega$ to 0. The next observation says that the theories $\{\delta_j\} \cup T_j$ are categorical for expansions of $\mathbb{Z}^*_p$.

Proposition 7. *Let $\mathfrak{A}_\omega$ and $\mathfrak{B}_\omega$ be any two expansions of $\mathbb{Z}^*_p$ to $\{+, 0\} \cup \tau$-structures which are both models of $\{\delta_j\} \cup T_j$. Then $\mathfrak{A}_\omega$ and $\mathfrak{B}_\omega$ are isomorphic.*

Proof. The universes of both $\mathfrak{A}_\omega$ and $\mathfrak{B}_\omega$ are the same as for $\mathbb{Z}_p^*$. Fix an enumeration $\boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots$ of this set, and define a sequence $(f_n)_{n\in\omega}$ of partial isomorphisms from $\mathfrak{A}_\omega$ to $\mathfrak{B}_\omega$ as follows. Let $f_0 = \{(\mathbf{0}, \mathbf{0})\}$. Since both $\mathfrak{A}_\omega$ and $\mathfrak{B}_\omega$ are models of δ_j, this is indeed a partial isomorphism. Suppose now that, for $k \geq 0$, f_k has already been defined, with domain $\mathrm{span}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k)$, and image $\mathrm{span}(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k)$. Since f_k is a partial isomorphism $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k)$ and $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k)$ realise the same atomic type $s(\bar{x})$.

For even k, let $\boldsymbol{a}_{k+1}$ be the first element in the enumeration $\boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots$ that does not appear in the domain of f_k, and let $t(\bar{x}, x_{k+1})$ be the atomic type realised by $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, \boldsymbol{a}_{k+1})$. Since $\mathfrak{B}_\omega \models \mathrm{ext}_{s,t}$ the tuple $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k)$ can be extended by a suitable element $\boldsymbol{b}_{k+1}$ to a realisation of $t(\bar{x}, x_{k+1})$. This defines an extension of f_k to a partial isomorphism f_{k+1} from $\mathrm{span}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{k+1})$ to $\mathrm{span}(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{k+1})$.

For odd k we proceed similarly, by choosing for $\boldsymbol{b}_{k+1}$ the first element in the enumeration of the universe that is not contained in the image of f_k. Since the appropriate extension axiom holds in $\mathfrak{A}_\omega$ the element $\boldsymbol{b}_{k+1}$ can then be matched by an element $\boldsymbol{a}_{k+1}$ to provide the extension f_{k+1}.

The union $f = \bigcup_{k\in\omega} f_k$ is then the desired isomorphism between $\mathfrak{A}_\omega$ and $\mathfrak{B}_\omega$. □

4 Limit Laws for Other Structures

In this section, we show that the following structures also have a limit law:

- Structures only equipped with monadic relation symbols.
- Permutation structures, i.e. structures equipped with unary function symbols that are interpreted by bijective functions.

In Section 5, we see concrete examples of two structures that have a non-convergence law instead. One of them has a binary relation symbol, while the another one has only unary function symbols that are interpreted by certain non-bijective functions.

4.1 Atomic Types and Extension Axioms in General Structures

We say that an element $a \in A$ is *uniformly* definable in some logic $\mathcal{L}$, if there exists an $\mathcal{L}$-formula $\varphi(x)$ such that

$$\mathfrak{A}^n \models \varphi(\bar{b}) \iff \bar{b} = (a, a, \ldots, a)$$

for every $n \in \mathbb{N}$ and $\bar{b} \in A^n$. For example, if $c \in \sigma$ is a constant symbol, then $\varphi(x) \coloneqq x = c$ is such a uniform definition of $c^{\mathfrak{A}}$. Another example is the formula $\varphi_0(x) \coloneqq \forall y(x + y = y)$ which uniformly defines the neutral element in a group $(G, +)$.

Now let $\bar{u} = (u_1, \ldots, u_p)$ be an enumeration of all uniformly FO-definable elements of $\mathfrak{A}$. Let $\sigma_{\mathfrak{A}} \coloneqq \{c_1, \ldots, c_p\}$ be a vocabulary containing constant symbols for these uniformly definable elements and let $\mathfrak{A}'$ be the $(\sigma \cup \sigma_{\mathfrak{A}})$-expansion with $c_i^{\mathfrak{A}'} = u_i$. Furthermore, let $\delta_1, \dots, \delta_m$ be the atomic $(\sigma_{\mathfrak{A}} \cup \tau)$-types (with no variables) of possible $(\sigma \cup \sigma_{\mathfrak{A}} \cup \tau)$-expansions of $\mathfrak{A}'$.

Similar to Definition 2, we define the sets AT_j (for each $j \leq m$) consisting of all atomic types $t(x_1, \dots, x_k)$ over the vocabulary $\sigma \cup \sigma_{\mathfrak{A}} \cup \tau$ with the following properties:

(1) t is consistent, i.e. realisable in some $\mathfrak{B} \in S_\tau^n(\mathfrak{A}')$.
(2) $t \models \delta_j$
(3) $t \models x_i \neq h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ where h is a $(\sigma \cup \sigma_{\mathfrak{A}})$-term.

Again, for every $j \in \{1, \dots, m\}$, we let

$$T_j := \{\mathrm{ext}_{s,t} : s, t \in \mathrm{AT}_j, t \text{ is an extension type of } s\}$$

where $\mathrm{ext}_{s,t} := \forall \bar{x}(s(\bar{x}) \to \exists x_{k+1} t(\bar{x}, x_{k+1}))$.

4.2 Only Monadic Relations

Now we investigate the case that the vocabulary σ contains only monadic relation symbols. Thus, $\mathfrak{A}'$ has the form $(A, (P^{\mathfrak{A}})_{P \in \sigma}, (c_i^{\mathfrak{A}'})_{i=1,\dots,p})$ where $P^{\mathfrak{A}} \subseteq A$ for every relation symbol $P \in \sigma$ and the $c_1, \dots, c_p$ are interpreted by the uniformly FO-definable elements of $\mathfrak{A} := (A, (P^{\mathfrak{A}})_{P \in \sigma})$. If $|A| = 1$, then $\mathfrak{A}^n \cong \mathfrak{A}$ and the limit law holds due to trivial reasons. Therefore, we consider structures $\mathfrak{A}$ with $|A| \geq 2$.

Proposition 8. *Let $\mathcal{L}$ be a logic with* FO $\leq \mathcal{L}$. *An element a of $\mathfrak{A}$ is uniformly definable in $\mathcal{L}$ if, and only if, there are some relation symbols $P_1, \dots, P_k \in \sigma$ with $\bigcap_{i=1}^k P_i^{\mathfrak{A}} = \{a\}$.*

Proof. For the direction "$\Leftarrow$", we prove that $\bigcap_{i=1}^k P_i^{\mathfrak{A}} = \{a\}$ implies that the first-order formula $\varphi(x) := \bigwedge_{i=1}^k P_i x$ is in fact a uniform definition of a. Towards this end, let $\mathfrak{A}^n \models \varphi(\bar{b})$ for some $\bar{b} = (b_1, \dots, b_n) \in A^n$. Then $b_j \in \bigcap_{i=1}^k P_i^{\mathfrak{A}} = \{a\}$ and, hence, $b_j = a$ for every j as desired.

"$\Rightarrow$": Now assume that some $\mathcal{L}$-formula $\varphi(x)$ is a uniform definition of a. Let $\sigma' := \{P \in \sigma : P \text{ monadic relation symbol and } a \in P^{\mathfrak{A}}\}$. Towards a contradiction, assume the existence of some $b \in \bigcap_{P \in \sigma'} P^{\mathfrak{A}} \setminus \{a\}$. Since $\varphi(x)$ is a uniform definition of a, we have $\mathfrak{A}^n \models \varphi((a, a, \dots, a))$ but $\mathfrak{A}^n \not\models \varphi((b, a, \dots, a))$. However, this is not possible, because the function $\pi \colon A^n \to A^n$ that swaps $(a, a, \dots, a)$ with $(b, a, \dots, a)$ (and maps every other element onto itself) is an isomorphism of $\mathfrak{A}^n$ for $n \geq 2$. Indeed, π is clearly bijective and for every relation symbol $P \in \sigma$, we can distinguish between the following two cases:

- $P \in \sigma'$: Then $(a, a, \dots, a) \in P^{\mathfrak{A}^n}$ and $(b, a, \dots, a) \in P^{\mathfrak{A}^n}$, because $a, b \in P^{\mathfrak{A}}$.
- $P \notin \sigma'$: Then $a \notin P^{\mathfrak{A}}$ and, hence, $(a, a, \dots, a) \notin P^{\mathfrak{A}^n}$ and $(b, a, \dots, a) \notin P^{\mathfrak{A}^n}$, because a occurs in both tuples at the second position. □

Proposition 9. *Let $\mathrm{ext}_{s,t} \in T_j$. Then $\mu(\mathrm{ext}_{s,t}) = 1$.*

Proof. Let $\mathrm{ext}_{s,t} = \forall \bar{x}(s(\bar{x}) \rightarrow \exists y t(\bar{x}, y))$. Furthermore, let $\bar{a} = (a_1, \dots, a_k)$ be a realisation of the atomic type $s(\bar{x}) \in \mathrm{AT}_j$ in some randomly chosen expansion $\mathfrak{B}$ of $(\mathfrak{A}')^n$. Let σ^+ resp. σ^- be the set of all monadic relation symbols P such that Py resp. $\neg Py$ occurs in t. Because t is realisable, it cannot happen that $\bigcap_{P\in\sigma^+} P^{\mathfrak{A}} = \emptyset$. Furthermore, we must have $|\bigcap_{P\in\sigma^+} P^{\mathfrak{A}}| \geq 2$, because otherwise t could only be realised by a tuple consisting of some uniformly definable element, but then t violates (3). Choose $a, b \in \bigcap_{P\in\sigma^+} P^{\mathfrak{A}}$ with $a \neq b$. Let $\{P_1, \dots, P_r\}$ be a complete enumeration of σ^- without repetitions. For every such $P_j \in \sigma^-$ there must be some $b_j \in \bigcap_{P\in\sigma^+} P^{\mathfrak{A}} \setminus P_j^{\mathfrak{A}}$, because of the same reason that t would not be realizable otherwise.

Now consider the tuples $\bar{d} \coloneqq (b_1, \dots, b_r, \bar{d}')$ where $\bar{d}' \in \{a, b\}^{n-r}$. Every such tuple that is different from $a_1, \dots, a_k$ is a candidate for t, because the conditions (1)-(3) from the definition of AT_j are satisfied. Thus, there are at least $2^{n-r} - k$ many elements that might extend $\bar{a} = (a_1, \dots, a_k)$ to a realisation of t. Each of them has a probability of 2^{-q} of being a realisation of t where q is the number of τ-literals in $t(\bar{x}, y)$ with y. Therefore, the probability that $\bar{a}$ cannot be extended to a realisation of t is at most $(1 - 2^{-q})^{2^{n-r}-k}$. There are at most $|A|^{nk}$ many realisations of s. The probability that one of them cannot be extended is at most

$$|A|^{nk} \cdot (1 - 2^{-q})^{2^{n-r}-k}$$

which tends to 0 exponentially fast as n goes to infinity. Thus $\mu(\mathrm{ext}_{s,t}) = 1$. □

By following the proof of Theorem 1, we obtain an analogous result for the case where the base structure $\mathfrak{A}$ exhibits only monadic relations.

Theorem 10. *Let σ be a vocabulary consisting only of monadic relation symbols, τ be some relational vocabulary and $\mathfrak{A}$ be some finite σ-structure. For every sentence $\psi \in L^\omega(\sigma \cup \tau)$,*

$$\lim_{n\to\infty} \mu_n(\psi) = \frac{r}{2^\ell}, \text{ for some } r \leq 2^\ell$$

where $\mu_n(\psi)$ denotes the probability that a random τ-expansion of $\mathfrak{A}^n$ satisfies ψ. The number ℓ is the number of τ-structures with p elements where p is the number of uniformly definable elements of $\mathfrak{A}$.

In Section 5, we shall see a counterexample to the limit law for the case where $\mathfrak{A}$ is allowed to have binary relations.

4.3 Permutation Structures

Let σ be a vocabulary consisting only of unary function symbols and τ be any relational vocabulary. We say that $\mathfrak{A}$ is a permutation structure, if every $s^{\mathfrak{A}}$ (for $s \in \sigma$) is a permutation of A.

We shall prove that permutation structures admit a limit law. First of all, we observe that $\mathfrak{A}^n$ can be decomposed into disjoint copies of finitely many finite structures that only depend on $\mathfrak{A}$.

Theorem 11. *Let $\mathfrak{A}$ be a finite permutation structure. Then there are finitely many pairwise non-isomorphic finite structures $\mathfrak{B}_1, \dots, \mathfrak{B}_q$ such that every $\mathfrak{A}^n$ is isomorphic to a disjoint union of copies from $\{\mathfrak{B}_1, \dots, \mathfrak{B}_q\}$.*

Proof. Consider some element $a \in A^n$ of $\mathfrak{A}^n$ and let $\mathfrak{A}^n(a)$ be the substructure of $\mathfrak{A}^n$ that is generated by a. Let $\#(a) := |\{a(i) : i = 1, \dots, n\}|$ be the number of pairwise different elements occurring in a. Clearly, we have $1 \leq \#(a) \leq \ell := |A|$. Choose some $b \in A^{\#(a)}$ such that $\{b(1), \dots, b(\#(a))\} = \{a(1), \dots, a(n)\}$ and for every $i \in \{1, \dots, n\}$, let $\iota(i) \in \{1, \dots, \#(a)\}$ be chosen such that $b(\iota(i)) = a(i)$. Notice that b must consist of pairwise different elements. Let $\mathfrak{A}^{\#(a)}(b)$ be the substructure of $\mathfrak{A}^{\#(a)}$ generated by b. We claim that $\mathfrak{A}^n(a)$ is isomorphic to the structure $\mathfrak{A}^{\#(a)}(b)$. Let $A^n(a)$ and $A^{\#(a)}(b)$ denote the universe of $\mathfrak{A}^n(a)$ resp. $\mathfrak{A}^{\#(a)}(b)$. $A^{\#(a)}(b)$ contains all elements of the form $[\![t(b)]\!]^{\mathfrak{A}^{\#(a)}}$, while $A^n(a)$ consists of all $[\![t(a)]\!]^{\mathfrak{A}^n}$ where $t(x)$ is some σ-term. By definition of $\mathfrak{A}^n$ resp. $\mathfrak{A}^{\#(a)}$, we have that $[\![t(b)]\!]^{\mathfrak{A}^{\#(a)}}(i) = [\![t(b(i))]\!]^{\mathfrak{A}}$ and $[\![t(a)]\!]^{\mathfrak{A}^n}(j) = [\![t(a(j))]\!]^{\mathfrak{A}}$. Since $\mathfrak{A}$ is a permutation structure, we can thus conclude that the equality type of a is the same of $[\![t(a)]\!]^{\mathfrak{A}^n}$ and that every $[\![t(b)]\!]^{\mathfrak{A}^{\#(a)}} \in B$ consists of pairwise different elements. Furthermore, we obtain in particular that $[\![t(a)]\!]^{\mathfrak{A}^n}(i) = [\![t(a(i))]\!]^{\mathfrak{A}} = [\![t(b(\iota(i)))]\!]^{\mathfrak{A}} = [\![t(b)]\!]^{\mathfrak{A}^{\#(a)}}(\iota(i))$. As a result, the mapping $\pi \colon A^{\#(a)}(b) \to A^n(a), (c_1, \dots, c_{\#(a)}) \mapsto (c_{\iota(1)}, \dots, c_{\iota(n)})$ is an isomorphism between $\mathfrak{A}^{\#(a)}(b)$ and $\mathfrak{A}^n(a)$. Thus, $\mathfrak{A}$ can be decomposed into disjoint copies of $\mathfrak{A}^k(c)$ where $c \in A^k$ consists of pairwise different elements and $1 \leq k \leq \ell = |A|$. So, there are (up to isomorphism) $q \leq \sum_{k=1}^{\ell} \binom{\ell}{k}$ many such (finite) structures $\mathfrak{B}_1, \dots, \mathfrak{B}_q$ that allow the decomposition of any $\mathfrak{A}^n$. □

For every $i = 1, \dots, q$ let $\#_i(n)$ be the number of disjoint copies of $\mathfrak{B}_i$ occurring in $\mathfrak{A}^n$. If $\#_i(r) \geq 2$ for some $r \in \mathbb{N}$, then there are two different tuples $\bar{a}_1, \bar{a}_2 \in A^r$ with $\mathfrak{B}_i \cong \mathfrak{A}^r(\bar{a}_1)$ and $\mathfrak{B}_i \cong \mathfrak{A}^r(\bar{a}_2)$. Now for $n = k \cdot r$, consider tuples of the form $(\bar{b}_1, \dots, \bar{b}_k) \in A^n$ with $\bar{b}_i \in \{\bar{a}_1, \bar{a}_2\}$ for every $i \geq 1$. Clearly, we have $\mathfrak{B}_i \cong \mathfrak{A}^n(\bar{b}_1, \dots, \bar{b}_k)$ for every such tuple, of which there are at least 2^k many. Thus, $\#_i(n) \geq 2^{\lfloor \frac{n}{2r} \rfloor}$ grows exponentially in n.

Now consider the case where $\#_i(r) \leq 1$ for every $r \in \mathbb{N}$. Clearly, there must be at least one $r \in \mathbb{N}$ with $\#_i(r) = 1$ (otherwise $\mathfrak{B}_i$ would not have been included in the list $\mathfrak{B}_1, \dots, \mathfrak{B}_q$) and, consequently, there exists some tuple $\bar{a} \in A^r$ with $\mathfrak{B}_i \cong \mathfrak{A}^r(\bar{a})$. If $\bar{a}$ would consist of two different elements, then tuples with the same elements as $\bar{a}$ but with different equality type would induce more (even disjoint) copies $\mathfrak{B}_i$ as substructures. This implies that $\bar{a} = (a, \dots, a)$ for some $a \in A$ and, thus, $\mathfrak{B}_i \cong \mathfrak{A}(a) \cong \mathfrak{A}^n(a, \dots, a)$ for every $n \geq 1$. Therefore, we actually have that $\#_i(n) = 1$ for every $n \geq 1$. Furthermore, it must also be the case that $\pi(a) = a$ for every automorphism π of $\mathfrak{A}$, because otherwise we would again find more than one copy of $\mathfrak{B}_i$. Since $\mathfrak{B}_i$ occurs exactly once as a copy in every $\mathfrak{A}^n$ and since a is a fixed point of every automorphism of $\mathfrak{A}$, there is a first-order formula that locates the (unique) copy of $\mathfrak{B}_i$ and defines $(a, \dots, a)$ in it, i.e. a must be uniformly definable.

Thus, for every $i = 1, \dots, q$ we have

(i) either $\#_i(n)$ grows exponentially in n, or
(ii) $\#_i(n) = 1$ for every n and $\mathfrak{B}_i \cong \mathfrak{A}(a) \cong \mathfrak{A}^n(a, \dots, a)$ for some uniformly FO-definable element $a \in A$.

Proposition 12. *Let $\mathfrak{A}$ be a finite permutation structure. Then $\mu(\mathrm{ext}_{s,t}) = 1$ for every $\mathrm{ext}_{s,t} \in T_j, j \leq m$.*

Proof. Let $\mathrm{ext}_{s,t} = \forall \bar{x}(s(\bar{x}) \to \exists y t(\bar{x}, y))$. Furthermore, let $\bar{a} = (a_1, \dots, a_k)$ be a realisation of the atomic type $s(\bar{x}) \in \mathrm{AT}_j$ in some randomly chosen expansion $\mathfrak{B}$ of $(\mathfrak{A}')^n$. (Recall that $\mathfrak{A}'$ is the expansion of $\mathfrak{A}$ with names for the uniformly definable elements.) Let $b \in (\mathfrak{A}')^n$ be an element that satisfies the $(\sigma \cup \sigma_{\mathfrak{A}})$-part of t, i.e. we have $(\mathfrak{A}')^n \models t(\bar{a}, b) \cap \mathrm{FO}(\sigma \cup \sigma_{\mathfrak{A}})$. Such an element must exist (for n sufficiently large), because t is consistent (see also condition (1)). As in the proof of Theorem 11, there must be an index $i \in \{1, \dots, q\}$ such that $\mathfrak{A}^n(b) \cong \mathfrak{B}_i$. It is not possible that $\#_i(n) = 1$ for every $n \in \mathbb{N}$, since otherwise b would have to be a tuple consisting only of some FO-definable element $a = c_j^{\mathfrak{A}'}$ for some $c_j \in \sigma_{\mathfrak{A}}$, but then we would have $y = c_j \in t$ in contradiction to condition (3). Therefore, $\#_i(n)$, the number of occurrences of $\mathfrak{B}_i$, grows exponentially in n. Let m_t be the number of τ-literals in $t(\bar{x}, y)$ with y. The probability that $\mathfrak{B} \models t(\bar{a}, b)$ is 2^{-m_t} and, therefore, the probability that $\bar{a}$ *cannot* be extended to a realisation of t is at most $(1 - 2^{-m_t})^{\#_i(n)-k}$. Since there are at most $|A|^{nk}$ realisations of s, the probability that one of them cannot be extended to a realisation of t is at most

$$|A|^{nk} \cdot (1 - 2^{-m_t})^{\#_i(n)-k}$$

which tends to 0 as n goes to infinity, because $\#_i(n)$ grows exponentially in n. Thus $\mu(\mathrm{ext}_{s,t}) = 1$. □

Again, by following the proof of Theorem 1, we obtain a limit law for permutation structures.

Theorem 13. *Let $\mathfrak{A}$ be a finite permutation structure of vocabulary σ. Then there exists a number m such that for every sentence $\psi \in L^\omega(\sigma \cup \tau)$,*

$$\lim_{n \to \infty} \mu_n(\psi) = \frac{r}{m}, \text{ for some } r \leq m.[3]$$

5 Nonconvergence for Linear Orders

The limit law for random expansions of products of the Abelian groups $\mathbb{Z}_p$ raised the question whether such a limit law could be proved for random expansions of products of *any* finite structure $\mathfrak{A}$.

However, this fails dramatically. Even in the very simple case where $\mathfrak{A} = (\{0, 1\}, \leq)$ we can establish a nonconvergence law, based on Kaufmann's proof

[3] Please recall that m is still the number of atomic $(\tau \cup \sigma_{\mathfrak{A}})$-types that are realisable in $(\sigma \cup \sigma_{\mathfrak{A}} \cup \tau)$-expansions of $\mathfrak{A}'$. These types $\delta_1, \dots, \delta_m$ have been defined in Section 4.1.

of the nonconvergence law for monadic second-order logic on random finite structures [9]. The heart of Kaufmann's argument is the construction of a formula which almost surely defines a linear ordering.

Proposition 14 (Kaufmann). *There exists a first-order formula $\varphi_<(x,y)$ of a vocabulary $\tau \cup \{Y_1,\dots,Y_m\}$ (where τ consists of four binary predicates and the Y_i are monadic) such that on randomly chosen τ-structures with universe $[n]$, the probability that, for some interpretation of $Y_1,\dots Y_m$, the formula $\varphi_<(x,y)$ defines a linear order, converges to 1 as n goes to infinity.*

To obtain an analogous first-order formula on random expansions of $\mathfrak{A}^n$, for $\mathfrak{A} = (\{0,1\},\leq)$, we observe that $\mathfrak{A}^n$ is isomorphic to $(\mathcal{P}([n]),\subseteq)$. A random expansion of $\mathfrak{A}^n$ to a $(\{\leq\}\cup\tau)$-structure $\mathfrak{B}_n$ can thus be equivalently viewed as a random $(\tau\cup\{\subseteq\})$-structure $\mathfrak{C}_n$ with universe $\mathcal{P}([n])$. By restricting the τ-relations of $\mathfrak{C}_n$ to singleton sets we further get a random structure $\mathfrak{D}_n$ with universe $[n]$ where, for each each $R\in\tau$ and $i_1,\dots,i_k\in[n]$,

$$\mathfrak{D}_n \models Ri_1\dots i_k \quad\Longleftrightarrow\quad \mathfrak{C}_n \models R\{i_1\}\dots\{i_k\}.$$

Further, let

$$\mathrm{sing}(x) := \exists z(z \neq x \land \forall y(z \leq y \land (y \leq x \rightarrow (y = z \lor y = x))).$$

Clearly, for $a=(a_1,\dots,a_n)\in\{0,1\}^n$ we have that $\mathfrak{A}^n \models \mathrm{sing}(a)$ if, and only if, $a_i = 1$ for exactly one i, which means that a represents a singleton set of $\mathcal{P}([n])$.

We now translate arbitrary sentences $\varphi\in\mathrm{FO}(\tau\cup\{Y_1,\dots,Y_m\})$ into formulae $\varphi^*(y_1,\dots,y_m)\in\mathrm{FO}(\{\leq\}\cup\tau)$ by the following operations

- replace the set predicates Y_i by new element variables y_i;
- relativise every first-order quantifier Qz to $\mathrm{sing}(z)$, i.e. replace every subformula $\exists z\vartheta$ by $\exists z(\mathrm{sing}(z)\land\vartheta)$ and every subformula $\forall z\vartheta$ by $\forall z(\mathrm{sing}(z)\rightarrow\vartheta)$;
- replace atoms Y_iz by $z\leq y_i$.

By induction on φ, one easily proves the following correspondence.

Lemma 15. *For any expansion of $\mathfrak{A}^n$ to a $(\{\leq\}\cup\tau)$-structure $\mathfrak{B}_n$ and the corresponding τ-structure $\mathfrak{D}_n$ over $[n]$, and for all sets $Y_1,\dots Y_m\subseteq[n]$ we have that*

$$\mathfrak{D}_n \models \varphi(Y_1,\dots,Y_m) \iff \mathfrak{B}_n \models \varphi^*(f(Y_1),\dots,f(Y_m))$$

where $f:\mathcal{P}([n])\rightarrow\{0,1\}^n$ is the above mentioned bijection witnessing the isomorphism between $(\mathcal{P}([n],\subseteq)$ and $\mathfrak{A}^n$.

By applying this translation to the $\mathrm{MSO}(\tau)$-sentence

$$\psi := \exists Y_1\dots\exists Y_m(\text{“}\varphi_<(x,y)\text{ defines a linear order”})$$

we get a first-order sentence $\psi^*\in\mathrm{FO}(\{\leq\}\cup\tau)$ (using first-order quantifiers $\exists y_1\dots\exists y_m$ to simulate $\exists Y_1\dots\exists Y_m$). Since ψ has asymptotic probability 1 on random τ-structures, it follows ψ^* has asymptotic probability 1 on random expansions of $\mathfrak{A}^n$. Notice that the linear order defined by $\varphi^*_<$ in ψ^* is not on the universe of $\mathfrak{A}^n$, but on those elements representing singleton sets in $\mathcal{P}([n])$. By standard constructions we now can get sentences that have no asymptotic probability. For instance, consider instead of ψ the MSO-sentence ψ_{odd}, with an additional existentially quantified set variable Z, saying that

- $\varphi_<(x, y)$ defines a linear order,
- Z contains precisely the even elements of this order,
- the minimal and the maximal element of the order belong to Z.

Translating ψ_{odd} as above results in a sentence $\psi^*_{\text{odd}} \in \text{FO}(\{\leq\} \cup \tau)$ such that $\mu_{2n}(\psi^*_{\text{odd}}) = 0$ for all n, and $\lim_{n\to\infty} \mu_{2n+1}(\psi^*_{\text{odd}}) = 1$. We thus have established the following nonconvergence law.

Theorem 16. *There exists a first-order sentence $\psi^* \in \text{FO}(\{\leq\} \cup \tau)$ such that on random expansions of products of $\mathfrak{A} = (\{0, 1\}, \leq)$, the sequence of probabilities $\mu_n(\psi^*)$ does not converge.*

6 Transferring (Non-)Convergence to Other Structures

In this section we present a method that allows us to transfer (non)convergence laws for structures such as $(\{0, 1\}, \leq)$ to other structures. This method is based on special logical interpretations, only using *positive primitive* formulae. A formula $\varphi \in \text{FO}(\sigma)$ is called *positive primitive*, if it consists only of $\exists, \wedge$ and σ-atoms. The following lemma is an immediate corollary of [8, Lemma 9.1.4].

Lemma 17. *Let $\mathfrak{A}$ be a σ-structure and $\varphi(x_1, \dots, x_r) \in \text{FO}(\sigma)$ a positive primitive formula. Then for every n and every $\bar{a}_1, \dots, \bar{a}_r \in A^n$,*

$$\mathfrak{A}^n \models \varphi(\bar{a}_1, \dots, \bar{a}_r) \iff \mathfrak{A} \models \varphi(\bar{a}_1(i), \dots, \bar{a}_r(i)) \text{ for every } i \in [n].$$

Let σ_1, σ_2 be vocabularies where σ_2 is relational. A *positive primitive interpretation* from σ_1 to σ_2 (of arity k) is a first-order interpretation $\mathcal{I}$ consisting only of positive primitive formulae (and without congruence formula). More precisely, $\mathcal{I}$ is a sequence $(\delta, (\psi_S)_{S\in\sigma_2})$ of positive primitive $\text{FO}(\sigma_1)$-formulae where

- $\delta = \delta(\bar{x})$ is the domain formula, and
- $\psi_S = \psi_S(\bar{x}_1, \dots, \bar{x}_{\text{ar}(S)})$ are the relation formulae for $S \in \sigma_2$.

Here, the tuples $\bar{x}, \bar{y}, \bar{x}_1, \dots$ are of length k respectively. We also write $\text{ar}(\mathcal{I})$ to denote the arity of $\mathcal{I}$, which is here the number k. For the sake of simplicity we always assume σ_2 to be relational, but it is not difficult to generalise these concepts to arbitrary vocabularies.

We say that $\mathcal{I}$ interprets a σ_2-structures $\mathfrak{B}$ in a σ_1-structure $\mathfrak{A}$ (and write $\mathcal{I}(\mathfrak{A}) \cong \mathfrak{B}$) if and only if there exists a bijection h, called the coordinate map, which maps $\delta^{\mathfrak{A}} = \{\bar{a} \in A^k : \mathfrak{A} \models \delta(\bar{a})\}$ to B such that for all $S \in \sigma_2$ and $\bar{a}_1, \ldots, \bar{a}_{\mathrm{ar}(S)} \in \delta^{\mathfrak{A}}$ holds

$$\mathfrak{A} \models \psi_S(\bar{a}_1, \ldots, \bar{a}_{\mathrm{ar}(S)}) \iff (h(\bar{a}_1), \ldots, h(\bar{a}_{\mathrm{ar}(S)})) \in S^{\mathfrak{B}}.$$

This coordinate map $h\colon \delta^{\mathfrak{A}} \to B$ induces coordinate maps $h_n\colon \delta^{\mathfrak{A}^n} \to B^n$ witnessing $\mathcal{I}(\mathfrak{A}^n) \cong \mathfrak{B}^n$. To see this, recall that $\delta^{\mathfrak{A}^n} = \{(\bar{a}_1, \ldots, \bar{a}_k) \in (A^n)^k : \mathfrak{A}^n \models \delta(\bar{a}_1, \ldots, \bar{a}_k)\}$. For every $(\bar{a}_1, \ldots, \bar{a}_k) \in \delta^{\mathfrak{A}^n}$ and every $i \in [n]$, let

$$(h_n(\bar{a}_1, \ldots, \bar{a}_k))(i) \coloneqq h(\bar{a}_1(i), \ldots, \bar{a}_k(i)).$$

Using Lemma 17, it is straightforward (but technical) to verify that this is indeed the definition of a coordinate map for $\mathcal{I}(\mathfrak{A}^n) \cong \mathfrak{B}^n$.

Proposition 18. *Let $\mathcal{I}$ be a positive primitive interpretation with $\mathcal{I}(\mathfrak{A}) \cong \mathfrak{B}$. Then $\mathcal{I}(\mathfrak{A}^n) \cong \mathfrak{B}^n$ for every $n \geq 1$.*

A positive primitive interpretation $\mathcal{I} = (\delta, (\psi_S)_{S \in \sigma_2})$ not only defines copies of σ_2-structures inside σ_1-structures, but it also can be used to convert σ_2-formulae $\varphi(x_1, \ldots, x_\ell)$ into σ_1-formulae $\varphi^{\mathcal{I}}(\bar{x}_1, \ldots, \bar{x}_\ell)$ as follows:

- Replace every variable x by a new k-tuple of variables, denoted by $\bar{x}$.
- Equalities $x = y$ are turned into $\bigwedge_{1 \leq \leq k} x_i = y_i$.
- Turn atoms like $Sx_1 \ldots x_{\mathrm{ar}(S)}$ for $S \in \sigma_2$ into $\psi_S(\bar{x}_1, \ldots, \bar{x}_{\mathrm{ar}(S)})$.
- Replace $\exists x \eta$ and $\forall x \eta$ by $\exists \bar{x}(\delta(\bar{x}) \wedge \eta^{\mathcal{I}})$ resp. $\forall \bar{x}(\delta(\bar{x}) \to \eta^{\mathcal{I}})$.

The connection between φ and $\varphi^{\mathcal{I}}$ is made precise in the following well-known interpretation lemma, which can be adapted for many different logics.

Lemma 19 (Interpretation Lemma for FO**).** *Let $\mathcal{I}(\mathfrak{A}) \cong \mathfrak{B}$ with coordinate map $h\colon \delta^{\mathfrak{A}} \to B$, $\varphi(x_1, \ldots, x_\ell) \in \mathrm{FO}(\sigma_2)$ and $\bar{a}_1, \ldots, \bar{a}_\ell \in \delta^{\mathfrak{A}}$. Then*

$$\mathfrak{A} \models \varphi^{\mathcal{I}}(\bar{a}_1, \ldots, \bar{a}_\ell) \iff \mathfrak{B} \models \varphi(h(\bar{a}_1), \ldots, h(\bar{a}_\ell)).$$

Now let τ be another finite, relational vocabulary disjoint from $\sigma_1 \cup \sigma_2$. A positive primitive interpretation also serves as a bridge between random $(\sigma_2 \cup \tau)$-expansions of $\mathfrak{B}$ and of $(\sigma_1 \cup \tau^\star)$-expansions of $\mathfrak{A}^n$. Here, we use $\tau^\star \coloneqq \{R^\star : R \in \tau\}$ where $R^\star$ is a new relation symbol of arity $\mathrm{ar}(\mathcal{I}) \cdot \mathrm{ar}(R) = k \cdot \mathrm{ar}(R)$ in order to account for the fact that $\mathcal{I}$ operates on k-tuples over $\mathfrak{A}$. Furthermore, let $\mathcal{I}_\tau$ be the result of adding the formulae $\psi_R(\bar{x}_1, \ldots, \bar{x}_{\mathrm{ar}(R)}) \coloneqq R^\star(\bar{x}_1, \ldots, \bar{x}_{\mathrm{ar}(R)})$ for $R \in \tau$ to $\mathcal{I}$.

Using this new interpretation we can now translate a given $(\sigma_2 \cup \tau)$-sentence φ into a $(\sigma_1 \cup \tau^\star)$-sentence $\varphi^{\mathcal{I}_\tau}$. We write $\mu_n^{\mathfrak{B},\tau}(\varphi)$ to denote the probability that a random $(\tau \cup \sigma_2)$-expansion of $\mathfrak{B}^n$ satisfies φ, while $\mu_n^{\mathfrak{A},\tau^\star}(\varphi^{\mathcal{I}_\tau})$ is defined analogously. The connection between $\mu_n^{\mathfrak{B},\tau}(\varphi)$ and $\mu_n^{\mathfrak{A},\tau^\star}(\varphi^{\mathcal{I}_\tau})$ is clarified in the following theorem.

Theorem 20. *Let $\mathcal{I}(\mathfrak{A}) \cong \mathfrak{B}$ for a positive primitive interpretation $\mathcal{I}$ from σ_1 to σ_2. For every sentence $\varphi \in \mathrm{FO}(\sigma_2 \cup \tau)$ and every $n \geq 1$ it holds that $\mu_n^{\mathfrak{A},\tau^\star}(\varphi^{\mathcal{I}_\tau}) = \mu_n^{\mathfrak{B},\tau}(\varphi)$.*

Proof. As in Proposition 18, we have $\mathcal{I}(\mathfrak{A}^n) \cong \mathfrak{B}^n$ witnessed by a coordinate map $h_n \colon \delta^{\mathfrak{A}^n} \to B^n$ for every $n \geq 1$.

For a randomly chosen $(\sigma_1 \cup \tau^\star)$-expansion $\mathfrak{C}$ of $\mathfrak{A}^n$ we obtain a corresponding $(\sigma_2 \cup \tau)$-expansion $\mathfrak{D}$ of $\mathfrak{B}^n$ by setting

$$R^{\mathfrak{D}} \coloneqq \{(h_n(\bar{a}_1), \dots, h_n(\bar{a}_k)) : (\bar{a}_1, \dots, \bar{a}_{\mathrm{ar}(R)}) \in (R^{\mathcal{I}})^{\mathfrak{C}} \cap (\delta^{\mathfrak{A}^n})^{\mathrm{ar}(R)}\}.$$

Then we have $\mathcal{I}_\tau(\mathfrak{C}) \cong \mathfrak{D}$ with coordinate map h_n and, by the interpretation lemma (Lemma 19), it follows that $\mathfrak{C} \models \varphi^{\mathcal{I}_\tau} \iff \mathfrak{D} \models \varphi$. Furthermore, this $(\sigma_2 \cup \tau)$-structure $\mathfrak{D}$ is already uniquely determined by $\mathfrak{C}$.

Conversely, for a randomly chosen $(\sigma_2 \cup \tau)$-expansion $\mathfrak{D}$ of $\mathfrak{B}^n$, we can define a corresponding $(\sigma_1 \cup \tau^\star)$-expansion $\mathfrak{C}$ of $\mathfrak{A}^n$ by setting $(R^\star)^{\mathfrak{C}} \coloneqq h_n^{-1}(R^{\mathfrak{D}})$ where

$$h_n^{-1}(R^{\mathfrak{D}}) \coloneqq \{(\bar{a}_1, \dots, \bar{a}_{\mathrm{ar}(R)}) \in (\delta^{\mathfrak{A}^n})^{\mathrm{ar}(R)} : (h_n(\bar{a}_1), \dots, h_n(\bar{a}_{\mathrm{ar}(R)})) \in R^{\mathfrak{D}}\}$$

for $R \in \tau$. Again, we have $\mathcal{I}_\tau(\mathfrak{C}) \cong \mathfrak{D}$ with coordinate map h_n and, because of the interpretation lemma, we again have $\mathfrak{C} \models \varphi^{\mathcal{I}_\tau} \iff \mathfrak{D} \models \varphi$. Please notice that we could also define $\mathfrak{C}$ differently in the case that $\delta^{\mathfrak{A}^n} \neq A^n$, because then $(R^\star)^{\mathfrak{C}}$ could theoretically contain tuples with elements from $A^n \setminus \delta^{\mathfrak{A}^n}$. However, for every $\mathfrak{D}$ we would have exactly the same number of possibilities. Thus, $\mu_n^{\mathfrak{A},\tau^\star}(\varphi^{\mathcal{I}_\tau}) = \mu_n^{\mathfrak{B},\tau}(\varphi)$ follows. □

Please recall that a finite σ-structure $\mathfrak{A}$ has a *limit law*, if for every finite relational vocabulary τ and every sentence $\varphi \in \mathrm{FO}(\sigma \cup \tau)$, $\mu^{\mathfrak{A},\tau}(\varphi) \coloneqq \lim_{n\to\infty} \mu_n^{\mathfrak{A},\tau}(\varphi)$ exists. Otherwise, we say that $\mathfrak{A}$ has no limit law.

Corollary 21. *Let σ_1, σ_2 be vocabularies where σ_2 is relational. Let $\mathfrak{A}, \mathfrak{B}$ be finite structures with $\mathcal{I}(\mathfrak{A}) \cong \mathfrak{B}$ for a positive primitive logical interpretation $\mathcal{I}$ from σ_1 to σ_2 without equality formula. Then:*

(i) If $\mathfrak{A}$ has a limit law, then $\mathfrak{B}$ has a limit law.
(ii) If $\mathfrak{B}$ has no limit law, then neither does $\mathfrak{A}$.

Proof. Since (ii) is just the contraposition of (i), it suffices to prove only one of these items. Towards proving (ii), assume that $\mathfrak{B}$ does not have a limit law. Thus, for some finite, relational vocabulary τ there exists a sentence $\psi \in \mathrm{FO}(\sigma_2 \cup \tau)$ such that $\mu^{\mathfrak{B},\tau}(\psi) = \lim_{n\to\infty} \mu_n^{\mathfrak{B},\tau}(\psi)$ does not exist. By Theorem 20, it follows that $\mu^{\mathfrak{A},\tau^\star}(\psi^{\mathcal{I}_\tau}) \coloneqq \lim_{n\to\infty} \mu_n^{\mathfrak{A},\tau^\star}(\psi) = \lim_{n\to\infty} \mu_n^{\mathfrak{B},\tau}(\psi)$ does not exist as well. Therefore, $\mathfrak{A}$ has no limit law. □

The following two examples demonstrate how Corollary 21 can be used to transfer nonconvergence laws to other structures.

Example 22. The structure $\mathfrak{A}_1 \coloneqq (\{0,1\}, f_{\leq}^{\mathfrak{A}_1})$ with $f_{\leq}^{\mathfrak{A}_1}(a,b) = 1 \iff a \leq b$ inherits the nonconvergence law of $(\{0,1\}, \leq)$, because $\mathcal{I} \coloneqq (\delta(x), \psi_{\leq}(x,y))$ where

$$\delta(x) \coloneqq x = x$$
$$\psi_{\leq}(x,y) \coloneqq f_{\leq}(x,y) = f_{\leq}(x,x)$$

is a positive primitive logical interpretation (without equality formula) with

$$\mathcal{I}(\mathfrak{A}_1) \cong (\{0,1\}, \leq).$$

By Corollary 21(ii), it follows that $\mathfrak{A}_1$ has no limit law.

The next example shows that even structures that are only equipped with unary functions may have a nonconvergence law.

Example 23. Consider $\mathfrak{A}_2 := (A_2, (s^{\mathfrak{A}_2}_{a\mapsto b})_{a,b\in A_2, a\neq b}, s^{\mathfrak{A}_2}_{\{0,1\}})$ where

- $A_2 := \{0, 1, 0'\}$,
- $s^{\mathfrak{A}_2}_{a\mapsto b}(a) := b$ and $s^{\mathfrak{A}_2}_{a\mapsto b}(c) := c$ for every $a, b, c \in A_2$ with $c \neq a$,
- $s^{\mathfrak{A}_2}_{\{0,1\}}(a) = 1$ for $a \in \{0, 1\}$ while $s^{\mathfrak{A}_2}_{\{0,1\}}(0') = 0$.

Here is a positive primitive logical interpretation $\mathcal{I} = (\delta(x), \psi_{\leq})$ with $\mathcal{I}(\mathfrak{A}_2) \cong (\{0,1\}, \leq)$:

$$\begin{aligned}\delta(x) &:= s_{0'\mapsto 0}(x) = x \\ \psi_{\leq}(x,y) &:= \exists z(s_{0'\mapsto 0}(z) = x \wedge s_{0'\mapsto 0}(s_{0\mapsto 1}(z)) = y).\end{aligned}$$

Applying Corollary 21(ii) yields that $\mathfrak{A}_2$ has no limit law.

7 Future Work

While we have analysed some structures with respect to limit laws and introduced a new method to transfer (non)convergence laws between structures, the question of what structures have such a limit law is not fully settled. There are many structures for which we do not know whether or not they have a limit law and this paper is a first step towards a complete characterisation of structures with limit laws.

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